A new proof of a theorem of H. C. Wang

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Abstract. Using the result that the group of isometries of a Finsler space is a Lie group, we reprove an important theorem of H.C. Wang. It turns out that our proof is simpler and more direct than H.C. Wang's original one.

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In [6], H. C. Wang proved the following important theorem:

If an $n\,(n > 2, n \neq 4)$ *dimensional Finsler space* (M, F) *admits a group* G *of motions depending on* $r > \frac{1}{2}n(n-1) + 1$ *essential parameters, then* (M, F) *is a Riemannian space of constant curvature.*

Note that although in H. C. Wang's paper [6], Finsler metrics are assumed to be reversible, the proof is actually valid to the non-reversible case. Recently, Deng et al. proved in [2] that the group of isometries of a Finsler space is a Lie group. The purpose of this paper is to use this result to reprove the above result. It is clear that the result can be restated as:

Theorem *Let* (M, F) *be a n-dimensional* $(n > 2, n \neq 4)$ *Finsler space (not necessary reversible). If the group of isometries* $I(M, F)$ *has dimension* $> \frac{1}{2}n(n-1) + 1$*, then* (M, F) *is a Riemannian space of constant curvature.*

The original proof of H. C. Wang is elegant but needs some complicated reasoning. The proof in this paper is simpler and more direct. For more information about Finsler metrics, we refer to [1, 3].

Proof of the theorem. Let x be an arbitrary point in M, $I_x(M)$ be the subgroup of $I(M)$ which leaves x fixed. Then $I(M)$ is a Lie transformation group of M with respect to the compact-open topology and $I_x(M)$ is a compact subgroup of $I(M)$ ([2]). Each $\phi \in I_x(M)$ induces a linear isometry $d\phi_x$ on the Minkowski space $T_x(M)$ ([2]). It is obvious that the correspondence $\phi \to d\phi_x$ is a one-to-one homomorphism from $I_x(M)$ to $GL(T_x(M))$. Denote by $I_x^*(M)$ the group consisting of the image of this homomorphism. Let $I \cdot x$ be the orbit of x under the action of $I(M)$. If $\dim I(M) > \frac{1}{2}n(n-1) + 1$, then

$$\begin{aligned} \dim I_x^*(M) &= \dim I_x(M) \geq \dim I(M) - \dim(I \cdot x) \\ &> \frac{1}{2}n(n-1) + 1 - n = \frac{1}{2}(n-1)(n-2). \end{aligned}$$

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Now fix a base of the linear space $T_x(M)$. Then $I_x^*(M)$ is a compact subgroup of $GL(n,\mathbb{R})$. We assert that the unit component $(I_x^*(M))_e$ of $I_x^*(M)$ is a subgroup of $SL(n,\mathbb{R})$. In fact, the determinant function is continuous on $GL(n,\mathbb{R})$, hence must be bounded on the compact subgroup $(I_x^*(M))_e$. Therefore each element in $(I_x^*(M))_e$ has determinant ≤ 1. On the other hand, if $g \in (I_x^*(M))_e$ has determinant < 1 (and of course > 0), then $g^{-1} \in (I_x^*(M))_e$ has determinant > 1, which is impossible. This proves our assertion. Now $SL(n,\mathbb{R})$ is a connected semisimple Lie group and $SO(n)$ is a maximal subgroup. By the conjugacy of maximal subgroups ([4]), there exists $g \in SL(n,\mathbb{R})$ such that $g^{-1}(I_x^*(M))_e g \subset SO(n)$. Now $\dim(I_x^*(M))_e > \frac{1}{2}(n-1)(n-2)$. According to a Lemma of Montgomery and Samelson ([5]), $O(n)$ contains no proper subgroup of dimension $> \frac{1}{2}(n-1)(n-2)$ other then $SO(n)$. Therefore $g^{-1}(I_x^*(M))_e g = SO(n)$. Now consider the hypersurface $g \cdot S^n$ of $T_x(M)$ (S^n is defined by the inner product defined by assuming the above base to be orthonormal). The group $(I_x^*(M))_e$ acts transitively on it. Hence F is constant on this surface. Therefore $F|_{T_x(M)}$ comes from an inner product of $T_x(M)$. Since x is arbitrary, F is Riemannian. Now $\dim(I_x^*(M))_e = \dim SO(n) = \frac{1}{2}n(n-1)$. So $(I_x^*(M))_e$ acts transitively on the set of the planes in $T_x(M)$. Therefore (M,F) is of constant curvature. $\square$

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References

[1] B. Bidabad, *Complete Finsler manifolds and adapted coordinates*, Balkan J. Geom. Appl., 14 (2) (2009), 21-29.

[2] S. Deng, Z. Hou, *The group of isometries of a Finsler space*, Pacific J. Math. 207 1 (2002), 149-155.

[3] S. Deng, X. Wang, *The S-curvature of homogeneous (α,β) metrics*, Balkan J. Geom. Appl., 15 (2) (2010), 39-48.

[4] S. Helgason, *Differential Geometry, Lie groups, and Symmetric Spaces*, Academic Press, 1978.

[5] D. Montgomery, H. Samelson, *Transformation groups of spheres*, Ann. Math. 44 (3) (1943), 454-470.

[6] H. C. Wang, *Finsler spaces with completely integrable equations of Killing*, J. London Math. Soc. 22 (1947), 5-9.

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