On some families of linear connections

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Abstract. On a Weyl manifold $(M, \widehat{g}, w)$, we consider $\overset{\circ}{\nabla}$ the Levi-Civita connection associated to a metric $g \in \widehat{g}$, ∇ the symmetric connection, compatible with the Weyl structure w and the family of linear connections $\mathcal{C} = \{\overset{\lambda}{\nabla} := \overset{\circ}{\nabla} + \lambda(\nabla - \overset{\circ}{\nabla}) \mid \lambda \in \mathbb{R}\}$. For $\overset{\lambda}{\nabla} \in \mathcal{C}$, we investigate some properties of the deformation algebra $\mathcal{U}(M, \overset{\lambda}{\nabla} - \overset{\circ}{\nabla})$. Next, we study the case when $\overset{\circ}{\nabla}$ and $\overset{\lambda}{\nabla}$ determine the same Ricci tensor and the case when the curvature tensors of the connections $\overset{\circ}{\nabla}$ and $\overset{\lambda}{\nabla}$ are proportional.

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1 Introduction

The general problem of "Comparison geometry" is: *to what extent topological, differential, differential affine, metric, etc invariants determine some structure on a manifold, up to a homeomorphism, diffeomorphism, isometry, etc.* Usually, one compares a given manifold with some "standard" manifolds, such as space forms, Einstein spaces, or special product manifolds.

The topic originated in the Erlangen Program of F. Klein and was founded explicitely in the work of E. Cartan, under the so-called "equivalence problem"([3]). In the first half of the 20-th century, local methods were developed by S. Chern, G. Vranceanu ([12]) and others (see [4] for a modern review). Global methods arrose in the second half of the 20-th century, especially in global Riemannian geometry.

Our paper finds sufficient conditions for the curvature tensor (or the Ricci tensor) determines the Levi-Civita connection. On a Weyl manifold, we consider some special connections and, with them, we construct several deformation algebras. Properties of these deformation algebras will determine "how far apart" will be the involved curvature tensors, or the Ricci tensors. In particular, we characterize the situation when the curvature tensors, or the Ricci tensors, coincide.

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2 Definitions and notations

Let M be an $n-$dimensional C^{∞} differentiable manifold. We denote by $\mathcal{F}(M)$ the ring of real-valued functions of class C^{∞} defined on M and by $\mathcal{T}_s^r(M)$ the $\mathcal{F}(M)-$module of tensor fields of type (r,s). In particular, for $\mathcal{T}_0^1(M)$ (respectively $\mathcal{T}_1^0(M)$) we use the notation $\mathcal{X}(M)$ (respectively $\Lambda^1(M)$).

Let $A \in \mathcal{T}_2^1(M)$.The $\mathcal{F}(M)-$module $\mathcal{X}(M)$ becomes an $\mathcal{F}(M)-$algebra, denoted by $\mathcal{U}(M,A)$, with the product of two vector fields X and Y defined by

$$X \circ Y = A(X,Y). \tag{2.1}$$

In particular, if $A = \overline{\nabla} - \nabla$, where ∇ and $\overline{\nabla}$ are two arbitrary linear connections on M, then $\mathcal{U}\left(M, \overline{\nabla} - \nabla\right)$ is called the deformation algebra associated to $\left(\nabla, \overline{\nabla}\right)$.

Let $A \in \mathcal{T}_2^1(M)$ and $m > 0$, $m \in \mathbb{Z}$. An object $X \in \mathcal{U}(M,A)$ is called almost m-principal field if there exists an $1-$form $\omega \in \Lambda^1(M)$ and a function $f \in \mathcal{F}(M)$ such that ([7])

$$A\left(Z, X^{(m)}\right) = fZ + \omega(Z)X, \ \forall Z \in \mathcal{X}(M), \tag{2.2}$$

where $X^{(m)} = X^{(m-1)} \circ X$, $X^{(1)} = X$.

If $m = 1$, then (2.2) shows that X is an almost principal field in the algebra $\mathcal{U}(M,A)$. If $m = 1$ and $f = 0$, then X is a principal field in the algebra $\mathcal{U}(M,A)$. If $m = 1$ and $\omega = 0$, then X is almost special field. If $m = 1$, $f = 0$ and $\omega = 0$, then X is special field. If $A(X,X) = 0$, then X is $2-$nilpotent field.

Let (M,g) be now an $n-$dimensional semi-Riemannian manifold and let $\widehat{g}$ the conformal structure generated by g, i.e. $\widehat{g} = \{e^u g \mid u \in \mathcal{F}(M)\}$.

Let w be a Weyl structure on the conformal manifold $(M, \widehat{g})$, i.e. an application $w : \widehat{g} \longrightarrow \Lambda^1(M)$, which verifies ([1], [5], [6]) $w(e^u g) = w(g) - du$, $\forall u \in \mathcal{F}(M)$. The triple $(M, \widehat{g}, w)$ is called a *Weyl manifold*.

Let $(M, \widehat{g}, w)$ be a Weyl manifold. A linear connection ∇ on M is called *compatible with the Weyl structure* w if $\nabla_X g + w(g)(X)g = 0$, for all $X \in \mathcal{X}(M)$.

It is well known that there exists an unique symmetric linear connection ∇, defined on M, compatible with the Weyl structure ([10]). This linear connection ∇ is called the *Weyl conformal connection* and is defined by ([9])

$$\begin{aligned} 2g(\nabla_X Y, Z) \;=\; & X(g(Y,Z)) + Y(g(X,Z)) - Z(g(X,Y)) + w(g)(X)g(Y,Z) + \\ & + w(g)(Y)g(X,Z) - w(g)(Z)g(X,Y) + g([X,Y],Z) + \\ & + g([Z,X],Y) - g([Y,Z],X), \end{aligned}$$

for all $X,Y,Z \in \mathcal{X}(M)$. Let $\overset{\circ}{\nabla}$ be the Levi-Civita connection associated to g and let $A = \nabla - \overset{\circ}{\nabla} \in \mathcal{T}_2^1(M)$. We have

$$2g(A(X,Y),Z) = w(g)(X)g(Y,Z) + w(g)(Y)g(X,Z) - w(g)(Z)g(X,Y), \tag{2.3}$$

for all $X,Y,Z \in \mathcal{X}(M)$. A natural problem is to deduce properties of two semi-Riemannian manifolds from properties of the deformation algebra of their Levi-Civita connections; a remarkable particular case is when the latter ones coincide (see also [2]).

Let g_{ij}, A^i_{jk} and u_i the components, in a system of local coordinates, of g, A and of the 1−form $\frac{1}{2}w\,(g)$, respectively. Then (2.3) can be written

$$A^i_{jk} = \delta^i_j u_k + \delta^i_k u_j - g_{jk} u^i. \tag{2.4}$$

We define the family of linear connections ([8])

$$\mathcal{C} = \left\{ \overset{\lambda}{\nabla} := \overset{\circ}{\nabla} + \lambda A \mid \lambda \in \mathbb{R} \right\}.$$

(In the affine space of all linear connections on M, $\mathcal{C}$ is the "affine line" passing through the "point" $\overset{\circ}{\nabla}$ and of "direction" A).

Let $\overset{\circ}{R}$ and $\overset{\lambda}{R}$ be the curvature tensor field of the connection $\overset{\circ}{\nabla}$ and $\overset{\lambda}{\nabla}$, respectively.

3 The main results

Theorem 3.1. *Let M be an $n-$dimensional connected manifold, $n > 3$. The following assertions are equivalent:*

(i) $\overset{\lambda}{\nabla} = \overset{\circ}{\nabla}$;

(ii) $\overset{\lambda}{R} = \overset{\circ}{R}$, *if* $\overset{\circ}{R}_p : T_pM \times T_pM \times T_pM \longrightarrow T_pM$ *is surjective,* $\forall p \in M$;

(iii) $\overset{\lambda}{\nabla}$ *and* $\overset{\circ}{\nabla}$ *admit the same geodesics;*

(iv) the algebra $\mathcal{U}\left(M, \overset{\lambda}{\nabla} - \overset{\circ}{\nabla}\right)$ *is associative;*

(v) if $\overset{\circ}{R}_p : T_pM \times T_pM \times T_pM \longrightarrow T_pM$ *is surjective,* $\forall p \in M$ *and the* 1−*form* $w\,(g)$ *is exact, then* $\overset{\lambda}{\nabla}$ *and* $\overset{\circ}{\nabla}$ *have the same Ricci tensor.*

Proof. (i)⟹(ii), (i)⟹(iii), (i)⟹(iv), (i)⟹(v) are trivial. (ii)⟹(i). From (ii), $\overset{\lambda}{\nabla}_X \overset{\lambda}{R} = \overset{\lambda}{\nabla}_X \overset{\circ}{R}$, $\forall X \in \mathcal{X}(M)$. Then, $\forall X, Y, Z, V \in \mathcal{X}(M)$,

$$\begin{aligned}(\overset{\lambda}{\nabla}_X \overset{\lambda}{R})(Y,Z,V) \;&=\; (\overset{\circ}{\nabla}_X \overset{\circ}{R})(Y,Z,V) + \overset{\lambda}{A}(X, \overset{\circ}{R}(Y,Z)V) - \overset{\circ}{R}(\overset{\lambda}{A}(X,Y),Z)V \\ &\quad - \overset{\circ}{R}(Y, \overset{\lambda}{A}(X,Z))V - \overset{\circ}{R}(Y,Z)\overset{\lambda}{A}(X,V),\end{aligned} \tag{3.1}$$

where we denoted by $\overset{\lambda}{A} = \overset{\lambda}{\nabla} - \nabla = \lambda A$. Similarly, we obtain

$$\begin{aligned}(\overset{\lambda}{\nabla}_Y \overset{\lambda}{R})(Z,X,V) \;&=\; (\overset{\circ}{\nabla}_Y \overset{\circ}{R})(Z,X,V) + \overset{\lambda}{A}(Y, \overset{\circ}{R}(Z,X)V) - \overset{\circ}{R}(\overset{\lambda}{A}(Y,Z),X)V \\ &\quad - \overset{\circ}{R}(Z, \overset{\lambda}{A}(Y,X))V - \overset{\circ}{R}(Z,X)\overset{\lambda}{A}(Y,V),\end{aligned} \tag{3.2}$$

$$(\overset{\lambda}{\nabla}_Z \overset{\lambda}{R})(X,Y,V) = (\overset{\circ}{\nabla}_Z \overset{\circ}{R})(X,Y,V) + \overset{\lambda}{A}(Z,\overset{\circ}{R}(X,Y)V) - \overset{\circ}{R}(\overset{\lambda}{A}(Z,X),Y)V$$

$$-\overset{\circ}{R}(X,\overset{\lambda}{A}(Z,Y))V - \overset{\circ}{R}(X,Y)\overset{\lambda}{A}(Z,V). \tag{3.3}$$

Using the Bianchi identities, from (3.1), (3.2) and (3.3), it follows

$$\lambda \Big\{ A\left(X, \overset{\circ}{R}(Y,Z)\,V\right) + A\left(Y, \overset{\circ}{R}(Z,X)\,V\right) + A\left(Z, \overset{\circ}{R}(X,Y)\,V\right)$$

$$-\overset{\circ}{R}(Y,Z)\,A(X,V) - \overset{\circ}{R}(Z,X)\,A(Y,V) - \overset{\circ}{R}(X,Y)\,A(Z,V)\Big\} = 0.$$

From this, we deduce $\lambda = 0$, so we proved (i); hence

$$A\left(X, \overset{\circ}{R}(Y,Z)\,V\right) + A\left(Y, \overset{\circ}{R}(Z,X)\,V\right) + A\left(Z, \overset{\circ}{R}(X,Y)\,V\right)$$

$$-\overset{\circ}{R}(Y,Z)\,A(X,V) - \overset{\circ}{R}(Z,X)\,A(Y,V) - \overset{\circ}{R}(X,Y)\,A(Z,V) = 0. \tag{3.4}$$

In local coordinates, (3.4) is written

$$\left(\delta^s_i \overset{\circ}{R}{}^r_{ljk} + \delta^s_k \overset{\circ}{R}{}^r_{lij} + \delta^s_j \overset{\circ}{R}{}^r_{lki}\right) u_r + \left(g_{il}\overset{\circ}{R}{}^s_{rjk} + g_{jl}\overset{\circ}{R}{}^s_{rki} + g_{kl}\overset{\circ}{R}{}^s_{rij}\right) u^r = 0. \tag{3.5}$$

We make $s := i$ and sum with respect to i; it follows

$$(n-2)\,\overset{\circ}{R}{}^r_{ljk} u_r + \left(\overset{\circ}{R}_{lrjk} - g_{jl}\overset{\circ}{R}_{rk} + g_{kl}\overset{\circ}{R}_{rj}\right) u^r = 0, \tag{3.6}$$

where $\overset{\circ}{R}_{jl} = \overset{\circ}{R}{}^i_{jil}$ are the components of Ricci tensor. Multiplying (3.6) by g^{jl} and summing with respect to j and l, it follows

$$(n-2)\,\overset{\circ}{R}_{rk} u^r = 0. \tag{3.7}$$

From (3.6) and (3.7) we get

$$(n-3)\,\overset{\circ}{R}{}^r_{ljk} u_r = 0. \tag{3.8}$$

Because we supposed $n > 3$, from (3.8) we have

$$\overset{\circ}{R}{}^r_{ljk} u_r = 0. \tag{3.9}$$

The relations (3.9) show that, for every $p \in M$,

$$(w\,(g))_p \left(\overset{\circ}{R}_p\,(X_p, Y_p)\,Z_p\right) = 0,\ \forall X_p, Y_p, Z_p \in T_pM. \tag{3.10}$$

Because $\overset{\circ}{R}_p : T_pM \times T_pM \times T_pM \longrightarrow T_pM$ is surjective, from (3.10) we obtain $w\,(g))_p = 0$, $\forall p \in M$, so $w\,(g) = 0$ and using (2.3), we find

$$g\,(A\,(X,Y)\,,Z) = 0,\ \forall X,Y,Z \in \mathcal{X}\,(M)\,. \tag{3.11}$$

Because g is non-degenerate, from (3.11) it follows $A=0$, so $\overset{\lambda}{A}=\lambda A=0$, i.e. $\overset{\lambda}{\nabla}=\overset{\circ}{\nabla}$.

(iii)$\Longrightarrow$(i) The linear connections $\overset{\lambda}{\nabla}$ and $\overset{\circ}{\nabla}$ are symmetric, hence they admit the same geodesics if and only if there exists an $1-$form on M such that

$$\overset{\lambda}{\nabla}_X Y=\overset{\circ}{\nabla}_X Y+\omega\left(X\right)Y+\omega\left(Y\right)X,\ \forall X,Y\in\mathcal{X}\left(M\right). \tag{3.12}$$

From (3.12) and from $\overset{\lambda}{A}=\lambda A$, we obtain

$$\lambda A\left(X,X\right)=2\omega\left(X\right)X,\ \forall X\in\mathcal{X}\left(M\right). \tag{3.13}$$

For $Y=X$, from (2.3) it follows

$$2g\left(A\left(X,X\right),Z\right)=2w\left(g\right)\left(X\right)g\left(X,Z\right)-w\left(g\right)\left(Z\right)g\left(X,X\right),\ \forall X,Z\in\mathcal{X}\left(M\right). \tag{3.14}$$

From (3.13) and (3.14), we obtain

$$4\omega\left(X\right)g\left(X,Z\right)-2\lambda w\left(g\right)\left(X\right)g\left(X,Z\right)=-\lambda w\left(g\right)\left(Z\right)g\left(X,X\right), \tag{3.15}$$

for all $X,Z\in\mathcal{X}\left(M\right)$.

Because $n>3$, for every $p\in M$ and every $Z_p\in T_pM-\{\mathbf{0}\}$, there exists a vector $X_p\in T_pM-\{\mathbf{0}\}$ such that we have

$$g_p\left(X_p,Z_p\right)=0,\ g_p\left(X_p,X_p\right)\neq 0. \tag{3.16}$$

From (3.15) and (3.16), we get

$$\left(w\left(g\right)\right)_p\left(Z_p\right)=0,\ \forall Z_p\in T_pM-\{\mathbf{0}\},\ \forall p\in M,$$

so $w\left(g\right)=0$ and from (2.3) we obtain that $A=0$, so $\overset{\lambda}{A}=0$, i.e. $\overset{\lambda}{\nabla}=\overset{\circ}{\nabla}$.

(iv)$\Longrightarrow$(i) For $\lambda=0$ is trivial. We suppose $\lambda\neq 0$. Because the algebra $\mathcal{U}\left(M,\overset{\lambda}{\nabla}-\overset{\circ}{\nabla}\right)$ is commutative, it follows that $\mathcal{U}\left(M,\overset{\lambda}{\nabla}-\overset{\circ}{\nabla}\right)$ is associative if and only if we have

$$A\left(X,A\left(Y,Z\right)\right)=A\left(Y,A\left(X,Z\right)\right),\ \forall X,Y,Z\in\mathcal{X}\left(M\right). \tag{3.17}$$

In local coordinates, (3.17) becomes

$$A^i_{sk}A^s_{jl}-A^i_{sl}A^s_{jk}=0. \tag{3.18}$$

From (3.18), using (2.4), we can write

$$\delta^i_k u_j u_l-\delta^i_l u_j u_k+g_{jl}u^i u_k-g_{jk}u^i u_l+\delta^i_l g_{jk}u_s u^s-\delta^i_k g_{jl}u^s u_s=0. \tag{3.19}$$

In (3.19), we make $i:=k$ and sum with respect to i; it follows

$$\left(n-2\right)\left(u_j u_l-g_{jl}u_s u^s\right)=0.$$

Because $n>3$, we obtain

$$u_j u_l-g_{jl}u^s u_s=0. \tag{3.20}$$

Multiplying (3.20) by g^{jl} and summing with respect to j and l, we get

$$u^s u_s = 0. \tag{3.21}$$

From (3.20) and (3.21) we obtain $u_j u_l = 0$, $\forall j, l \in \{1, 2, ..., n\}$. Therefore $w(g) = 0$ and, from (2.3), it follows $A = 0$, so $\overset{\lambda}{\nabla} = \overset{\circ}{\nabla}$.

(v)$\Longrightarrow$(i) In local coordinates, the conformal Weyl connection has the components

$$\Gamma^i_{jk} = \left|{}^{\,i}_{jk}\right| + \delta^i_j u_k + \delta^i_k u_j - g_{jk} u^i, \tag{3.22}$$

where $\left|{}^{\,i}_{jk}\right|$ are the Christoffel symbols of the second kind, constructed using the metric components g_{ij}. From (3.22) and from $\overset{\lambda}{\nabla} = \overset{\circ}{\nabla} + \lambda\left(\nabla - \overset{\circ}{\nabla}\right)$, we obtain the components of the linear connection $\overset{\lambda}{\nabla}$:

$$\overset{\lambda}{\Gamma}{}^i_{jk} = \left|{}^{\,i}_{jk}\right| + \delta^i_j \psi_k + \delta^i_k \psi_j - g_{jk} \psi^i, \tag{3.23}$$

where $\psi_i = \lambda u_i$. The curvature tensor of the connection $\overset{\lambda}{\nabla}$ has the components

$$\overset{\lambda}{R}{}^i_{jkl} = \overset{\circ}{R}{}^i_{jkl} + \delta^i_j (\psi_{kl} - \psi_{lk}) + \delta^i_k \psi_{jl} - \delta^i_l \psi_{jk} - g_{jk} \psi^i_l + g_{jl} \psi^i_k, \tag{3.24}$$

where we denoted

$$\psi_{jl} = \frac{\partial \psi_j}{\partial x^l} + \left|{}^{\,r}_{jl}\right| \psi_r + \psi_j \psi_l - \frac{1}{2}\phi^2 g_{jl},$$

$$\psi^i_l = g^{ij}\psi_{jl},\ \phi^2 = g_{rs}\psi^r\psi^s = \psi_i \psi^i.$$

If we assign $i = k$ and sum, from (3.24) we obtain

$$\overset{\lambda}{R}_{jl} = \overset{\circ}{R}_{jl} + (n-1)\,\psi_{jl} - \psi_{lj} + g_{jl}\varphi, \tag{3.25}$$

where $\overset{\lambda}{R}_{jl} = \overset{\lambda}{R}{}^i_{jil}$, $\varphi = \psi^i_i = g^{jl}\psi_{jl}$, $\overset{\circ}{R}_{jl} = \overset{\circ}{R}{}^i_{jil}$. Because the 1$-$form $w(g)$ is exact, it follows that $\psi_{jl} = \psi_{lj}$. From (3.25), we have

$$\overset{\lambda}{R}_{jl} = \overset{\circ}{R}_{jl} + (n-2)\,\psi_{jl} + g_{jl}\varphi. \tag{3.26}$$

Using the hypothesis, $\overset{\lambda}{R}_{jl} = \overset{\circ}{R}_{jl}$. Therefore, from (3.26) we get

$$(n-2)\,\psi_{jl} + g_{jl}\varphi = 0. \tag{3.27}$$

Multiplying (3.27) by g^{jl} and summing with respect to j and l, it follows

$$\varphi = 0. \tag{3.28}$$

From (3.27) and (3.28), we have

$$\psi_{jl} = 0,\ \psi^i_l = 0. \tag{3.29}$$

The relations (3.28) and (3.29) show that

$$\overset{\lambda}{R}{}^{i}_{jkl} = \overset{\circ}{R}{}^{i}_{jkl}.$$

From the last relation, we find (i). $\square$

Remark 3.1. For $n = 3$, the theorem remains true if we replace the condition "$\overset{\circ}{R}_p : T_pM \times T_pM \times T_pM \longrightarrow T_pM$ is surjective, $\forall p \in M$" by "the Ricci tensor of the semi-Riemannian manifold (M, g) is non-degenerate". This is easy to see from (3.7).

Theorem 3.2. *Let $\mathcal{U}\left(M, \overset{\lambda}{\nabla} - \overset{\circ}{\nabla}\right)$ be the deformation algebra as above. We suppose that the manifold M is connected and $n > 2$. The following assertions are equivalent:*

- *(i)* $\overset{\lambda}{\nabla} = \overset{\circ}{\nabla}$;
- *(ii) all elements of the deformation algebra $\mathcal{U}\left(M, \overset{\lambda}{\nabla} - \overset{\circ}{\nabla}\right)$ are almost principal fields;*
- *(iii) all elements of the deformation algebra $\mathcal{U}\left(M, \overset{\lambda}{\nabla} - \overset{\circ}{\nabla}\right)$ are principal fields;*
- *(iv) all elements of the deformation algebra $\mathcal{U}\left(M, \overset{\lambda}{\nabla} - \overset{\circ}{\nabla}\right)$ are almost special fields;*
- *(v) all elements of the deformation algebra $\mathcal{U}\left(M, \overset{\lambda}{\nabla} - \overset{\circ}{\nabla}\right)$ are special fields;*
- *(vi) all elements of the deformation algebra $\mathcal{U}\left(M, \overset{\lambda}{\nabla} - \overset{\circ}{\nabla}\right)$ are $2-$ nilpotent fields.*

Proof. (i)$\Longrightarrow$(ii), (i)$\Longrightarrow$(iii), (i)$\Longrightarrow$(iv), (i)$\Longleftrightarrow$(v), (i)$\Longleftrightarrow$(vi) are obvious.

(iv)$\Longrightarrow$(i). Because $\overset{\lambda}{A}(Z, X) = f_X Z$, $\forall X, Z \in \mathcal{X}(M)$, it follows that there exists an $1-$form θ on M such that $\theta(X) = f_X$. We have, therefore $\overset{\lambda}{A}(Z, X) = \theta(X) Z$, $\forall X, Z \in \mathcal{X}(M)$. Because $\overset{\lambda}{A}$ is symmetric, we get $\theta(X) Z = \theta(Z) X$, $\forall X, Z \in \mathcal{X}(M)$. In local coordinates, the last equality is written $\theta_i \delta^j_k = \theta_k \delta^j_i$. If we make $j = i$ and sum, we obtain $(n-1)\theta_k = 0$, so $\theta = 0$ and, finally, $\overset{\lambda}{\nabla} = \overset{\circ}{\nabla}$.

(iii)$\Longrightarrow$(i). We have $\overset{\lambda}{A}(Z, X) = \omega(Z) X$, $\forall X, Z \in \mathcal{X}(M)$. Because $\overset{\lambda}{A}(Z, X) = \overset{\lambda}{A}(X, Z)$, $\forall X, Z \in \mathcal{X}(M)$, we obtain $\omega(Z) X = \omega(X) Z$, $\forall X, Z \in \mathcal{X}(M)$, which implies $\omega = 0$, i.e. $\overset{\lambda}{\nabla} = \overset{\circ}{\nabla}$.

(ii)$\Longrightarrow$(i). Using the hypothesis, we have

$$\overset{\lambda}{A}(Z, X) = f_X Z + \omega(Z) X, \forall X, Z \in \mathcal{X}(M).$$

From here, it follows that there exists an $1-$form η, defined on M, such that $\eta(X) = f_X$. Therefore, we have $\overset{\lambda}{A}(Z,X) = \eta(X)Z + \omega(Z)X$, $\forall X, Z \in \mathcal{X}(M)$. Because $\overset{\lambda}{A}$ is symmetric, we find that

$$(\eta(X) - \omega(X))Z = (\eta(Z) - \omega(Z))X, \; \forall X, Z \in \mathcal{X}(M).$$

In local coordinates, the last equality may be written

$$(\eta_i - \omega_i)\,\delta_k^j - (\eta_k - \omega_k)\,\delta_i^j = 0.$$

If we assign $j = k$ and sum, we get

$$(n-1)(\eta_i - \omega_i) = 0, \; \forall i \in \{1, 2, ..., n\}.$$

Therefore $\eta = \omega$ and we have

$$\overset{\lambda}{\nabla}_Z X = \overset{\circ}{\nabla}_Z X + \eta(X)Z + \eta(Z)X, \; \forall X, Z \in \mathcal{X}(M).$$

The last equality shows that the symmetric linear connections $\overset{\lambda}{\nabla}$ and $\overset{\circ}{\nabla}$ admit the same geodesics. Using Theorem 3.1, we obtain $\overset{\lambda}{\nabla} = \overset{\circ}{\nabla}$. □

Remark 3.2. In the following, we will denote by $C \in \mathcal{T}_3^1(M)$ the curvature conformal Weyl tensor. Let C^i_{jkl} the components of C in a system of local coordinates. Then, we have ([11], [12])

$$\begin{aligned} C^i_{jkl} &= \overset{\circ}{R}{}^i_{jkl} - \frac{1}{n-2}\left(\delta^i_k \overset{\circ}{R}_{jl} - \delta^i_l \overset{\circ}{R}_{jk} g_{jk} g^{is} \overset{\circ}{R}_{sl} + g_{jl} g^{is} \overset{\circ}{R}_{sk}\right) \\ &\quad + \frac{g^{rs}\overset{\circ}{R}_{rs}}{(n-1)(n-2)}\left(\delta^i_k g_{jl} - \delta^i_l g_{jk}\right). \end{aligned}$$

Theorem 3.3. *With the above notations, we suppose that*

(i) the $1-$form $w(g)$ is exact;

(ii) $\overset{\circ}{R}_p : T_pM \times T_pM \times T_pM \longrightarrow T_pM$ is surjective, $\forall p \in M$;

(iii) the curvature conformal Weyl tensor is nowhere vanishing, i.e. $C_p \neq 0$, $\forall p \in M$.

If there exists a function $f \in \mathcal{F}(M)$, $f(p) \neq 0$, $\forall p \in M$, such that $\overset{\lambda}{R} = f\overset{\circ}{R}$, then $\overset{\lambda}{\nabla} = \overset{\circ}{\nabla}$.

Proof. In local coordinates, the linear connection $\overset{\lambda}{\nabla}$ has the components

$$\overset{\lambda}{\Gamma}{}^i_{jk} = \left|{}^{\,i}_{jk}\right| + \delta^i_j \psi_k + \delta^i_k \psi_j - g_{jk}\psi^i,$$

where $\left|{i \atop jk}\right|$ are the Christoffel symbols of the second kind, constructed using the metric components g_{ij} and ψ_i are the components of $1-$form $\frac{\lambda}{2}w\left(g\right)$. Because the $1-$form $w\left(g\right)$ is exact, the last equalities imply

$$\overset{\lambda}{R}{}^{i}_{jkl} = \overset{\circ}{R}{}^{i}_{jkl} + \delta^i_k \psi_{jl} - \delta^i_l \psi_{jk} - g_{jk}\psi^i_l + g_{jl}\psi^i_k, \tag{3.30}$$

where

$$\psi_{jl} = \frac{\partial \psi_j}{\partial x^l} + \left|{r \atop jl}\right| \psi_r + \psi_j \psi_l - \frac{1}{2} g_{jl}\psi^s\psi_s = \psi_{lj},\ \psi^i_k = g^{ij}\psi_{jk}. \tag{3.31}$$

From (3.30), we have

$$\overset{\lambda}{R}_{jl} = \overset{\circ}{R}_{jl} + \left(n-2\right)\psi_{jl} + g_{jl} g^{rs}\psi_{rs}, \tag{3.32}$$

where $\overset{\lambda}{R}_{jl} = \overset{\lambda}{R}{}^{k}_{jkl} = \overset{\lambda}{R}_{lj}$. Multiplying (3.32) by g^{jl} and summing with respect to j and l, we obtain

$$g^{jl}\overset{\lambda}{R}_{jl} = g^{jl}\overset{\circ}{R}_{jl} + 2\left(n-1\right) g^{rs}\psi_{rs}. \tag{3.33}$$

Because $n > 3$, from (3.32) and (3.33) we deduce that

$$g^{rs}\psi_{rs} = \frac{1}{2\left(n-1\right)}\left(\overset{\lambda}{R}_{rs} - \overset{\circ}{R}_{rs}\right) g^{rs} \tag{3.34}$$

$$\psi_{jl} = \frac{1}{n-2}\left(\overset{\lambda}{R}_{jl} - \overset{\circ}{R}_{jl}\right) - \frac{g_{jl}}{2\left(n-1\right)\left(n-2\right)} g^{rs}\left(\overset{\lambda}{R}_{rs} - \overset{\circ}{R}_{rs}\right). \tag{3.35}$$

Introducing (3.34), (3.35) in (3.30), we find

$$\overset{\lambda}{R}{}^{i}_{jkl} - \frac{1}{n-2}\left(\delta^i_k \overset{\lambda}{R}_{jl} - \delta^i_l \overset{\lambda}{R}_{jk} + g_{jk} g^{is}\overset{\lambda}{R}_{sl} + g_{jl} g^{is}\overset{\lambda}{R}_{sk}\right)$$

$$+\frac{g^{rs}\overset{\lambda}{R}_{rs}}{\left(n-1\right)\left(n-2\right)}\left(\delta^i_k g_{jl} - \delta^i_l g_{jk}\right) = C^i_{jkl}, \tag{3.36}$$

where

$$\begin{aligned} C^i_{jkl} \quad &= \quad \overset{\circ}{R}{}^{i}_{jkl} - \frac{1}{n-2}\left(\delta^i_k \overset{\circ}{R}_{jl} - \delta^i_l \overset{\circ}{R}_{jk} - g_{jk} g^{is}\overset{\circ}{R}_{sl} + g_{jl} g^{is}\overset{\circ}{R}_{sk}\right) \\ &\quad +\frac{g^{rs}\overset{\circ}{R}_{rs}}{\left(n-1\right)\left(n-2\right)}\left(\delta^i_k g_{jl} - \delta^i_l g_{jk}\right) \end{aligned}$$

are the components of the curvature conformal Weyl tensor C. On the other hand, $\overset{\lambda}{R} = f\overset{\circ}{R}$ implies that

$$\overset{\lambda}{R}{}^{i}_{jkl} = f\overset{\circ}{R}{}^{i}_{jkl},\ \overset{\lambda}{R}_{jl} = f\overset{\circ}{R}_{jl},\ g^{rs}\overset{\lambda}{R}_{rs} = f g^{rs}\overset{\circ}{R}_{rs},$$

$$\overset{\lambda}{R}{}^{i}_{jkl}-\frac{1}{n-2}\left(\delta^i_k\overset{\lambda}{R}_{jl}-\delta^i_l\overset{\lambda}{R}_{jk}-g_{jk}g^{is}\overset{\lambda}{R}_{sl}+g_{jl}g^{is}\overset{\lambda}{R}_{sk}\right)$$

$$+\frac{g^{rs}\overset{\lambda}{R}_{rs}}{(n-1)(n-2)}\left(\delta^i_k g_{jl}-\delta^i_l g_{jk}\right)=fC^i_{jkl}. \tag{3.37}$$

Because $C_p \neq 0$, $\forall p \in M$, it follows that the functions C^i_{jkl} are nowhere vanishing. Taking this into the account, the relations (3.36) and (3.37) show that we have $f=1$, so $\overset{\lambda}{R}=\overset{\circ}{R}$. Using Theorem (3.1), we obtain $\overset{\lambda}{\nabla}=\overset{\circ}{\nabla}$. □

Theorem 3.4. *Let (M,g) be a connected $n-$dimensional semi-Riemannian manifold, with $n\geq 3$. We consider two Weyl structures w and w' on the conformal manifold $(M,\widehat{g})$. Let ∇ (resp. ∇') be the symmetric conformal Weyl connection, compatible with the Weyl structure w (resp. w'). Define the family of linear connections*

$$\widetilde{\mathcal{C}}=\left\{\nabla+\lambda\left(\nabla'-\nabla\right)\mid \lambda\in\mathbb{R}\right\}.$$

For $\lambda\in\mathbb{R}$, we consider the linear connection $\overset{\lambda}{\nabla}=\nabla+\lambda\left(\nabla'-\nabla\right)\in\widetilde{\mathcal{C}}$. The following assertions are equivalent:

- *(i)* $\overset{\lambda}{\nabla}=\nabla$;
- *(ii) the deformation algebra* $\mathcal{U}\left(M,\overset{\lambda}{\nabla}-\nabla\right)$ *is associative;*
- *(iii)* $\overset{\lambda}{\nabla}$ *and* ∇ *admit the same geodesics;*
- *(iv) all the elements of the deformation algebra* $\mathcal{U}\left(M,\overset{\lambda}{\nabla}-\nabla\right)$ *are almost principal fields.*

Proof. (i)$\Longrightarrow$(ii), (i)$\Longrightarrow$(iii), (i)$\Longrightarrow$(iv) are obvious.

(ii)$\Longrightarrow$(i) The algebra $\mathcal{U}\left(M,\overset{\lambda}{\nabla}-\nabla\right)$ is commutative. It follows that $\mathcal{U}\left(M,\overset{\lambda}{\nabla}-\nabla\right)$ is associative if and only if we have

$$\overset{\lambda}{A}\left(X,\overset{\lambda}{A}(Y,Z)\right)-\overset{\lambda}{A}\left(Y,\overset{\lambda}{A}(X,Z)\right)=0,\ \forall X,Y,Z\in\mathcal{X}(M), \tag{3.38}$$

where $\overset{\lambda}{A}=\overset{\lambda}{\nabla}-\nabla=\lambda A'$, $A'=\overset{\lambda}{\nabla}-\nabla$.

For $\lambda=0$ the assertion is trivial. We suppose $\lambda\neq 0$. The linear connection ∇' is defined by

$$\begin{aligned}2g(\nabla'_X Y,Z) \quad = \quad & X(g(Y,Z))+Y(g(X,Z))-Z(g(X,Y))+w'(g)(X)g(Y,Z)+\\ & +w'(g)(Y)g(X,Z)-w'(g)(Z)g(X,Y)+g([X,Y],Z)+\\ & +g([Z,X],Y)-g([Y,Z],X),\ \forall X,Y,Z\in\mathcal{X}(M).\end{aligned}$$

From here, it follows that A' is given by

$$g\left(A'\left(X,Y\right),Z\right)=\varphi\left(X\right)g\left(Y,Z\right)+\varphi\left(Y\right)g\left(X,Z\right)-\varphi\left(Z\right)g\left(X,Y\right),\quad \forall X,Y,Z\in\mathcal{X}\left(M\right), \tag{3.39}$$

where we used the notation $2\varphi=w'\left(g\right)-w\left(g\right)$.

Let A'^{i}_{jk} (resp. φ_i) the components of A' (resp. φ) in a system of local coordinates. Then we can rewrite (3.39) as

$$A'^{i}_{jk}=\delta^i_j\varphi_k+\delta^i_k\varphi_j-g_{jk}\varphi^i, \tag{3.40}$$

where $\varphi^i=g^{ik}\varphi_k$. In local coordinates, (3.38) can be written

$$\left(\delta^i_k\varphi_l-\delta^i_l\varphi_k\right)\varphi_j+\left(\delta^i_lg_{jk}-\delta^i_kg_{jl}\right)\varphi_s\varphi^s+\left(g_{jl}\varphi_k-g_{jk}\varphi_l\right)\varphi^i=0.$$

For $i=k$, we sum and obtain

$$\left(n-2\right)\left(\varphi_j\varphi_l-g_{jl}\varphi_s\varphi^s\right)=0. \tag{3.41}$$

Because $n\geq 3$, from (3.41) we get

$$\varphi_j\varphi_l-g_{jl}\varphi_s\varphi^s=0. \tag{3.42}$$

Multiplying (3.42) by g^{jl} and summing with respect to j and l, we have $\varphi_s\varphi^s=0$. Taking into account this, from (3.42) follows $\varphi_j\varphi_l=0$, $\forall j,l\in\{1,2,...,n\}$. We obtain $\varphi=0$, i.e. $w=w'$. From (3.39) we find $A'=0$, so $\overset{\lambda}{A}=0$, i.e. $\overset{\lambda}{\nabla}=\nabla$.

(iii)$\Longrightarrow$(i). Because the linear connections $\overset{\lambda}{\nabla}$ and ∇ are symmetric, it follows that they admit the same geodesics if and only if there exists an $1-$form σ, defined on M, such that

$$\overset{\lambda}{\nabla}_XY=\nabla_XY+\sigma\left(X\right)Y+\sigma\left(Y\right)X,\ \forall X,Y\in\mathcal{X}\left(M\right). \tag{3.43}$$

From (3.43) we get

$$\lambda A'\left(X,X\right)=2\sigma\left(X\right)X,\ \forall X\in\mathcal{X}\left(M\right). \tag{3.44}$$

For $\lambda=0$, we obtain $\sigma=0$ and from (3.34) we have $\overset{\lambda}{\nabla}=\nabla$. We shall consider the case for $\lambda\neq 0$. For $Y=X$, from (3.39), we find

$$g\left(A'\left(X,X\right),Z\right)=2\varphi\left(X\right)g\left(X,Z\right)-\varphi\left(Z\right)g\left(X,X\right),\ \forall X,Z\in\mathcal{X}\left(M\right). \tag{3.45}$$

From (3.44) and (3.45), we get

$$\left\{2\sigma\left(X\right)-2\lambda\varphi\left(X\right)\right\}g\left(X,Z\right)+\lambda\varphi\left(Z\right)g\left(X,X\right)=0,\ \forall X,Z\in\mathcal{X}\left(M\right). \tag{3.46}$$

Because $n\geq 3$, for any $p\in M$ and any $Z_p\in T_pM-\{0\}$, there exists $X_p\in T_pM-\{0\}$ such that we have

$$g_p\left(X_p,Z_p\right)=0,\ g_p\left(X_p,X_p\right)\neq 0. \tag{3.47}$$

From (3.46) and (3.47) we obtain $\varphi_p(Z_p) = 0$, for any $Z_p \in T_pM - \{0\}$ and any $p \in M$. Therefore, $\varphi = 0$, so $w' = w$. From (3.39) follows $g(A'(X,Y), Z) = 0$, $\forall X, Y, Z \in \mathcal{X}(M)$. Because the metric g is non-degenerate, from the last equality we find $A'(X,Y) = 0$, $\forall X, Y \in \mathcal{X}(M)$, so $A' = 0$. In conclusion, $\overset{\lambda}{A} = \lambda A' = 0$, i.e. $\overset{\lambda}{\nabla} = \nabla$.

(iv)$\Longrightarrow$(i). Because all elements of the algebra $\mathcal{U}\left(M, \overset{\lambda}{\nabla} - \nabla\right)$ are almost principal fields, it follows that for any $X \in \mathcal{X}(M)$ there exists a function $f_X \in \mathcal{F}(M)$ and an $1-$form $\omega \in \Lambda^1(M)$ such that

$$\overset{\lambda}{A}(Z,X) = f_X Z + \omega(Z)X, \forall Z, X \in \mathcal{X}(M). \tag{3.48}$$

From (3.48), there exists an $1-$form $\eta \in \Lambda^1(M)$ such that $\eta(X) = f_X$. From (3.48), we obtain

$$\overset{\lambda}{A}(Z,X) = \eta(X)Z + \omega(Z)X, \forall Z, X \in \mathcal{X}(M). \tag{3.49}$$

Because the algebra $\mathcal{U}\left(M, \overset{\lambda}{\nabla} - \nabla\right)$ is abelian, from the last equality follows $\eta = \omega$ and from (3.49) we obtain

$$\overset{\lambda}{\nabla}_Z X = \nabla_Z X + \eta(X)Z + \eta(Z)X, \forall Z, X \in \mathcal{X}(M).$$

The last equality shows that the symmetric and linear connections $\overset{\lambda}{\nabla}$ and ∇ admit the same geodesics. From here, $\overset{\lambda}{\nabla} = \nabla$. □

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