Conformally symmetric manifolds and quasi conformally recurrent Riemannian manifolds

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Abstract. In order to give a new proof of a theorem concerned with conformally symmetric Riemannian manifolds due to Roter and Derdzinsky [8], [9] and Miyazawa [15], we have adopted the technique used in a theorem about conformally recurrent manifolds with harmonic conformal curvature tensor in [3]. In this paper, we also present a new proof of a successive refined version of a theorem about conformally recurrent manifolds with harmonic conformal curvature tensor. Moreover, as an extension of theorems mentioned above we prove some theorems related to quasi conformally recurrent Riemannian manifolds with harmonic quasi conformal curvature tensor.

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1 Introduction

Let M be a non flat $n(\geq 4)$ dimensional Riemannian manifold with metric g_{ij} and Riemannian connection ∇. It is said to be conformally recurrent if the conformal curvature tensor satisfies $\nabla_i C_{jk\ell}{}^m = \lambda_i C_{jk\ell}{}^m$ (See [1], [3] and [11]), where λ_i is some non null covector and the components of the conformal curvature tensor [16] are given by :

$$C_{jk\ell}{}^m = R_{jk\ell}{}^m + \frac{1}{n-2}(\delta_j^m R_{k\ell} - \delta_k^m R_{j\ell} + R_j^m g_{k\ell} - R_k^m g_{j\ell}) - \frac{R}{(n-1)(n-2)}(\delta_j^m g_{k\ell} - \delta_k^m g_{j\ell}). \tag{1.1}$$

Here we have defined the Ricci tensor to be $R_{k\ell} = -R_{mk\ell}{}^m$ [23] and the scalar curvature $R = g^{ij}R_{ij}$. The recurrence properties of Weyl's tensor has been analized also in [13]. If $\nabla_i C_{jk\ell}{}^m = 0$, the manifold is said to be conformally symmetric (See [5], [8],[10] and [18]). If $\nabla_m C_{jk\ell}{}^m = 0$, the manifold is said to have harmonic curvature

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tensor (See [4]). If $C_{jk\ell}{}^{m} = 0$, the manifold is called conformally flat (See [16]). In [13] the properties of some class of conformally flat manifolds are pointed out. It may be scrutinized that the conformal curvature tensor vanishes identically if $n = 3$ and if M is a space of constant curvature. A manifold is said to be Ricci recurrent if its non null Ricci tensor is recurrent, i.e. if $\nabla_k R_{ij} = \beta_k R_{ij}$ (See [11]) where β_k is another non null covector.
Recently a theorem concerning conformally recurrent Riemannian or semi-Riemannian manifolds with harmonic curvature tensor was introduced in [3] (Theorem 3.4) and [19]. We refer to it as:

Theorem 1.1. *Let M be an $n(\geq 4)$ dimensional Riemannian manifold with Riemannian connection ∇. Assume that M is conformally recurrent and has the harmonic conformal curvature tensor. If the scalar curvature is constant ($\nabla_j R = 0$), then M is conformally symmetric, conformally flat or Ricci recurrent.*

This theorem was used in [3] to give a complete classification of conformally recurrent Riemannian manifolds with harmonic curvature tensor. In the same reference it was stated another Theorem ([3], Theorem 3.6) that refines Theorem 1.1. We refer to it as:

Theorem 1.2. *Let M be an $n(\geq 4)$ dimensional Riemannian manifold with Riemannian connection ∇. Assume that M is conformally recurrent and has the harmonic conformal curvature tensor. If the scalar curvature is non zero constant, then M is conformally flat or locally symmetric.*

In [19] the authors extended Theorem 1.2 to the case of semi-Riemannian manifolds. Moreover they also pointed out that the assumption of a constant scalar curvature may be dropped in the case of a definite metric and stated the following (see [19] Remark 3.3) :

Theorem 1.3. *Let M be an $n(\geq 4)$ dimensional Riemannian manifold with Riemannian connection ∇. Assume that M is conformally recurrent and has the harmonic conformal curvature tensor. Then M is conformally symmetric.*

In this paper we give a new proof of a classical theorem about conformally symmetric Riemannian manifolds using a technique adopted in [3] for Theorem 1.1. Now we assert the following :

Theorem 1.4. *An $n(\geq 4)$ dimensional conformally symmetric manifold is conformally flat or locally symmetric.*

This result is fulfilled on a manifold with positive definite metrics. Miyazawa proved this statement with the extra assumption of $n > 4$ in [15]. A proof of the general case $n > 3$ was pointed out by Derdzinski and Roter in [9]. In section 2 of this paper we reobtain Theorem 1.4 by a correction of the procedure employed in the proof of Theorem 1.1 used in [3]. In section 3 we give an alternative proof of Theorem 1.3 and provide extensions of Theorems 1.1, 1.3 and 1.4 related to quasi-conformal symmetric or quasi-conformal recurrent Rimannian manifold.

Moreover, combining the results of Theorems 1.3 and 1.4, we can state another theorem as follows:

Theorem 1.5. *Let M be an $n(\geq 4)$ dimensional Riemannian manifold with Riemannian connection ∇. Assume that M is conformally recurrent and has the harmonic conformal curvature tensor. Then M is conformally flat or locally symmetric.*

2 The proof of Theorem B

In this section the procedure adopted in [3] is pursued to obtain a proof of Theorem 1.4. It is worth to notice that the assumption of constant scalar curvature mentioned in Theorem 1.1 and employed in [3] is not used here in the proof of Theorem 1.4. Let M be an n dimensional conformally symmetric manifold. Then the following relation is fulfilled:

$$\begin{aligned}\nabla_i R_{jk\ell}{}^m = &-\frac{1}{n-2}(\delta_j^m \nabla_i R_{k\ell} - \delta_k^m \nabla_i R_{j\ell} + \nabla_i R_j^m g_{k\ell} - \nabla_i R_k^m g_{j\ell}) \\ &+ \frac{\nabla_i R}{(n-1)(n-2)}(\delta_j^m g_{k\ell} - \delta_k^m g_{j\ell}).\end{aligned} \tag{2.1}$$

From the previous result we can state the following

Remark 2.1. *Any conformally symmetric manifold with parallel Ricci tensor is symmetric in the sense of Cartan, that is, $\nabla_i R_{jk\ell}{}^m = 0$ (See [12], [16] and [18]).*

From the notion of conformally symmetric manifold one easily gets $(\nabla_b \nabla_a - \nabla_a \nabla_b) C_{jk\ell m} = 0$. Then by the Ricci identity [23], we can write the following equation:

$$R_{baj}{}^p C_{pk\ell m} + R_{bak}{}^p C_{jp\ell m} + R_{ba\ell}{}^p C_{jkpm} + R_{bam}{}^p C_{jk\ell p} = 0. \tag{2.2}$$

Performing the covariant derivative of equation (2.2) and taking account that $\nabla_i C_{jk\ell}{}^m = 0$, one obtains:

$$\nabla_i R_{baj}{}^p C_{pk\ell m} + \nabla_i R_{bak}{}^p C_{jp\ell m} + \nabla_i R_{ba\ell}{}^p C_{jkpm} + \nabla_i R_{bam}{}^p C_{jk\ell p} = 0. \tag{2.3}$$

From (2.3) and the fact that the manifold is conformally symmetric we obtain:

$$\begin{aligned}&(\nabla_i R_{bj} C_{ak\ell m} + \nabla_i R_{bk} C_{ja\ell m} + \nabla_i R_{b\ell} C_{jkam} + \nabla_i R_{bm} C_{jk\ell a}) \\ &-(\nabla_i R_{aj} C_{bk\ell m} + \nabla_i R_{ak} C_{jb\ell m} + \nabla_i R_{a\ell} C_{jkbm} + \nabla_i R_{am} C_{jk\ell b}) \\ &+\frac{1}{n-1}\nabla_i R(g_{aj} C_{bk\ell m} + g_{ak} C_{jb\ell m} + g_{a\ell} C_{jkbm} + g_{am} C_{jk\ell b}) \\ &-\frac{1}{n-1}\nabla_i R(g_{bj} C_{ak\ell m} + g_{bk} C_{ja\ell m} + g_{b\ell} C_{jkam} + g_{bm} C_{jk\ell a}) \\ &-\nabla_i R_b^p (g_{aj} C_{pk\ell m} + g_{ak} C_{jp\ell m} + g_{a\ell} C_{jkpm} + g_{am} C_{jk\ell p}) \\ &+\nabla_i R_a^p (g_{bj} C_{pk\ell m} + g_{bk} C_{jp\ell m} + g_{b\ell} C_{jkpm} + g_{bm} C_{jk\ell p}) = 0.\end{aligned} \tag{2.4}$$

Now transvecting the last equation with g^{jb} taking account of the first Bianchi identity for the conformal curvature tensor we have:

$$\begin{aligned}&(n-2)\nabla_i R_{ab} C_{m\ell k}{}^b + \nabla_i R_{bk} C_{m\ell a}{}^b + \nabla_i R_{b\ell} C_{mak}{}^b + \nabla_i R_{bm} C_{a\ell k}{}^b \\ &\quad - (g_{a\ell} C_{mpk}{}^b + g_{am} C_{p\ell k}{}^b)\nabla_i R_b^p = 0.\end{aligned} \tag{2.5}$$

Again the previous equation is transvected with g^{im} to obtain:

$$\begin{aligned}(n-2)\nabla^m R_{ab}C_{m\ell k}{}^b + \nabla^m R_{bk}C_{m\ell a}{}^b + \nabla^m R_{b\ell}C_{mak}{}^b + \frac{1}{2}(\nabla_b R)C_{a\ell k}{}^b \\ - g_{a\ell}C_{mpk}{}^b\nabla^m R^p_b - C_{p\ell k}{}^b\nabla_a R^p_b = 0.\end{aligned} \tag{2.6}$$

Now it is well known that the divergence of the conformal curvature is given by ([8] and [9]):

$$\nabla_m C_{jk\ell}{}^m = \frac{n-3}{n-2}\Big[\nabla_k R_{j\ell} - \nabla_j R_{k\ell} + \frac{1}{2(n-1)}\{(\nabla_j R)g_{k\ell} - (\nabla_k R)g_{j\ell}\}\Big]. \tag{2.7}$$

So if the manifold is conformally symmetric, it is easily seen that:

$$\nabla_j R_{k\ell} - \nabla_k R_{j\ell} = \frac{1}{2(n-1)}\{(\nabla_j R)g_{k\ell} - (\nabla_k R)g_{j\ell}\}. \tag{2.8}$$

This result allows us to examine the last two terms contained in equation (2.6). The first term vanishes; in fact:

$$\begin{aligned}g_{a\ell}C_{mpk}{}^b\nabla^m R^p_b &= \frac{1}{2}g_{a\ell}C_{mpk}{}^b(\nabla^m R^p_b - \nabla^p R^m_b) \\ &= \frac{1}{2}g_{a\ell}C^{mp}{}_k{}^b(\nabla_m R_{pb} - \nabla_p R_{mb}) \\ &= \frac{1}{4(n-1)}g_{a\ell}C^{mp}{}_k{}^b\{(\nabla_m R)g_{pb} - (\nabla_p R)g_{mb}\} \\ &= 0.\end{aligned} \tag{2.9}$$

Moreover with similar procedure the last term results to be:

$$\begin{aligned}C_{p\ell k}{}^b\nabla_a R^p_b &= C^p{}_{\ell k}{}^b\nabla_a R_{pb} \\ &= C^p{}_{\ell k}{}^b\Big[\nabla_p R_{ab} + \frac{1}{2(n-1)}\{(\nabla_a R)g_{pb} - (\nabla_p R)g_{ab}\}\Big] \\ &= C^p{}_{\ell k}{}^b\nabla_p R_{ab} - \frac{1}{2(n-1)}C^p{}_{\ell k}{}^b(\nabla_p R)g_{ab} \\ &= C_{m\ell k}{}^b\nabla^m R_{ab} - \frac{1}{2(n-1)}(\nabla_m R)C^m{}_{\ell ka}.\end{aligned} \tag{2.10}$$

So equation (2.6) can be rewritten in the following form:

$$\begin{aligned}(n-3)\nabla^m R_{ab}C_{m\ell k}{}^b + \nabla^m R_{bk}C_{m\ell a}{}^b + \nabla^m R_{b\ell}C_{mak}{}^b + \frac{1}{2}(\nabla_m R)C_{a\ell k}{}^m \\ + \frac{1}{2(n-1)}(\nabla_m R)C^m{}_{\ell ka} = 0.\end{aligned} \tag{2.11}$$

Now in [3] an interesting Lemma is pointed out (See also [9]):

Lemma 2.2. *Let M be an n dimensional conformally symmetric manifold. Then the following equations hold:*

$$\begin{aligned}R_{ab}C_{m\ell k}{}^b + R_{mb}C_{\ell ak}{}^b + R_{\ell b}C_{amk}{}^b = 0, \\ \nabla_s R_{ab}C_{m\ell k}{}^b + \nabla_s R_{mb}C_{\ell ak}{}^b + \nabla_s R_{\ell b}C_{amk}{}^b = 0.\end{aligned} \tag{2.12}$$

Transvecting the last of the previous relations with g^{sm} one obtains:

$$\nabla^m R_{ab} C_{m\ell k}{}^b - \nabla^m R_{\ell b} C_{mak}{}^b = \frac{1}{2}(\nabla_m R) C_{a\ell k}{}^m. \tag{2.13}$$

The equivalent relation $-2\nabla^m R_{ba} C_{m\ell k}{}^b = -2\nabla^m R_{b\ell} C_{mak}{}^b - (\nabla_m R) C_{a\ell k}{}^m$ is then substituted in equation (2.11) to obtain:

$$\begin{aligned}(n-1)\nabla^m R_{ab} C_{m\ell k}{}^b + \nabla^m R_{bk} C_{m\ell a}{}^b - \nabla^m R_{b\ell} C_{mak}{}^b - \frac{1}{2}(\nabla_m R) C_{a\ell k}{}^m \\ + \frac{1}{2(n-1)}(\nabla_m R) C^m{}_{\ell ka} = 0.\end{aligned} \tag{2.14}$$

Again employing Lemma 2.2 with indices k and a exchanged gives:

$$\nabla^m R_{bk} C_{m\ell a}{}^b - \nabla^m R_{b\ell} C_{mka}{}^b = \frac{1}{2}(\nabla_m R) C_{k\ell a}{}^m. \tag{2.15}$$

So equation (2.14) takes the form:

$$\begin{aligned}(n-1)\nabla^m R_{ab} C_{m\ell k}{}^b + \nabla^m R_{b\ell}(C_{mka}{}^b - C_{mak}{}^b) + \frac{1}{2}(\nabla_m R)(C_{k\ell a}{}^m + C_{\ell ak}{}^m) \\ + \frac{1}{2(n-1)}(\nabla_m R) C^m{}_{\ell ka} = 0.\end{aligned} \tag{2.16}$$

Recalling that $C_{mka}{}^b + C_{kam}{}^b + C_{amk}{}^b = 0$, the previous equation may be written in the following form:

$$(n-1)\nabla^m R_{ab} C_{m\ell k}{}^b + \nabla^m R_{b\ell} C_{akm}{}^b + \frac{1}{2}(\nabla_m R) C_{ka\ell}{}^m + \frac{1}{2(n-1)}(\nabla_m R) C^m{}_{\ell ka} = 0. \tag{2.17}$$

Now recalling that $\nabla_m R_{b\ell} - \nabla_b R_{m\ell} = \frac{1}{2(n-1)}\{(\nabla_m R) g_{b\ell} - (\nabla_b R) g_{m\ell}\}$, the second term of the previous equation satisfies the following identities chain:

$$\begin{aligned}\nabla^m R_{b\ell} C_{akm}{}^b &= \nabla_m R_{b\ell} C_{ak}{}^{mb} = \frac{1}{2}\, C_{ak}{}^{mb}(\nabla_m R_{b\ell} - \nabla_b R_{m\ell}) \\ &= \frac{1}{4(n-1)} C_{ak}{}^{mb}(\nabla_m R g_{b\ell} - \nabla_b R g_{m\ell}) \\ &= \frac{1}{2(n-1)}\Big[(\nabla^m R) C_{akm\ell} - (\nabla_b R) C_{ak\ell}{}^b\Big] \\ &= \frac{1}{4(n-1)}(\nabla^m R)\Big[C_{akm\ell} - C_{ak\ell m}\Big] = \frac{(\nabla^m R)}{2(n-1)} C_{akm\ell}.\end{aligned} \tag{2.18}$$

So equation (2.16) takes the form:

$$\begin{aligned}(n-1)\nabla^m R_{ab} C_{m\ell k}{}^b + \frac{1}{2(n-1)}(\nabla^m R) C_{akm\ell} \\ + \frac{1}{2}(\nabla_m R) C_{ka\ell}{}^m + \frac{1}{2(n-1)}(\nabla^m R) C_{m\ell ka} = 0,\end{aligned} \tag{2.19}$$

or better:

$$(n-1)\nabla^m R_{ab} C_{m\ell k}{}^b + \frac{1}{2}(\nabla^m R) C_{ka\ell m} = 0. \tag{2.20}$$

Now one can observe that $\nabla_m R_{ab} = \nabla_a R_{mb} + \dfrac{1}{2(n-1)}\{(\nabla_m R) g_{ab} - (\nabla_a R) g_{mb}\}$ and thus we can write:

$$\begin{aligned}\nabla^m R_{ab} C_{m\ell k}{}^b &= \nabla_m R_{ab} C^m{}_{\ell k}{}^b \\ &= \nabla_a R_{mb} C^m{}_{\ell k}{}^b + \frac{1}{2(n-1)}\Big[(\nabla_m R) C^m{}_{\ell k}{}^b g_{ab} - (\nabla_a R) C^m{}_{\ell k}{}^b g_{mb}\Big].\end{aligned} \tag{2.21}$$

This fact implies that:

$$\nabla_m R_{ab} C^m{}_{\ell k}{}^b = \nabla_a R_{mb} C^m{}_{\ell k}{}^b + \frac{1}{2(n-1)}(\nabla^m R) C_{m\ell ka}. \tag{2.22}$$

If the equivalent relation $(n-1)\nabla_m R_{ab} C^m{}_{\ell k}{}^b = (n-1)\nabla_a R_{mb} C^m{}_{\ell k}{}^b + \frac{1}{2}(\nabla^m R) C_{m\ell ka}$ is substituted in equation (2.20), one obtains that the following holds:

$$(n-1)\nabla_a R_{mb} C^m{}_{\ell k}{}^b = 0. \tag{2.23}$$

At last equation (2.5) takes the form:

$$(n-2)\nabla_i R_{ab} C_{m\ell k}{}^b + \nabla_i R_{bk} C_{m\ell a}{}^b + \nabla_i R_{b\ell} C_{mak}{}^b + \nabla_i R_{bm} C_{a\ell k}{}^b = 0. \tag{2.24}$$

Now Lemma 2.2 is again employed in the form $\nabla_i R_{mb} C_{a\ell k}{}^b + \nabla_i R_{ab} C_{\ell mk}{}^b + \nabla_i R_{\ell b} C_{mak}{}^b = 0$ to equation (2.24) to obtain:

$$(n-1)\nabla_i R_{ab} C_{m\ell k}{}^b = -\nabla_i R_{bk} C_{m\ell a}{}^b. \tag{2.25}$$

Now exchanging the indices k and a in the previous result gives immediately:

$$(n-1)\nabla_i R_{kb} C_{m\ell a}{}^b = -\nabla_i R_{ab} C_{m\ell k}{}^b. \tag{2.26}$$

This implies that $(n-1)^2 \nabla_i R_{ab} C_{m\ell k}{}^b = \nabla_i R_{ab} C_{m\ell k}{}^b$ and so as in [3] and [19] that:

$$\nabla_i R_{bk} C_{m\ell a}{}^b = 0. \tag{2.27}$$

Transvecting the previous result with g^{ik} it follows immediately that:

$$\frac{1}{2}\nabla_b R C_{m\ell a}{}^b = 0. \tag{2.28}$$

Transvecting (2.4) with $\nabla_i R_{bj}$ or with $C_{ak\ell m}$ and applying (2.27), one can obtain the following results:

$$\nabla_i R_{bj} \nabla^i R^{bj} C_{ak\ell m} = 0 \quad \text{or} \quad \nabla_i R_{bj} C_{ak\ell m} C^{ak\ell m} = 0\,.$$

In fact if equation (2.4) is transvected with $\nabla^i R^{bj}$ one obtains:

$$\left(\nabla^i R^{bj} \nabla_i R_{bj} - \frac{1}{n-1}\nabla^i R \nabla_i R\right) C_{ak\ell m} = 0. \tag{2.29}$$

On the other hand if equation (2.4) is transvected with $C^{ak\ell m}$ one easily obtains:

$$\begin{aligned}\nabla_i R_{bj} C_{ak\ell m} C^{ak\ell m} &+ \frac{1}{n-1}\nabla_i R\{g_{aj}C_{bk\ell m} \\ &- g_{bj}C_{ak\ell m} - g_{bk}C_{ja\ell m} - g_{b\ell}C_{jkam} - g_{bm}C_{jk\ell a}\}C^{ak\ell m} = 0.\end{aligned} \tag{2.30}$$

Transvecting this last result with g^{ij} and making use of equation (2.28) one comes to the following:

$$\frac{n-3}{2(n-1)}\nabla_b R C_{akm\ell} C^{akm\ell} = 0. \tag{2.31}$$

Thus we obtain that the manifold is conformally flat or the manifold has constant scalar curvature and employing (2.30) it is Ricci symmetric. In this way we have proved that the following Theorem holds:

Theorem 2.3. *Let M be an n dimensional conformally symmetric manifold. Then it is Ricci symmetric or conformally flat.*

Now recalling Remark 2.1 and Theorem 2.3, we have just proved that $\nabla_i R_{jk\ell}{}^m = 0$ or $C_{jk\ell}{}^m = 0$.

Remark 2.4. *It is worth to notice that from Theorem 2.3 we recover a result of Tanno ([20], Theorem 6): any non conformally flat conformally symmetric manifold has constant scalar curvature. This result was used in [9] for the proof of Theorem 1.4. In the present paper it has been recovered in our main argument.*

3 An alternative proof of Theorem 1.3 and generalizations of Theorems 1.1, 1.3 and 1.4

In this section we provide an alternative proof of Theorem 1.3 given in [19] and consider a possible generalization of Theorems 1.1, 1.3 and 1.4.

Theorem 1.3. *Let M be an $n(\geq 4)$ dimensional Riemannian manifold with Riemannian connection ∇. Assume that M is conformally recurrent and has the harmonic conformal curvature tensor. Then M is conformally symmetric or conformally flat.*

Proof. It is well known ([1] eq. 3.7) that the second Bianchi identity for the conformal curvature tensor may be written in the following form:

$$\begin{aligned}\nabla_i C_{jk\ell}{}^m &+ \nabla_j C_{ki\ell}{}^m + \nabla_k C_{ij\ell}{}^m \\ &= \frac{1}{n-3}\Big[\delta_j^m \nabla_p C_{ki\ell}{}^p + \delta_k^m \nabla_p C_{ij\ell}{}^p + \delta_i^m \nabla_p C_{jk\ell}{}^p \\ &\quad + g_{k\ell}\nabla_p C_{ji}{}^{mp} + g_{i\ell}\nabla_p C_{kj}{}^{mp} + g_{j\ell}\nabla_p C_{ik}{}^{mp}\Big].\end{aligned} \tag{3.1}$$

Thus on a manifold with harmonic conformal curvature tensor [4], the second Bianchi identity reduces to:

$$\nabla_i C_{jk\ell}{}^m + \nabla_j C_{ki\ell}{}^m + \nabla_k C_{ij\ell}{}^m = 0\,. \tag{3.2}$$

If the manifold is also conformally recurrent, i.e. $\nabla_i C_{jk\ell}{}^m = \lambda_i C_{jk\ell}{}^m$, the last equation takes the form:

$$\lambda_i C_{jk\ell}{}^m + \lambda_j C_{ki\ell}{}^m + \lambda_k C_{ij\ell}{}^m = 0\,. \tag{3.3}$$

We note also that if the manifold has the harmonic conformal curvature tensor, i.e. $\nabla_m C_{jk\ell}{}^m = 0$, then $\lambda_m C_{jk\ell}{}^m = 0$. Now equation (3.3) is multiplied by λ^i to obtain the following result:

$$\lambda^i \lambda_i C_{jk\ell}{}^m + \lambda^i \lambda_j C_{ki\ell}{}^m + \lambda^i \lambda_k C_{ij\ell}{}^m = 0\,. \tag{3.4}$$

In the previous equation the second and the last terms vanish. In fact for example one easily obtains $\lambda^i \lambda_j C_{ki\ell}{}^m = g^{mp} \lambda_j \lambda^i C_{ki\ell p} = g^{mp} \lambda_j \lambda^i C_{\ell pki} = 0$. Then equation (3.4) give the following result:

$$\lambda^i \lambda_i C_{jk\ell}{}^m = 0\,. \tag{3.5}$$

We have thus obtained that the manifold is confromally flat. In the same manner equation (3.3) is multiplied by $C^{jk\ell}{}_m$ and the following is fulfilled:

$$\lambda_i C_{jk\ell}{}^m C^{jk\ell}{}_m + \lambda_j C_{ki\ell}{}^m C^{jk\ell}{}_m + \lambda_k C_{ij\ell}{}^m C^{jk\ell}{}_m = 0\,. \tag{3.6}$$

Thus following the same procedure employed previously we have that the relation:

$$\lambda_i C_{jk\ell}{}^m C^{jk\ell}{}_m = 0\,. \tag{3.7}$$

So we have $\lambda_i = 0$ and the manifold is conformally symmetric. □

It is worth to notice that the class of conformally symmetric spaces includes the class of conformally flat spaces. The version of Theorem 1.3 proved in the present paper is slightly different from [19].

Now we consider a possible generalization of Theorems 1.1, 1.3 and 1.4 in the direction of quasi-conformal symmetric or quasi-conformal recurrent Riemannian manifold. In order to do this, first we need the definition of the concircular curvature tensor (See [17] and [21]), that is:

$$\tilde{C}_{jk\ell}{}^m = R_{jk\ell}{}^m + \frac{R}{n(n-1)}(\delta_j^m g_{k\ell} - \delta_k^m g_{j\ell}). \tag{3.8}$$

Contracting m with j gives the so called Z tensor, i.e. $Z_{k\ell} = -\tilde{C}_{mk\ell}{}^m$, that is:

$$Z_{k\ell} = R_{k\ell} - \frac{R}{n} g_{k\ell}. \tag{3.9}$$

It may be noted from (3.8) that the vanishing of the concircular tensor implies the manifold to be a space of constant curvature and from (3.9) that the vanishing of the Z tensor implies the manifold to be an Einstein space. So the concircular tensor is a measure of the deviation of a manifold from a space of constant curvature and the Z tensor is a measure of the deviation from an Einstein space (See [14]).

In 1968 Yano and Sawaki [22] defined and studied a tensor $W_{jk\ell}{}^m$ on a Riemannian manifold of dimension n, which includes both the conformal curvature tensor $C_{jk\ell}{}^m$

and the concircular curvature tensor $\tilde{C}_{jk\ell}{}^m$ as particular cases. This tensor is known as quasi conformal curvature tensor and its components are given by:

$$W_{jk\ell}{}^m = -(n-2)b\, C_{jk\ell}{}^m + \big[a + (n-2)b\big]\tilde{C}_{jk\ell}{}^m. \tag{3.10}$$

In the previous equation $a \neq 0$, $b \neq 0$ are constants and $n > 3$ since the conformal curvature tensor vanishes identically for $n = 3$. A non flat manifold is said to be quasi-conformally recurrent if $\nabla_i W_{jk\ell}{}^m = \alpha_i W_{jk\ell}{}^m$ for a non null covector α_i. It is said to be quasi-conformally symmetric if $\nabla_i W_{jk\ell}{}^m = 0$ and has the harmonic quasi conformal curvature tensor if $\nabla_m W_{jk\ell}{}^m = 0$. Z recurrency or Z symmetry are defined in analogous ways. Clearly the class of quasi conformally recurrent Riemannian manifolds includes all the class of quasi conformally symmetric and quasi conformally flat manifolds. In [2] Amur and Maralabhavi proved that a quasi conformally flat Riemannian manifold is either conformally flat or Einstein. A similar remark can be proved for quasi conformally symmetric manifolds.

Remark 3.1. *Let M be an $n(\geq 4)$ dimensional quasi conformally symmetric Riemannian manifold. Then it is either conformally symmetric or Ricci symmetric.*

Proof. In fact the condition $\nabla_i W_{jk\ell}{}^m = 0$ implies:

$$(n-2)b\nabla_i C_{jk\ell}{}^m = \big[a + (n-2)b\big]\nabla_i \tilde{C}_{jk\ell}{}^m. \tag{3.11}$$

Contracting m with j in the previous equation gives $\big[a + (n-2)b\big] = 0$ or $\nabla_i Z_{k\ell} = 0$, that is, by the equation (3.11) the manifold is conformally symmetric or Z symmetric. Now Z symmetric implies $\nabla_i R_{k\ell} = \frac{1}{n}(\nabla_i R)g_{k\ell}$ and transvecting with g^{ik} one gets $\nabla_l R = 0$ and thus $\nabla_i R_{k\ell} = 0$ □

We note that the class of Z symmetric spaces includes the class of Einstein spaces. The previous remark allows us to state a modified version of Theorem 1.4 whose proof follows immediately from Remark 3.1 and Theorem 1.4 itself:

Theorem 3.2. *Let M be an $n(\geq 4)$ dimensional quasi-conformally symmetric manifold. Then it is conformally flat or locally symmetric.*

The statement of the previous theorem is due to the fact that local symmetry implies Ricci symmetry. We can also state the following modified version of Theorem 1.1.

Theorem 3.3. *Let M be an $n(\geq 4)$ dimensional Riemannian manifold of with Riemannian connection ∇. Assume that M is quasi-conformally recurrent and has the harmonic quasi conformal curvature tensor. Then M is conformally symmetric, conformally flat or generalized Ricci recurrent [6].*

Proof. If $\nabla_i W_{jk\ell}{}^m = \alpha_i W_{jk\ell}{}^m$, then one has:

$$-(n-2)b\nabla_i C_{jk\ell}{}^m + \big[a+(n-2)b\big]\nabla_i \tilde{C}_{jk\ell}{}^m = -(n-2)b\alpha_i C_{jk\ell}{}^m + \big[a+(n-2)b\big]\alpha_i \tilde{C}_{jk\ell}{}^m. \tag{3.12}$$

Contracting m with j in the previous equation gives:

$$\big[a + (n-2)b\big]\nabla_i Z_{k\ell} = \big[a + (n-2)b\big]\alpha_i Z_{k\ell}. \tag{3.13}$$

That is, the manifold is Z recurrent or $\big[a+(n-2)b\big]=0$. In this case we get from (3.10) that:

$$\nabla_m W_{jk\ell}{}^m = -(n-2)b\nabla_m C_{jk\ell}{}^m + \big[a+(n-2)b\big]\nabla_m \tilde{C}_{jk\ell}{}^m. \tag{3.14}$$

This fact implies that $\nabla_m W_{jk\ell}{}^m = -(n-2)b\nabla_m C_{jk\ell}{}^m$ and hence that $\nabla_m C_{jk\ell}{}^m = 0$ because $\nabla_m W_{jk\ell}{}^m = 0$. From (3.12) we have also in the same case $\big[a+(n-2)b\big]=0$ that:

$$-(n-2)b\nabla_i C_{jk\ell}{}^m = -(n-2)b\alpha_i C_{jk\ell}{}^m. \tag{3.15}$$

That is the manifold is conformally recurrent.

On the other hand, if the covariant derivative with respect to the index m is applied on the definition of quasi conformal curvature tensor, one obtains straightforwardly

$$\nabla_m W_{jk\ell}{}^m = [a+b]\nabla_m R_{jk\ell}{}^m + \frac{2a-b(n-1)(n-4)}{2n(n-1)}\Big[(\nabla_j R)g_{kl} - (\nabla_k R)g_{jl}\Big]. \tag{3.16}$$

Now if $\nabla_m W_{jk\ell}{}^m = 0$, transvecting the previous equation with $g^{k\ell}$ after some calculations it follows that

$$(n-2)\frac{a+b(n-2)}{n}\nabla_j R = 0. \tag{3.17}$$

This means that $\nabla_j R = 0$ if $a+(n-2)b \neq 0$ or $a+(n-2)b=0$. Inserting the latter case in (3.16) we obtain the following

$$\nabla_m R_{jk\ell}{}^m = \frac{1}{2(n-1)}\Big[(\nabla_k R)g_{jl} - (\nabla_j R)g_{kl}\Big]. \tag{3.18}$$

From this, we recover obviously $\nabla_m C^m_{jk\ell} = 0$. Now if the conditions $\nabla_i W^m_{jk\ell} = \alpha_i W^m_{jk\ell}$ and $\nabla_m W^m_{jk\ell} = 0$ are taken in conjunction, we have two cases. One is obtained from (3.12) that $\nabla_i C^m_{jkl} = \alpha_i C^m_{jkl}$ with $\nabla_m C^m_{jkl} = 0$ when $a+b(n-2)=0$. The other case can be given by (3.13) that $\nabla_i Z_{kl} = \alpha_i Z_{kl}$ with $\nabla_j R = 0$ when $a+(n-2)b \neq 0$.

In the first case we are in the hypothesis of Theorem 1.3. Accordingly, M is conformally symmetric or conformally flat.

In the second case, we have a Z-recurrent manifold with $\nabla_j R = 0$ and thus $\nabla_i R_{kl} = \alpha_i(R_{kl} - \frac{R}{n}g_{kl})$, that is, a generalized Ricci recurrent manifold [6]. □

Combining the results of Theorems 3.3 and 1.5, we can state the following modified version of Theorem 1.3:

Theorem 3.4. *Let M be an $n(\geq 4)$ dimensional Reimannian manifold of with Riemannian connection ∇. Assume that M is quasi-conformally recurrent and has the harmonic quasi conformal curvature tensor. Then M is conformally flat, locally symmetric, or generalized Ricci recurrent.*

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