Tzitzeica and sine-Gordon solitons

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Abstract. Our paper studies the soliton Tzitzeica PDE and soliton sine-Gordon PDE. Section 1 poves that Tzitzeica solitons are expressed either by elliptic integrals or in parametric forms. Section 2 shows that the Tzitzeica soliton PDE and the sine-Gordon PDE are generated respectively by a flow and an appropriate Lorentzian metric. Section 3 introduces and studied a generalization of Tzitzeica PDE. Section 4 rises the question of generated soliton PDEs.

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1 Soliton solutions of Tzitzeica PDE

In a series of papers between 1907 and 1910, the Romanian geometer Tzitzeica [23]-[24] investigated a particular class of surfaces associated with a nonlinear wave PDE

$$(\ln h)_{uv} = h - \frac{1}{h^2},$$

known in our days as *Tzitzeica hyperbolic nonlinear PDE.* The Tzitzeica surfaces are the analogues of spheres in affine differential geometry [10]. That is why, according to Nomizu and Sasaki, the origins of affine differential geometry reside in the work of Tzitzeica.

The rediscovery of the Tzitzeica PDE in a solitonic context had to wait until some seventy years later [1]-[9], [12]-[22], [27] and further links between soliton theory and affine differential geometry are treated in [17].

Via the change $\ln h = \omega$ of the unknown function, the Tzitzeica equation is written as

$$\omega_{uv} = e^{\omega} - e^{-2\omega}. \tag{1.1}$$

For the Tzitzeica PDE (1.1), we seek for *soliton solutions* (solitary wave solutions) in the scalar form

$$\omega(u, v) = \phi(u + cv) = \phi(\xi),\ \xi = u + cv.$$

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These soliton solutions do not depend on (u, v) coordinates in explicit form. The case $c = 1$ was discussed by Tzitzeica [[17], pg. 88], asserting that this solitons are elliptic integrales.

The case $c = -1$ is the subject of the present paper. With foregoing choice, the PDE (1.1) becomes an ODE, $\phi'' = e^{-2\phi} - e^{\phi}$. We multiply both sides by $\phi'(\xi)$ and write the ODE as $\phi''\phi' = e^{-2\phi}\phi' - e^{\phi}\phi'$. Taking the primitive, we find a first integral and hence a new ODE

$$\phi'^{\,2} = k - e^{-2\phi} - 2e^{\phi}.$$

For $k = 3$ we have only the trivial solution $\phi(\xi) = 0$ (which corresponds to the Tzitzeica solution $h = 1$, equilibrium point). In rest, we have non-trivial solutions only for $k > 3$.

Denoting $e^{\phi} = H > 0$, we obtain the ODE

$$H'^2 = -2H^3 + kH^2 - 1. \tag{1.2}$$

Theorem 1.1. *The Tzitzeica solitons (solutions of ODE* (1.1) *via solutions of ODE* (1.2)*) exist only for* $k > 3$ *and are expressed either by elliptic integrals or in parametric forms.*

Remark 1.2. *In general, elliptic integrals cannot be expressed in terms of elementary functions. Exceptions to this general rule are in special conditions. However, with the appropriate reduction formula, every elliptic integral can be brought into a form that involves integrals over rational functions and the three Legendre canonical forms (i.e., the elliptic integrals of the first, second and third kind).*

Remark 1.3. *The parametric solutions can be realized by MAPLE symbolic resources.*

2 Tzitzeica PDE and sine-Gordon PDE generated by flows in the same family and a Lorentz metric

The sine-Gordon soliton PDE is

$$\omega_{uv} = sin\,\omega. \tag{2.1}$$

Now let us show that the Lorentzian metric $\{g^{11} = 0, g^{12} = g^{21} = 1, g^{22} = 0\}$ connects the theory of geometric dynamics to the theory of Tzitzeica solitons and of sine-Gordon solitons (see also [11, 25, 26]), i.e., the Tzitzeica soliton PDE respectively the sine-Gordon soliton PDE are generated by similar 2-flows. For that we introduce the source as a Lorentzian manifold $(R^2,\ g^{11} = 0,\ g^{12} = g^{21} = 1,\ g^{22} = 0)$ and the target as the Riemannian manifold $(R, g = 1)$. Let $\omega : R^2 \to R$, $(u, v) \to \omega(u, v)$ be a C^2 function. On the Riemannian manifold $(R, g = 1)$, we introduce a C^1 function F and two vector fields

$$X_1(\omega) = F(\omega),\ \ X_2(\omega) = -F(\omega).$$

These determine the 2-flow

$$\omega_u = F(\omega),\ \ \omega_v = -F(\omega).$$

The complete integrability condition $\omega_u + \omega_v = 0$ is automatically satisfied. The flow is characterized by the general solution $\omega(u,v) = \phi(u-v)$, where ϕ is a C^2 function. We build the Lagrangian $L = (\omega_u - F)(\omega_v + F)$ and the associated Euler-Lagrange PDE

$$\omega_{uv} = -\frac{1}{2}(F^2)_\omega. \tag{2.2}$$

The PDE (2.2) is just the Tzitzeica PDE iff $-\frac{1}{2}(F^2)_\omega = e^\omega - e^{-2\omega}$, which yields

$$F^2 = K - e^{-2\omega} - 2e^\omega, \; K \geq 3.$$

In this way, the Tzitzeica PDE is a prolongation of the 2-flow if the function F is a solution of the previous ODE.

Theorem 2.1. *The Tzitzeica soliton PDE is generated by a 2-flow and an appropriate Lorentzian metric.*

The PDE (2.2) is just the sine-Gordon PDE (2.1) iff $-\frac{1}{2}(F^2)_\omega = sin\,\omega$, which infers

$$F^2 = K + cos\,\omega, \; K \geq 1.$$

In this way, the sine-Gordon PDE is a prolongation of the 2-flow if the function F is a solution of the previous ODE.

Theorem 2.2. *The sine-Gordon soliton PDE is generated by a 2-flow and an appropriate Lorentzian metric.*

3 Soliton solutions of generalized Tzitzeica PDE

In the paper [26], we extended the Tzitzeica PDE via a Laurent series (power series which includes terms of negative degree) or via a Laurent polynomial (only finitely many coefficients a_n are non-zero),

$$\omega_{uv} = \sum_{n\in Z} a_n e^{n\omega}. \tag{3.1}$$

This ODE has the solution $\omega = 0$, as in the case of Tzitzeica PDE, iff $\sum_{n\in Z} a_n = 0$. Particularizing the sequence a_n, we find different interesting PDEs. For example, the sinh-Gordon equation is of type (3.1). If we have a Laurent series, then $e^\omega \in (0, R)$, $R \leq \infty$. We look for soliton solutions in the scalar form

$$\omega(u,v) = \phi(u + cv) = \phi(\xi),\, \xi = u + cv.$$

The PDE (3.1) reduces to the ODE $c\phi'' = -\sum_{n\in Z} a_n e^{n\phi}$. Multiplying both sides by $\phi'(\xi)$, we get $c\phi''\phi' = -\sum_{n\in Z} a_n e^{n\phi}\phi'$, and taking the primitive, it appears a new ODE

$$c{\phi'}^2 = k - 2\sum_{n\in Z\setminus\{0\}} \frac{a_n}{n} e^{n\phi} - 2a_0\phi. \tag{3.2}$$

For $k = 2\displaystyle\sum_{n\in Z\setminus\{0\}} \frac{a_n}{n}$, we find the equilibrium point $\phi(\xi) = 0$.

Open problem. Study the solutions of the ODEs (3.2), giving the coefficients a_n.

4 Conclusion

Although the Tzitzeica PDE and the sine-Gordon PDE have their roots in the differential geometry of surfaces, now they are the soul of soliton theory. More deeply they are generated by appropriate flows and Lorentzian metrics. But this remark rises the following question: does any soliton PDE can be an Euler-Lagrange prolongation of a flow?

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