Symmetry analysis of the cylindrical Laplace equation

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Abstract. The symmetry analysis for Laplace equation on cylinder is considered. Symmetry algebra, the structure of the Lie algebra of the symmetries and some related topics such as invariant solutions, one-parameter subgroups, one dimensional optimal system and differential invariants are given.

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1 Introduction

Pierre-Simon, marquis de Laplace (March 23, 1749 - March 5, 1827) was a French mathematician and astronomer whose work was pivotal to the development of mathematical astronomy. He summarized and extended the work of his predecessors in his five volume *Mcanique Cleste* (Celestial Mechanics) (1799-1825). This seminal work translated the geometric study of classical mechanics, used by Isaac Newton, to one based on calculus, opening up a broader range of problems. He formulated Laplace's equation, and invented the Laplace transform which appears in many branches of mathematical physics, a field that he took a leading role in forming. The Laplacian differential operator, widely used in applied mathematics, is also named after him.

In mathematics and physics, the *Laplace operator* or *Laplacian*, denoted by Δ or ∇^2 and named after Pierre-Simon de Laplace, is a differential operator, specifically an important case of an *elliptic operator*, with many applications. In physics, it is used in *modeling* of *wave propagation* (the wave equation is an important second-order linear partial differential equation that describes the propagation of a variety of waves, such as sound waves, light waves and water waves. It arises in fields such as acoustics, electromagnetics, and fluid dynamics. Historically, the problem of a vibrating string such as that of a musical instrument was studied by *Jean le Rond d'Alembert*, *Leonhard Euler*, *Daniel Bernoulli*, and *Joseph-Louis Lagrange*.), an important partial differential equation which describes the distribution of heat (or variation in temperature) in a given region over time. For a function of three spatial variables (x, y, z) and one time

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variable t; *heat flow*, forming the *Helmholtz equation*. It is central in *electrostatics* and *fluid mechanics*, anchoring in *Laplace's equation* and *Poisson's equation*. In *quantum mechanics*, it represents the *kinetic energy* term of the *Schrödinger equation*. The kinetic energy of an object is the extra energy which it possesses due to its motion. It is defined as the work needed to accelerate a body of a given mass from rest to its current velocity. Having gained this energy during its acceleration, the body maintains this kinetic energy unless its speed changes. Negative work of the same magnitude would be required to return the body to a state of rest from that velocity. In mathematics, functions with vanishing Laplacian are called *harmonic functions*; the Laplacian is at the core of *Hodge theory* and the results of *de Rham cohomology* [9].

The aim is to analysis the point symmetry structure of the Laplace equation on cylinder, i.e., cylindrical Laplace equation, which is

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\Big(r\frac{\partial u}{\partial r}\Big) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} = 0, \tag{1.1}$$

where u is a smooth function of (r, θ, z).

The symmetry algebra of (1.1) is determined in the next section and some results yield from the structure of the Lie algebra of symmetries are given.

2 Lie symmetries of the equation

The method of determining the classical symmetries of a partial differential equation is standard which is described in [1, 4, 5] and [8]. To obtain the symmetry algebra of the (1.1), we take an infinitesimal generator of symmetry algebra of the form

$$X = \xi_1(r,\theta,z,u)\frac{\partial}{\partial r} + \xi_2(r,\theta,z,u)\frac{\partial}{\partial \theta} + \xi_3(r,\theta,z,u)\frac{\partial}{\partial z} + \eta(r,\theta,z,u)\frac{\partial}{\partial u}.$$

Using the invariance condition, i.e., applying the second prolongation $X^{(2)}$ to (1.1), the following system of 18 determining equations yields:

$$\begin{array}{lll}
\xi_{2u} = 0, & \xi_{3rr} + \xi_{3zz} = 0, & \xi_1 + r(\xi_{2\theta} - \xi_{3z}) = 0,\\
\xi_{3u} = 0, & \xi_{3\theta} + r^2\xi_{2z} = 0, & \xi_{2\theta} + r(\xi_{2r\theta} - \xi_{3rz}) = 0,\\
\eta_{uu} = 0, & \xi_{3\theta} - r\xi_{3r\theta} = 0, & \xi_{2\theta\theta} - r\xi_{2r} - \xi_{3\theta z} = 0,\\
\xi_{3rzz} = 0, & \xi_{3rz} + 2\eta_{ru} = 0, & \xi_{3\theta\theta} + r(\xi_{3r} + r\xi_{3zz}) = 0,\\
\xi_{3\theta zz} = 0, & \xi_{3\theta z} + 2\eta_{\theta u} = 0, & \xi_{3\theta z} + r(2\xi_{2r} - r\xi_{2rr}) = 0,\\
\xi_{3zzz} = 0, & \xi_{3zz} + 2\eta_{zu} = 0, & \eta_{\theta\theta} + r(\eta_r + r\eta_{rr} + r\eta_{zz}) = 0.
\end{array}$$

The solution of the above system gives the following infinitesimals,

$$\begin{array}{rcl}
X_1 &=& \dfrac{\partial}{\partial \theta}, \qquad X_2 = \dfrac{\partial}{\partial z}, \qquad X_3 = u\dfrac{\partial}{\partial u},\\
X_4 &=& r\dfrac{\partial}{\partial r} + z\dfrac{\partial}{\partial z}, \qquad X_5 = rz\dfrac{\partial}{\partial r} + \dfrac{1}{2}(z^2 - r^2)\dfrac{\partial}{\partial z} - \dfrac{1}{2}zu\dfrac{\partial}{\partial u},\\
X_6 &=& \sin\theta\dfrac{\partial}{\partial r} + \dfrac{1}{r}\cos\theta\dfrac{\partial}{\partial \theta}, \qquad X_7 = \cos\theta\dfrac{\partial}{\partial r} - \dfrac{1}{r}\sin\theta\dfrac{\partial}{\partial \theta},
\end{array}$$

$$
\begin{aligned}
X_8 &= z\sin\theta\frac{\partial}{\partial r} + \frac{1}{r}z\cos\theta\frac{\partial}{\partial\theta} - r\sin\theta\frac{\partial}{\partial z},\\
X_9 &= z\cos\theta\frac{\partial}{\partial r} - \frac{1}{r}z\sin\theta\frac{\partial}{\partial\theta} - r\cos\theta\frac{\partial}{\partial z},\\
X_{10} &= \frac{1}{2}(z^2-r^2)\sin\theta\frac{\partial}{\partial r} + \frac{1}{2r}(z^2+r^2)\cos\theta\frac{\partial}{\partial\theta} - rz\sin\theta\frac{\partial}{\partial z} + \frac{1}{2}ru\sin\theta\frac{\partial}{\partial u},\\
X_{11} &= \frac{1}{2}(z^2-r^2)\cos\theta\frac{\partial}{\partial r} - \frac{1}{2r}(z^2+r^2)\sin\theta\frac{\partial}{\partial\theta} - rz\cos\theta\frac{\partial}{\partial z} + \frac{1}{2}ru\cos\theta\frac{\partial}{\partial u}.
\end{aligned}
$$

The commutation relations of the 11−dimensional Lie algebra g spanned by the vector fields $X_1, \cdots, X_{11}$, are shown in following table.

	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8	X_9	X_{10}	X_{11}
X_1	0	0	0	0	0	X_7	$-X_6$	X_9	$-X_8$	X_{11}	$-X_{10}$
X_2	0	0	0	X_2	$-\frac{1}{2}X_3+X_4$	0	0	X_6	X_7	X_8	X_9
X_3	0	0	0	0	0	0	0	0	0	0	0
X_4	0	$-X_2$	0	0	X_5	$-X_7$	$-X_7$	0	0	X_{10}	X_{11}
X_5	0	$\frac{1}{2}X_3-X_4$	0	$-X_5$	0	$-X_8$	$-X_9$	$-X_{10}$	$-X_{11}$	0	0
X_6	$-X_7$	0	0	X_7	X_8	0	0	$-X_2$	0	$\frac{1}{2}X_3-X_4$	$-X_1$
X_7	X_6	0	0	X_7	X_9	0	0	0	$-X_2$	X_1	$\frac{1}{2}X_3-X_4$
X_8	$-X_9$	$-X_6$	0	0	X_{10}	X_2	0	0	X_1	$-X_5$	0
X_9	X_8	$-X_7$	0	0	X_{11}	0	X_2	$-X_1$	0	0	$-X_5$
X_{10}	$-X_{11}$	$-X_8$	0	$-X_{10}$	0	$-\frac{1}{2}X_3+X_4$	$-X_1$	X_5	0	0	0
X_{11}	X_{10}	$-X_9$	0	$-X_{11}$	0	X_1	$-\frac{1}{2}X_3+X_4$	0	X_5	0	0

2.1 Differential invariants

Suppose that G is a transformation group. It is well known that a smooth real differential function $I : J^n \to \mathbb{R}$, where J^n is the corresponding $n-$th jet space, is a *differential invariant* for G if and only if for all $X \in$ g, its $n-$th prolongation annihilates I, i.e., $X^{(n)}(I) = 0$. Here we are going to find the differential invariants of order 0, 1 and 2 and some of their results. For more details and some practical examples see [3] and [8].

The symmetry group of the cylindrical Laplace equation has not any nonconstant ordinary differential invariants, i.e., it does not have any differential invariants of order zero, thus the orbit of generic points (the points which lies in the orbit of maximal dimension) has dimension four and its isotropy subgroup is seven dimensional.

The first prolonged action of the symmetry group has one differential invariant, which is a function of (u, u_r, u_θ, u_z), so the action has a generic orbit of dimension six, and its isotropy subgroup is five dimensional.

Finally the second prolonged action has another differential invariant which is a function of $(u, u_r, u_\theta, u_z, u_{rr}, u_{r\theta}, u_{rz}, u_{\theta\theta}, u_{\theta z}, u_{zz})$, consequently the dimension of the generic orbit is equal to the dimension of the group; thus, the second prolonged action is transitive and has a discrete isotropy subgroup.

According to the relations between above subjects and [4], we can obtain the number of functionally independent differential invariants up to order n.

Theorem 2.1. *The number of functionally independent differential invariants up to order $n \geq 3$ for the symmetry group of cylindrical Laplace equation is $\binom{3+n}{n} - 8$ and the number of strictly differential invariants which occur in each prolongation is $\binom{3+n}{n} - \binom{2+n}{n-1}$.*

Proof. According to the above discussion for $n \geq 3$ the action is transitive, so the number of independent differential invariants i_n, is

$$i_n = 3 + \binom{3+n}{n} - 11 = \binom{3+n}{n} - 8.$$

If j_n is the number of strictly differential invariants then,

$$j_n = \binom{3+n}{n} - \binom{3+n-1}{n-1} = \binom{3+n}{n} - \binom{2+n}{n-1}. \qquad \square$$

2.2 Reduction of the equation

The Eq. (1.1) maybe regarded as a submanifold M of the jet space $J^2(\mathbb{R}^3, \mathbb{R})$. Doing as section 3.1 of [5] and find (in a sense) the most general group-invariant solutions to the Eq. (1.1).

Theorem 2.2. *The one-parameter groups $g_i(t) : M \to M$ generated by the X_i, $i = 1, \cdots, 11$, are given in the following table:*

$$\begin{aligned}
g_1(t) &: (r,\theta,z,u) \longmapsto (r, \theta + t, z, u),\\
g_2(t) &: (r,\theta,z,u) \longmapsto (r, \theta, z + t, u),\\
g_3(t) &: (r,\theta,z,u) \longmapsto (r, \theta, z, ue^t),\\
g_4(t) &: (r,\theta,z,u) \longmapsto (re^t, \theta, ze^t, u),\\
g_5(t) &: (r,\theta,z,u) \longmapsto \Big(rzt + r, \theta, \frac{t}{2}(z^2 - r^2), -\frac{1}{2}zut + u\Big),\\
g_6(t) &: (r,\theta,z,u) \longmapsto \Big(t\sin\theta + r, \frac{t}{r}\cos\theta + \theta, z, u\Big),\\
g_7(t) &: (r,\theta,z,u) \longmapsto \Big(t\cos\theta + r, \frac{t}{r}\sin\theta + \theta, z, u\Big),\\
g_8(t) &: (r,\theta,z,u) \longmapsto \Big(tz\sin\theta + r, \frac{t}{r}z\cos\theta + \theta, -rt\sin\theta + z, u\Big),\\
g_9(t) &: (r,\theta,z,u) \longmapsto \Big(tz\cos\theta + r, -\frac{t}{r}z\sin\theta + \theta, -rt\cos\theta + z, u\Big),\\
g_{10}(t) &: (r,\theta,z,u) \longmapsto \bigg(\frac{t}{2}(z^2 - r^2)\sin\theta, \frac{t}{2}\Big(\frac{z^2}{r} + r\Big)\cos\theta + \theta,\\
&\qquad\qquad -rzt\sin\theta + z, \frac{1}{2}rut\sin\theta + u\bigg),\\
g_{11}(t) &: (r,\theta,z,u) \longmapsto \bigg(\frac{t}{2}(z^2 - r^2)\cos\theta, -\frac{t}{2}\Big(\frac{z^2}{r} + r\Big)\sin\theta + \theta,\\
&\qquad\qquad -rzt\cos\theta + z, \frac{1}{2}rut\cos\theta + u\bigg),
\end{aligned}$$

where entries give the transformed point $\exp(tX_i)(r,\theta,z,u) = (\tilde{r},\tilde{\theta},\tilde{z},\tilde{u})$.

The transformations g_1 and g_2 demonstrate the radius- and space-invariance and radius- and angle-invariance solutions of the equation. The one-parameter g_3 reflects the linearity of the cylindrical laplace equation, and the well-known scaling symmetry turns up in g_4. The most general one-parameter group of symmetries is obtained by considering a general linear combination of the given vector fields; the explicit formula for the group transformations is very complicated, but alternatively we can use theorem 2.52 in [4], and represent an arbitrary group transformation g as the composition of transformations in theorem (2.2).

Theorem 2.3. *Let $i = 1,\cdots,11$ and $t \in \mathbb{R}$. If $u = U(r,\theta,z)$ is a solution of Eq. (1.1) so are the functions $u^i(r,\theta,z) = U(r,\theta,z)$, where*

$$\begin{aligned}
u^1 &= U(r,\theta+t,z), \qquad u^2 = U(r,\theta,z+t),\\
u^3 &= e^{-t}U(r,\theta,z), \qquad u^4 = U(e^t r,\theta,e^t z),\\
u^5 &= \Big(1+\frac{1}{2}zt\Big)U(rzt+r,\theta,\frac{t}{2}(z^2-r^2)),\\
u^6 &= U(t\sin\theta+r,\frac{t}{r}\cos\theta+\theta,z), \qquad u^7 = U\Big(t\cos\theta+r,\frac{t}{r}\cos\theta+\theta,z\Big),\\
u^8 &= U\Big(tz\sin\theta+r,\frac{t}{r}z\cos\theta+\theta,-rt\sin\theta+z\Big),\\
u^9 &= U\Big(tz\cos\theta+r,-\frac{t}{r}z\sin\theta+\theta,-rt\cos\theta+z\Big),\\
u^{10} &= \Big(1-\frac{rt}{2}\sin\theta\Big)U\left(\frac{t}{2}(z^2-r^2)\sin\theta,\frac{t}{2r}(z^2+r^2)\cos\theta+\theta,z-rzt\sin\theta\right),\\
u^{11} &= \Big(1-\frac{rt}{2}\cos\theta\Big)U\left(\frac{t}{2}(z^2-r^2)\cos\theta,\frac{t}{2r}(z^2+r^2)\sin\theta+\theta,z-rzt\cos\theta\right).
\end{aligned}$$

For example, if $u(r,\theta,z) = 1$ be a constant solution of Eq. (1.1), we conclude that the trivial functions $g_i(t)\cdot 1 = 1$ for $i = 1,2,4,6,7,8,9$, $g_3(t)\cdot 1 = e^{-t}$ and $g_5(t)\cdot 1 = 2+zt$, and by applying $g_5\Big(\frac{2a}{b}\Big)\cdot g_3(\ln b)$ on $u = 1$, we conclude the linear solution $u = az+b$. The two nontrivial solutions for Eq. (1.1) are:

$$g_{10}(t)\cdot 1 = 2 - rt\sin\theta, \quad g_{11}(t)\cdot 1 = 2 - rt\cos\theta.$$

Now by summing g_1, $g_5\Big(\frac{-8+2c}{d}\Big)\cdot g_3\ln d\cdot 1$, $g_{10}(-a)\cdot 1$ and $g_{11}(-b)\cdot 1$, we conclude the following result:

Corollary 2.4. *The function $u(r,\theta,z) = ar\sin\theta + br\cos\theta + dz + c$, is a solution of Eq. (1.1), where a,b,c and d are arbitrary constants.*

2.3 Some invariant solutions

The first advantage of symmetry group method is to construct new solutions from known solutions. Neither the first advantage nor the second will be investigated here, but symmetry group method will be applied to the Eq. (1.1) to be connected directly to some order differential equations. To do this, a particular linear combinations of infinitesimals are considered and their corresponding invariants are determined.

The cylindrical Laplace equation expressed in the coordinates (r, θ, z), so to reduce this equation is to search for its form in specific coordinates. Those coordinates will be constructed by searching for independent invariants (y, v) corresponding to an infinitesimal generator. so using the chain rule, the expression of the equation in the new coordinate allows us to the reduced equation. Here we will obtain some invariant solutions with respect to symmetries.

1. First consider X_6. To determine the independent invariants I, we need to solve the first order partial differential equation $X_6\{I(r, \theta, z, u)\} = 0$, that is

$$\sin\theta\frac{\partial I}{\partial r} + \frac{\cos\theta}{r}\frac{\partial I}{\partial \theta} + 0\frac{\partial I}{\partial z} + 0\frac{\partial I}{\partial u} = 0,$$

which is a homogeneous first order PDE. Thus we solve the associated characteristic ordinary differential equations

$$\frac{dr}{\sin\theta} = \frac{\cos\theta d\theta}{r} = \frac{dz}{0} = \frac{du}{0}.$$

Hence we obtain two functionally independent invariants $y = \dfrac{z}{r\sin\theta}$ and $v = u$.

If we treat v as a function of y, we can compute formulae for the derivatives of u with respect to x and t in terms of y, v and the derivatives of v with respect to y, along with a single parametric variable, which we designate to be t, so that x will be the corresponding principal variable. Using the chain rule we find that if $u = v = v(y) = v(z/r\sin\theta)$, then

$$u_r = v_y y_r, \tag{2.1}$$
$$u_{rr} = y_{rr}v_y + v_{yy}y_r, \tag{2.2}$$
$$u_{\theta\theta} = y_{\theta\theta}v_y + v_{yy}y_\theta, \tag{2.3}$$
$$u_{zz} = y_{zz}v_y + v_{yy}y_z. \tag{2.4}$$

Substituting $u_r, u_{rr}, u_{\theta\theta}$ and u_{zz} in (1.1) we obtain an ODE with solution $u = C_1 \arctan\left(\frac{z}{r\sin\theta}\right) + C_2$, which is called an invariant solution for Eq. (1.1).

2. Similarly, the invariants of X_7 are $y = \dfrac{z}{r\cos\theta}$ and $v = u$, and the solution of reduced equation (invariant solution) by substituting (2.1), (2.2), (2.3) and (2.4) is $u = C_1 \arctan\left(\frac{z}{r\cos\theta}\right) + C_2$.

3. The invariants of X_8 are $\dfrac{r^2+z^2}{r\sin\theta}$ and $v = u$. The invariant solution constructed with (2.1), (2.2), (2.3) and (2.4) is $u = C_1\ \text{arctanh}\left(\frac{\sqrt{r^2+z^2}}{r\sin\theta}\right) + C_2$ obtained from ordinary differential equations

$$\frac{dr}{z\sin\theta} = \frac{rd\theta}{a\cos\theta} = \frac{dz}{-r\sin\theta} = \frac{du}{0}.$$

The above mentioned method in sections **1** and **2** can be followed for X_9 with invariants $\dfrac{r^2+z^2}{r\sin\theta}$ and $v = u$, and invariant solution $u = C_1 \operatorname{arctanh}\left(\frac{\sqrt{r^2+z^2}}{r\cos\theta}\right) + C_2$. We can proceed for other X_is as above.

2.4 The Lie algebra of symmetries

In this section, we determine the structure of full symmetry Lie algebra g, of Eq. (1.1).

Theorem 2.5. *The full symmetry Lie algebra* g*, of Eq. (1.1) has the following semidirect decomposition:*

$$\mathrm{g} = \mathbb{R} \ltimes \mathrm{g}_1, \tag{2.5}$$

where g_1 *is a simple Lie algebra.*

Proof: The center z of g is $\mathrm{Span}_{\mathbb{R}}\{X_3\}$. Therefore the quotient algebra $\mathrm{g}_1 = \mathrm{g}/\mathrm{z}$ is $\mathrm{Span}_{\mathbb{R}}\{Y_1, \cdots, Y_{10}\}$, where $Y_i := X_i + \mathrm{z}$ for $i = 1, \cdots, 10$. The commutator table of this quotient algebra is given in following table:

	Y_1	Y_2	Y_3	Y_4	Y_5	Y_6	Y_7	Y_8	Y_9	Y_{10}
Y_1	0	0	0	0	Y_7	$-Y_5$	Y_8	$-Y_7$	Y_{10}	$-Y_9$
Y_2	0	0	Y_2	Y_3	0	0	Y_5	Y_6	Y_7	Y_8
Y_3	0	$-Y_2$	0	Y_4	$-Y_5$	$-Y_6$	0	0	Y_9	Y_{10}
Y_4	0	$-Y_3$	$-Y_4$	0	$-Y_7$	$-Y_8$	$-Y_9$	$-Y_{10}$	0	0
Y_5	$-Y_7$	0	Y_5	Y_7	0	0	$-Y_2$	0	$-Y_3$	$-Y_1$
Y_6	Y_5	0	Y_6	Y_8	0	0	0	$-Y_2$	Y_1	$-Y_3$
Y_7	$-Y_8$	$-Y_5$	0	Y_9	Y_2	0	0	Y_1	$-Y_4$	0
Y_8	Y_7	$-Y_6$	0	Y_{10}	0	Y_2	$-Y_1$	0	0	$-Y_4$
Y_9	$-Y_{10}$	$-Y_7$	$-Y_9$	0	Y_3	$-Y_1$	Y_4	0	0	0
Y_{10}	Y_9	$-Y_8$	$-Y_{10}$	0	Y_1	Y_3	0	$-Y_4$	0	0

The Lie algebra g is non-solvable, because

$$\begin{aligned} \mathrm{g}^{(1)} &= [\mathrm{g}, \mathrm{g}] = \mathrm{Span}_{\mathbb{R}}\left\{X_1, X_2, \frac{1}{2}X_3 - X_4, X_5, \cdots, X_{11}\right\}, \\ \mathrm{g}^{(2)} &= [\mathrm{g}^{(1)}, \mathrm{g}^{(1)}] = \mathrm{g}^{(1)}. \end{aligned}$$

Similarly, g_1 is semi-simple and non-solvable, because

$$\mathrm{g}_1^{(1)} = [\mathrm{g}_1, \mathrm{g}_1] = \mathrm{Span}_{\mathbb{R}}\{Y_1, \cdots, Y_{10}\} = \mathrm{g}_1.$$

The Lie algebra g admits a *Levi decomposition* as the following semi-direct product $\mathrm{g} = \mathrm{r} \ltimes \mathrm{s}$, where $\mathrm{r} = \mathrm{Span}_{\mathbb{R}}\{X_3\}$ is the radical of g (the largest solvable ideal contained in g), and

$$\mathrm{s} = \left\{ X_1, X_2, \frac{1}{2}X_3 - X_4, X_5, \cdots X_{11} \right\}.$$

r is a one-dimensional subalgebra of g, therefore it is isomorphic to $\mathbb{R}$; Thus the identity $\mathrm{g} = \mathrm{r} \ltimes \mathrm{s}$ reduces to $\mathrm{g} = \mathbb{R} \ltimes \mathrm{g}_1$. □

2.5 Optimal system of sub-algebras

As is well known, the theoretical Lie group method plays an important role in finding exact solutions and performing symmetry reductions of differential equations. Since any linear combination of infinitesimal generators is also an infinitesimal generator, there are always infinitely many different symmetry subgroups for the differential equation. So, a mean of determining which subgroups would give essentially different types of solutions is necessary and significant for a complete understanding of the invariant solutions. As any transformation in the full symmetry group maps a solution to another solution, it is sufficient to find invariant solutions which are not related by transformations in the full symmetry group, this has led to the concept of an optimal system [6]. The problem of finding an optimal system of subgroups is equivalent to that of finding an optimal system of subalgebras. For one-dimensional subalgebras, this classification problem is essentially the same as the problem of classifying the orbits of the adjoint representation. This problem is attacked by the naive approach of taking a general element in the Lie algebra and subjecting it to various adjoint transformations so as to simplify it as much as possible. The idea of using the adjoint representation for classifying group-invariant solutions is due to [5] and [6].

The adjoint action is given by the Lie series

$$\mathrm{Ad}(\exp(tY_i)Y_j) = Y_j - t[Y_i, Y_j] + \frac{t^2}{2}[Y_i, [Y_i, Y_j]] - \cdots, \tag{2.6}$$

where $[Y_i, Y_j]$ is the commutator for the Lie algebra, t is a parameter, and $i, j = 1, \cdots, 10$. We can write the adjoint action for the Lie algebra g_1, and show that

Theorem 2.6. *A one dimensional optimal system of the* g_1 *is given by*

1) Y_1,	8) $aY_1 + bY_4$,	15) $aY_7 + bY_8$,
2) Y_3,	9) $aY_1 + bY_7$,	16) $aY_1 + bY_4 + cY_9$,
3) Y_4,	10) $aY_1 + bY_8$	17) $aY_1 + bY_3 + cY_8$,
4) Y_7,	11) $aY_1 + bY_9$,	18) $aY_1 + bY_7 + cY_8$,
5) Y_8,	12) $aY_3 + bY_7$,	19) $aY_3 + bY_7 + cY_8$,
6) Y_9,	13) $aY_3 + bY_8$,	20) $aY_1 + bY_3 + cY_7$,
7) $aY_1 + bY_3$,	14) $aY_4 + bY_9$,	21) $aY_1 + bY_3 + cY_7 + dY_8$,

where $a, b, c, d \in \mathbb{R}$ *are arbitrary numbers.*

Proof. $F_i^t : \mathrm{g}_1 \to \mathrm{g}_1$ defined by $Y \mapsto \mathrm{Ad}(\exp(tY_i)Y)$ is a linear map, for $i = 1, \cdots, 10$. The matrices M_i^t of F_i^t, $i = 1, \cdots, 10$, with respect to basis $\{Y_1, \cdots, Y_{10}\}$ are

$$
M_1^t = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & \sin t & 1-\cos t & 0 & 0 \\
0 & 0 & 0 & 0 & t & 1 & \cos t - 1 & t - \sin t & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \cos t & -\sin t & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \sin t & \cos t & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cos t & -\sin t \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sin t & \cos t
\end{pmatrix},
$$

$$
M_2^t = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -t & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & t^2/2 & -t & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -t & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -t & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & t^2/2 & 0 & -t & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & t^2/2 & 0 & -t & 0 & 1
\end{pmatrix},
$$

$$
M_3^t = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & e^t-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & e^{-t}-1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & e^t-1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & e^t-1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{-t}-1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{-t}-1
\end{pmatrix},
$$

$$
M_4^t = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & t & t^2/2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & t & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & t & 0 & t^2/2 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & t & 0 & t^2/2 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & t & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & t \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix},
$$

$$
M_5^t = \begin{pmatrix}
1 & t^2/2 & 0 & 0 & 0 & 0 & t & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & -t & 0 & 0 & 0 & 0 & 0 \\
0 & -t^2/2 & 0 & 1 & 0 & 0 & -t & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & t & 0 & -t^2/2 & 0 & 0 & 0 & 1 & 0 \\
t & t^3/6 & 0 & 0 & 0 & 0 & t^2/2 & 0 & 0 & 1
\end{pmatrix},
$$

$$M_6^t = \begin{pmatrix} 1 & 0 & 0 & 0 & -t & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -t & 0 & 0 & 0 & 0 \\ 0 & -t^2/2 & 0 & 1 & 0 & 0 & 0 & -t & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -t & 0 & 0 & 0 & t^2/2 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & t & 0 & 0 & t^2/2 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$M_7^t = \begin{pmatrix} \cos t & 0 & 0 & 0 & 0 & 0 & t & \sin t & 0 & 0 \\ 0 & \cos t & 0 & 0 & \sin t & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos t & 0 & 0 & 0 & 0 & -\sin t & 0 \\ 0 & -\sin t & 0 & 0 & \cos t & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -\sin t & 0 & 0 & 0 & 0 & 0 & 0 & \cos t & 0 & 0 \\ 0 & 0 & 0 & \sin t & 0 & 0 & 0 & 0 & \cos t & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$M_8^t = \begin{pmatrix} \cos t & 0 & 0 & 0 & 0 & 0 & -\sin t & 0 & 0 & 0 \\ 0 & \cos t & 0 & 0 & 0 & \sin t & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos t & 0 & 0 & 0 & 0 & 0 & -\sin t \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\sin t & 0 & 0 & 0 & \cos t & 0 & 0 & 0 & 0 \\ \sin t & 0 & 0 & 0 & 0 & 0 & \cos t & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \sin t & 0 & 0 & 0 & 0 & 0 & \cos t \end{pmatrix},$$

$$M_9^t = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & t \\ 0 & 1 & 0 & -t^2/2 & 0 & 0 & t & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & t & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -t & 0 & 1 & 0 & 0 & 0 & -t^2/2 & 0 \\ t & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & t^2/2 \\ 0 & 0 & 0 & -t & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$M_{10}^t = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -t & 0 \\ 0 & 1 & 0 & -t^2/2 & 0 & 0 & 0 & t & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & t \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -t & 0 & 0 & 0 & 1 & 0 & 0 & 0 & t^2/2 & 0 \\ 0 & 0 & -t & 0 & 0 & 1 & 0 & 0 & 0 & -t^2/2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -t & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Let $Y = \sum_{i=1}^{10} a_i Y_i$, then

$$F_{10}^{t_{10}} \circ F_9^{t_9} \circ \cdots \circ F_1^{t_1} : Y \mapsto \Bigg(\Big(\cos t_7 \cos t_8 + t_5 \sin t_8 +$$

$$\Big(\frac{1}{2}t_5^2 \cos t_7 + t_6 \sin t_7\Big) t_9 \sin t_8 - \Big(\frac{1}{2}t_5^2 \sin t_7 - t_6 \cos t_7\Big) t_{10}\Big) a_1 + \cdots +$$

$$\Big(\Big(-\Big(\frac{1}{2}t_5^2 \sin t_7 - t_6 \cos t_7\Big) t_9 t_{10}\Big) - \frac{1}{2}\Big(\frac{1}{2}t_5^2 \cos t_7 + t_6 \sin t_7\Big) t_{10}^2 \sin t_8 +$$

$$(\cos t_7 \cos t_8 + t_5 \sin t_8) t_9 + \frac{1}{2}\Big(\frac{1}{2}t_5^2 \cos t_7 + t_6 \sin t_7\Big) t_9^2 \sin t_8\Big) a_{10}\Bigg) Y_1$$

$$\vdots$$

$$+\Bigg\{\Bigg[\Big(\Big(\Big(\frac{1}{4}t_2^2 t_4^2 (e^{s_t} - 1)\cos t_1 - t_2 t_4 \cos t_1\Big) t_5 - \Big(\frac{1}{4}t_2^2 t_4^2 (e^{t_3} - 1)\sin t_1 -$$

$$t_2 t_4 \sin t_1 + (e^{-t_3} - 1)\sin t_1 + (e^{-t_3} - 1)\cos t_1\Big) t_6\Big) \cos t_7 -$$

$$\Big(\frac{1}{2}t_2^2 t_4 (e^{t_3} - 1)\cos t_1 - t_2 \cos t_1\Big) \sin t_7\Big) \cos t_8 + \cdots -$$

$$\Bigg(\frac{1}{6}\Big(\frac{1}{4}t_2^2 t_4^2 (e^{t_3} - 1)\cos t_1 - t_2 t_4 \cos t_1\Big) t_5^3 \sin t_7 +$$

$$\Big(-\Big(\frac{1}{4}t_2^2 t_4^2 (e^{t_3} - 1)\cos t_1 - t_2 t_4 \cos t_1\Big) t_5 t_6 + \frac{1}{2}t_2^2 (e^{t_3} - 1)\sin t_1 -$$

$$\frac{1}{2}\Big(\frac{1}{4}t_2^2 t_4^2 (e^{t_3} - 1)\sin t_1 - t_2 t_4 \sin t_1 + (e^{-t_3} - 1)\sin t_1 + (e^{-t_3} - 1)\cos t_1\Big) t_5^2 +$$

$$\frac{1}{2}\Big(\frac{1}{4}t_2^2 t_4^2 (e^{t_3} - 1)\sin t_1 - t_2 t_4 \sin t_1 + (e^{t_3} - 1)\sin t_1 +$$

$$(e^{-t_3} - 1)\cos t_1\Big) t_6^2\Big) \cos t_7\Bigg) t_{10}\Bigg] a_1 + \cdots + \Bigg[\Big(\Big(\frac{1}{4}t_2^2 t_4^2 (e^{t_3} - 1)\sin t_1 - t_2 t_4 \sin t_1 +$$

$$(e^{-t_3} - 1)\sin t_1 + (e^{-t_3} - 1)\cos t_1\Big) t_5 + \cdots + (e^{-t_3} - 1)\sin t_1 + \cdots\Bigg) t_{10} +$$

$$\cdots + \Big(\frac{1}{4}t_2^2 t_4^2 (e^{t_3} - 1)\sin t_1 - t_2 t_4 \sin t_1 + (e^{-t_3} - 1)\sin t_1 +$$

$$(e^{-t_3} - 1)\cos t_1 + \frac{1}{4}t_2^2 t_4^2 (e^{t_3} - 1)\cos t_1 - t_2 t_4 \cos t_1\Big) \cos t_8\Bigg] a_{10}\Bigg\} Y_{10}.$$

Then, if $a_i = 0$, for $i = 2, 3, 4, 5, 6, 9, 10$, and a_1, a_7, a_8, are non-zero then we can make the coefficients of Y_8, vanish, by $F_2^{t_1}$ and $F_3^{t_2}$. Scaling Y if necessary, we can assume that $a_1 = a_7 = a_8 = 1$. And Y is reduced to Case 1, 4, 5 and 18.

If $a_i = 0$, for $i = 2, 4, 5, 6, 9, 10$, and a_1, a_3, a_7, a_8, are non-zero then we can make the coefficients of Y_8, vanish, by $F_2^{t_1}$ and $F_3^{t_2}$. Scaling Y if necessary, we can assume that $a_3 = a_9 = 1$. And Y is reduced to Case 2, 7, 9, 12, 13 and 15.

If $a_i = 0$, for $i = 2, 3, 5, 6, 7, 10$, and a_1, a_4, a_8, a_9, are non-zero then we can make the coefficients of Y_4 and Y_8 vanish, by $F_2^{t_1}$ and $F_3^{t_2}$. Scaling Y if necessary, we can assume that $a_4 = 1$. And Y is reduced to Case 3, 6, 8, 11, 14 and 16.

If $a_i = 0$, for $i = 2, 4, 5, 6, 9, 10$, and a_1, a_3, a_7, a_8, are non-zero then we can make the coefficients of Y_8, vanish, by $F_2^{t_1}$ and $F_3^{t_2}$. Scaling Y if necessary, we can assume that $a_1 = 1$. And Y is reduced to Case 10, 17, 19, 20, and 21. □

According to our optimal system of one-dimensional subalgebras of the full symmetry algebra g_1, we need only find group-invariant solutions for 21 one-parameter subgroups generated by Y as Theorem (2.6).

3 Conclusions

In this paper by using the criterion of invariance of the equation under the infinitesimal prolonged infinitesimal generators, we find the most Lie point symmetry group of the cylindrical Laplace equation. Looking the adjoint representation of the obtained symmetry group on its Lie algebra, we have found the preliminary classification of group-invariants solutions.

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