Simplified single-time stochastic maximum principle

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Abstract. Many microeconomic and engineering problems can be formulated as stochastic optimization problems that are modelled by Itô evolution systems and by cost functionals expressed as stochastic integrals. Our paper studies some optimization problems constrained by stochastic evolution systems, giving original results on stochastic first integrals, adjoint stochastic processes and a version of simplified single-time stochastic maximum principle. It extends to the stochastic case the work of first author regarding the geometrical methods in optimal control, constrained by normal ODEs. More precisely, our Lagrangians and Hamiltonians are stochastic 1-forms. Physical and economic applications of the general results are discussed.

M.S.C. 2010: 93E20, 60H20.
Key words: stochastic optimal control, stochastic maximum principle, stochastic first integrals, adjoint process.

1 Introduction

The objective of this paper is to study the single-time stochastic optimal control problems by some crucial geometrical observations/intuitions. Besides mathematical curiosity, however, there are practical motivations for imposing new point of views on such problems.

The paper is organized as follows. Section 2 recalls some preliminaries results on Wiener processes. In Section 3 we start from Itô product formula and a variational stochastic differential system in order to introduce the adjoint (dual) Itô stochastic differential system. Section 4 combines the mathematical ingredients necessary to obtain an optimizing method by selecting a nonanticipative decision among the one satisfying all the constraints. In Theorem 4.1 one proves that if exists an optimal control $u^*(\cdot)$ which determines the stochastic optimal evolution $x(\cdot)$, then there exists an adapted dual process $p(t,\omega)_{t\in\Omega_{0T}}$ which verifies the adjoint (dual) stochastic differential system. This result is called single-time stochastic maximum principle. Section 5 presents the optimal feedback control of a continuously monitored spin. Section 6 analyses the Ramsey and Uzawa-Lucas stochastic models using our formulation

Balkan Journal of Geometry and Its Applications, Vol.16, No.2, 2011, pp. 155-173.

of stochastic maximum principle. Section 7 underlines the most important original contributions.

A still open problem for the stochastic optimal control is the stochastic multitime maximum principle [15]. The difficulties of this problem are involved in the definition of the multitime stochastic Itô evolution and in accepting a payoff as a curvilinear integral or as a multiple integral.

In this paper we formulate and prove a new single-time stochastic maximum principle, different from the classical maximum principles existing in the stochastic literature (e.g., [2], [10]). The advantage of our theory consists in the possibility of extending it to the multitime case. Consequently we are able to state a multitime stochastic maximum principle associated to curvilinear integral actions or multiple integral actions. We remark that one cannot build a multitime version starting from the classical statements of the stochastic maximum principles.

Since the stochastic optimal evolution, the variational optimal evolution and the adjoint optimal evolution are not generally given by explicit formulas, we show how is possible to apply numerical simulations for solving the control stochastic problems.

2 Wiener process

Let $0, T$ be fixed points in $\mathbb{R}_+$ and denote by Ω_{0T} the closed interval $0 \leq t \leq T$. Let $t \in \Omega_{0T}$ be the parameter of evolution or *time*.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space endowed with a *complete*, *increasing* and *right-continuous* filtration (a complete natural history)

$$\left\{(\mathcal{F}_t)_t : t \in \Omega_{0T}\right\}.$$

Such a probability space $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\Omega_{0T}}, \mathbb{P}\right)$ is called *filtred probability space.* Let us denote by $\mathcal{K}_1, \mathcal{K}_2 \subset \mathcal{F}$ two arbitrary $\sigma-$algebras with the property $\mathcal{K}_1 \subset \mathcal{K}_2$. We say that the filtration satisfies the *conditional independence* property if for all bounded random variables X, all $t \in \mathbb{R}_+$, we have

$$\mathbb{E}\left[X \mid \mathcal{K}_1\right] = \mathbb{E}\left[\mathbb{E}\left[X \mid \mathcal{K}_2\right] \mid \mathcal{K}_1\right]. \tag{2.1}$$

This property implies that the conditional expectations with respect to $\mathcal{K}_1$ and $\mathcal{K}_2$ *commutes.*

For a process $x = (x\,(t, \omega))_{t\in\Omega_{0T}}$, the *increment* of x on an interval $(t_1, t_2]$, $t_1 \leq t_2$, is given by

$$x\left((t_1, t_2]\right) = x\left(t_2, \omega\right) - x\left(t_1, \omega\right).$$

Definition 2.1. (Martingale) *Let $x = (x(t, \omega))_{t\in\Omega_{0T}}$ be an $\mathcal{F}_t$-adapted process.*

1. *The process x is called weak martingale if $\mathbb{E}\left[x\left((t, s]\right)(\omega) \mid \mathcal{F}_t\right] = 0$, for all $t, s \in \mathbb{R}_+$, such that $t \leq s$.*

2. *The process x is called martingale if $\mathbb{E}\left[x\left(s, \omega\right) \mid \mathcal{F}_t\right] = x_t$, for all $t, s \in \mathbb{R}_+$, such that $t \leq s$.*

3. *The process x is called strong martingale if $\mathbb{E}\left[x\left((t, s]\right)(\omega) \mid \mathcal{F}_t^*\right] = 0$, for all $t, s \in \mathbb{R}_+$, such that $t \leq s$.*

Obviously, every strong martingale and every martingale is a weak martingale.

Definition 2.2. (Wiener process) *A stochastic process* $(W(t,\omega) : t \in \Omega_{0T})$ *is called Wiener process (starting at zero) (or Brownian motion) if* $W(0,\omega) = 0$ *and* $W(t,\omega)$ *is a gaussian process with* $\mathbb{E}[W(t,\omega)] = 0$ *and for* $t_1, t_2 \in \mathbb{R}$*, we have*

$$\mathbb{E}[W(t_1,\omega)\, W(t_2,\omega)] = \min\{t_1, t_2\}.$$

Definition 2.3. *The stochastic process* $(W(t,\omega) : t \in \Omega_{0T})$ *is called* $\mathcal{F}_t-$ *Wiener process if, in addition,*

$$\mathbb{E}[W(s,\omega) \mid \mathcal{F}_t] = W(t,\omega),$$

for all $t, s \in \mathbb{R}_+$*, such that* $t \le s$*.*

A first example of martingale is the Wiener process.

Hypothesis RL *Suppose that a sample function* $x : \Omega_{0T} \to \mathbb{R}$ *is continuous from the right and limited from the left at every point. That means, for every* $t_0 \in T$*,* $t \downarrow t_0$*, implies* $x(t) \longrightarrow x(t_0)$ *and for* $t \uparrow t_0$*,* $\lim_{t \uparrow t_0} x(t)$ *exists, but need not be* $x(t_0)$*. We use only stochastic processes* x *where almost all sample paths have the* **RL** *property.*

3 Itô product formula and adjoint stochastic systems

Let Ω_{0T} be the closed interval $0 \le t \le T$ in $\mathbb{R}_+$ and $t \in \Omega_{0T}$ be the *time*. For any given Euclidean space H, we denote by $\langle \cdot, \cdot \rangle$ (resp. $|\cdot|$) the *inner product* (resp. *norm*) of H. Let $\mathcal{M}^2(\Omega_{0T}; H)$ denote the space of all $\mathcal{F}_t-$progressively processes $x(\cdot,\omega)$ with values in H such that

$$\mathbb{E}\int_{\Omega_{0T}} |x(t,\omega)|^2\, dt < \infty.$$

Given a filtred probability space $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}^m_+}, \mathbb{P}\right)$ satisfying the usual conditions, on which a Wiener process $W(\cdot,\omega)$ with values in $\mathbb{R}^d$ is defined, we consider a constraint as a *controlled stochastic system*

$$\left\{ \begin{array}{l} dx^i_t = \mu^i(t, x_t, u_t)\, dt + \sigma^i_a(t, x_t, u_t)\, dW^a_t, \\ x(0) = x_0 \in \mathbb{R}^n, \ a = \overline{1,d}, \ i = \overline{1,n}, \end{array} \right. \tag{3.1}$$

where

$$\mu(\cdot, x(\cdot,\omega), u(\cdot,\omega)) : \Omega_{0T} \times \mathbb{R}^n \times U \longrightarrow \mathbb{R}^n,$$

$$\sigma(\cdot, x(\cdot,\omega), u(\cdot,\omega)) : \Omega_{0T} \times \mathbb{R}^n \times U \longrightarrow \mathbb{R}^{n\times d}$$

and, for simplicity, we denote $x(t,\omega)$, respectively $u(t,\omega)$, by x_t and u_t. Here and in the whole paper we use Einstein summation convention. For a new viewpoint regarding the stochastic ODE, see [3], [4]. We assume:

(**H1**) μ, σ, and f are continuous in their arguments and continuously differentiable in (x,u);

(**H2**) the derivatives of μ and σ in (x,u) are bounded;

(**H3**) the derivatives of f in (x,u) are bounded by $C(1+|x|+|u|)$ and the derivative of h in x is bounded by $C(1+|x|)$.

The process $u(\cdot,\omega)$ is called *control (vector-valued) variable*. We assume that $u(t,\omega)$ has values in a given closed set in $\mathbb{R}^k$ and that $u(t,\omega)$ is satisfying the hypothesis RL. In addition we require that $u(t,\omega)$ gives rise to a unique solution $x(t)=x^{(u)}(t)$ of (3.1) for $t\in\Omega_{0T}$. This control is taken from the set

$$\mathcal{A}=\left\{u(\cdot,\omega)\mid u(\cdot,\omega)\in\mathcal{M}^2\left(\Omega_{0T},\mathbb{R}^k\right)\right\}.$$

Any $u(\cdot)\in\mathcal{A}$ is called a *feasible control*.

Definition 3.1. *Let* $\left(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\in\mathbb{R}_+},\mathbb{P}\right)$ *be given, satisfying the usual conditions and let* $W(t)$ *be a given standard* $(\mathcal{F}_t)_{t\in\mathbb{R}_+}-$*Wiener process with values in* $\mathbb{R}^d$. *A control* $u(\cdot,\omega)$ *is called admissible, and the pair* $(x(\cdot,\omega),u(\cdot,\omega))$ *is called admissible, if*

1. $u(\cdot,\omega)\in\mathcal{A}$;
2. $x(\cdot,\omega)$ *is the unique solution of system (3.1);*
3. *some additional convex constraint on the terminal state variable are satisfied, e.g.*
$$x(T,\omega)\in K,$$
where K *is a given nonempty convex subset in* $\mathbb{R}^n$*;*
4. $f(\cdot,x(\cdot,\omega),u(\cdot,\omega))\in L^1_{\mathcal{F}}(\Omega_{0T};\mathbb{R}^n)$ *and* $h(x(T,\omega))\in L^1_{\mathcal{F}}(\Omega;\mathbb{R})$.

The set of all admissible controls is denoted by $\mathcal{A}_{ad}$.

Taking into account hypothesis (**H1**)-(**H3**), for a given $u(\cdot,\omega)\in\mathcal{A}_{ad}$, there exists a unique solution

$$x(\cdot,\omega)\in\mathcal{M}^2(\Omega_{0T},\mathbb{R}^n)$$

of the system (3.1) (see [8] or [20]).

3.1 Itô product formula

In order to prove the single-time stochastic maximum principle using the ideas rising from the papers [15], [14], [17], [16], we need the following auxiliary result, which is a special case of the Itô formula ([2, Theorem 3.5.2, p. 265]). To prove this Lemma, one uses *Itô stochastic calculus rules*: $dW_t^a\,dW_t^b=\delta^{ab}dt$, $dW_t^a\,dt=dt\,dW_t^a=0$, respectively $dt^2=0$, for any $a,b=\overline{1,d}$, where the Kronecker symbol δ^{ab} represents the *correlation coefficient*.

Lemma 3.1. (Itô product formula) *Suppose the processes* $\left(x^i(t)\right)_{t\in\Omega_{0T}}$ *and* $(p_i(t))_{t\in\Omega_{0T}}$ *are solutions of the Itô stochastic systems*

$$\begin{cases} dx^i(t)=\mu^i(t,x(t,\omega),u(t,\omega))\,dt+\sigma^i_a(t,x(t,\omega),u(t,\omega))\,dW_t^a,\\ x(0,\omega)=x_0\in\mathbb{R}^n,\end{cases}$$

respectively,

$$\begin{cases} dp_i(t)=a_i(t,x(t,\omega),u(t,\omega))\,dt+q_{ia}(t,x(t,\omega),u(t,\omega))\,dW_t^a,\\ p(0,\omega)=p^0\in\mathbb{R}^n,\end{cases}$$

where the coefficients in both evolutions are predictable processes. Then

$$d\left(p_i(t)\, x^i(t)\right) = p_i dx^i + x^i dp_i + q_{ib}\sigma^i_a \delta^{ab} dt,\ i = \overline{1,n},$$

where the operator d is the stochastic differential.

3.2 Adjoint stochastic system

Definition 3.2. (Variational stochastic system) *Let $u(\cdot,\omega)$ be an admissible control. Let*

$$dx^i_t = \mu^i(t, x_t, u_t)\, dt + \sigma^i_a(t, x_t, u_t)\, dW^a_t$$

be a stochastic evolution whose coefficients μ^i and σ^i_a are of class C^1 in the second argument. Denote $\mu^i_{x^j} = \frac{\partial \mu^i}{\partial x^j}$, $\sigma^i_{ax^j} = \frac{\partial \sigma^i_a}{\partial x^j}$. The linear stochastic system

$$d\xi^i_t = \left(\mu^i_{x^j}(t, x_t, u_t)\, dt + \sigma^i_{ax^j}(t, x_t, u_t)\, dW^a_t\right) \xi^j_t \tag{3.2}$$

is called variational stochastic system with control $u(\cdot,\omega)$.

Definition 3.3. (Adjoint stochastic system) *Consider the stochastic evolution (3.2). A linear stochastic system of the form*

$$dp_j(t) = \left(a^i_j(t, x_t, u_t)\, dt + q^i_{bj}(t, x_t, u_t)\, dW^b\right) p_i(t),\ \ b = \overline{1,d},\ \ i,j = \overline{1,n} \tag{3.3}$$

is called the adjoint (dual) stochastic system of (3.2) if the scalar product $p_k(t)\,\xi^k(t)$ is a global stochastic first integral, i.e., $d(p_i(t)\xi^i(t)) = 0$.

Theorem 3.2. *The stochastic system*

$$dp_j(t) = [\left(-\mu^i_{x^j}(t, x_t, u_t) + \sigma^i_{ax^k}(t, x_t, u_t)\, \sigma^k_{bx^j}(t, x_t, u_t)\, \delta^{ab}\right) dt -$$

$$-\sigma^i_{ax^j}(t, x_t, u_t)\, dW^a_t]\ p_i(t)$$

is the adjoint stochastic system with respect to the variational stochastic system (3.2).

Proof. For simplicity, we will omit ω as argument of processes. Let

$$d\xi^i_t = \left(\mu^i_{x^j}(t, x_t, u_t)\, dt + \sigma^i_{ax^j}(t, x_t, u_t)\, dW^a\right) \xi^j_t,\ \ i,j = \overline{1,n},$$

be the linear variational stochastic system. Denote the adjoint stochastic system by

$$dp_j(t) = \left(a^i_j(t, x_t, u_t)\, dt + q^i_{bj}(t, x_t, u_t)\, dW^b\right) p_i(t),\ \ i,j = \overline{1,n}.$$

We determine the coefficients $a^i_j(t, x_t, u_t)$ and $q^i_{bj}(t, x_t, u_t)$ such that $p_k(t)\,\xi^k(t)$ be a *stochastic first integral*, i.e.,

$$d\left(p_k(t)\, \xi^k(t)\right) = 0,$$

where d is the *stochastic differential.* Imposing the identity,

$$p_i(t)\, \xi^i(t) = p_i(0)\, \xi^i(0),\ \text{ for any } t \in \Omega_{0T},$$

or

$$0 = d\left(p_k(t)\,\xi^k(t)\right) = p_i(t)\,\xi^j(t)\,(\mu^i_{x^j}(t, x_t, u_t) + a^i_j(t, x_t, u_t) +$$

$$+q^i_{ak}(t, x_t, u_t)\,\sigma^k_{bx^j}(t, x_t, u_t)\,\delta^{ab})dt + p_i(t)\,\xi^j(t)\left(\sigma^i_{ax^j}(t, x_t, u_t) + q^i_{aj}(t, x_t, u_t)\right)dW^a_t,$$

we obtain

$$a^i_j(t, x_t, u_t) = -\mu^i_{x^j}(t, x_t, u_t) - q^i_{ak}(t, x_t, u_t)\,\sigma^k_{bx^j}(t, x_t, u_t)\,\delta^{ab},$$

$$q^i_{aj}(t, x_t, u_t) = -\sigma^i_{ax^j}(t, x_t, u_t).$$

□

4 Optimization problems with stochastic integral functionals

Stochastic optimal control problems have some common features: there is a *constraint diffusion system*, which is described by an Itô stochastic differential system; there are some other *constraints* that the decisions and/or the state are subject to; there is a *criterion* that measures the performance of the decisions. The goal is to *optimize* the criterion by selecting a *nonanticipative* decision among the ones satisfying all the constraints.

Next, we introduce the *cost functional* as follows

$$J(u(\cdot)) = \mathbb{E}\left[\int_{\Omega_{0T}} f(t, x(t,\omega), u(t,\omega))\,dt + \Psi(x(T,\omega))\right], \tag{4.1}$$

where the *running cost* 1-form $f(t, x(t,\omega), u(t,\omega))\,dt$ has the coefficient

$$f(\cdot, x(\cdot,\omega), u(\cdot,\omega)) : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^k \longrightarrow \mathbb{R}$$

and $\Psi(x(\cdot,\omega)) : \mathbb{R}^n \longrightarrow \mathbb{R}$. The simplest stochastic optimal control problem (under strong formulation) can be stated as follows: *Find*

$$\max\ J(u(\cdot,\omega))\ \text{ over } \mathcal{A}_{ad}, \tag{4.2}$$

constrained by

$$\begin{cases} dx^i_t = \mu^i(t, x_t, u_t)\,dt + \sigma^i_a(t, x_t, u_t)\,dW^a_t, \\ x(0) = x_0 \in \mathbb{R}^n,\ a = \overline{1,d},\ i = \overline{1,n}. \end{cases} \tag{4.3}$$

The goal is to find $u^*(\cdot)$ (if it ever exists), such that

$$J(u^*(\cdot,\omega)) = \max_{u(\cdot,\omega)} J(u(\cdot,\omega)).$$

4.1 Simplified stochastic maximum principle

Let Ω_{0T} be the closed interval $0 \leq t \leq T$, in $\mathbb{R}_+$ and $t \in \Omega_{0T}$ be an arbitrary *time* in Ω_{0T} . Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. The information structure is given by a filtration $(\mathcal{F}_t)_{t\in\Omega_{0T}}$, satisfying the usual conditions, which is generated by a $\mathcal{F}_t$-Wiener process with values in $\mathbb{R}^d$, $W(\cdot) = (W_1(\cdot), ..., W_d(\cdot))$ and augmented by all the $\mathbb{P}-$null sets. Sometimes, for simplicity, we will omit ω as argument of processes.

In order to solve the problem (4.2), with state constraints (4.3), we introduce the *stochastic Lagrange multiplier*

$$p(t,\omega) \in L^2_{\mathcal{F}}(\Omega_{0T}, \mathbb{R}^n),$$

where $L^2_{\mathcal{F}}(\Omega_{0T}, \mathbb{R}^n)$ is the space of all $\mathbb{R}^n-$valued adapted processes such that

$$\mathbb{E}\int_{\Omega_{0T}} |\phi(t,\omega)|^2 \, dt < \infty.$$

To extend the methods of the first author ([15], [17]) to the stochastic control theory, let us suppose $(p_t)_{t\in\Omega_{0T}}$ as a stochastic Itô-process:

$$dp_i(t) = a_i(t, x_t, u_t)\, dt + q_{ia}(t, x_t, u_t)\, dW^a_t,$$

where $(a_i(t, x_t, u_t))_{t\in\Omega_{0T}}$, respectively, $(q_{ia}(t, x_t, u_t))_{t\in\Omega_{0T}}$ are predictable processes of the form (see (3.3))

$$a_i(t, x_t, u_t) = a^j_i(t, x_t, u_t)\, p_j(t),$$

$$q_{ia}(t, x_t, u_t) = q^j_{ia}(t, x_t, u_t)\, p_j(t),\ \ i, j = \overline{1,n},\ \ a = \overline{1,d}.$$

Now, we use the *Lagrangian stochastic* $1-$*form*

$$\mathcal{L}(t, x_t, u_t, p_t) = f(t, x_t, u_t)\, dt +$$

$$+ p_i(t)\left[\mu^i(t, x_t, u_t)\, dt + \sigma^i_a(t, x_t, u_t)\, dW^a_t - dx^i_t\right].$$

The adjoint process $p(t,\omega)$ is required to be $(\mathcal{F}_t)_{t\in\Omega_{0T}}-$adapted, for any $t \in \Omega_{0T}$.

The contact distribution with stochastic perturbations constrained optimization problem (4.2)-(4.3) can be changed into another free stochastic optimization problem

$$\max_{u(\cdot,\omega)\in\mathcal{A}_{ad}} \mathbb{E}\left[\int_{\Omega_{0T}} \mathcal{L}(t, x_t, u_t, p_t) + \Psi(x(T,\omega))\right], \tag{4.4}$$

subject to

$$p(t,\omega) \in \mathcal{P},\ \forall t \in \Omega_{0T},\ x(0,\omega) = x_0 \in \mathbb{R}^n,$$

where the set $\mathcal{P}$ will be defined as the set of adjoint stochastic processes. The problem (4.4) can be rewritten as

$$\max_{u(\cdot,\omega)\in\mathcal{A}_{ad}} \mathbb{E}\left\{\int_{\Omega_{0T}} \left[f(t, x_t, u_t) + p_i(t,\omega)\, \mu^i(t, x_t, u_t)\right] dt + \right. \tag{4.5}$$

$$+\int_{\Omega_{0T}} p_i(t,\omega)\,\sigma_a^i(t,x_t,u_t)\,dW_t^a - \int_{\Omega_{0T}} p_i(t,\omega)\,dx_t^i + \Psi(x(T,\omega))\Big\},$$

subject to

$$p(t,\omega)\in\mathcal{P},\ \forall t\in\Omega_{0T},\ x(0,\omega)=x_0\in\mathbb{R}^n,\ i=\overline{1,n}.$$

Due to properties of stochastic integrals [12], we have

$$\mathbb{E}\left(\int_{\Omega_{0T}} p_i(t,\omega)\,\sigma_a^i(t,x_t,u_t)\,dW_t^a\right)=0.$$

Evaluating

$$\int_{\Omega_{0T}} p_i(t,\omega)\,dx_t^i$$

via stochastic integration by parts, it appears the *control Hamiltonian stochastic* $1-$*form*

$$\mathcal{H}(t,x_t,u_t,p_t,q_t)=f(t,x_t,u_t)\,dt+$$

$$+\left[p_i(t)\,\mu^i(t,x_t,u_t)-p_i(t)\,\sigma_{ax^j}^i(t,x_t,u_t)\,\sigma_b^j(t,x_t,u_t)\,\delta^{ab}\right]dt. \tag{4.6}$$

It verifies

$$\mathcal{H}=\mathcal{L}+p_i(t)\ dx_t^i-p_i(t)\,\sigma_{ax^j}^i(t,x_t,u_t)\,\sigma_b^j(t,x_t,u_t)\,\delta^{ab}dt-p_i(t)\ \sigma_a^i(t,x_t,u_t)\,dW_t^a,$$

(modified stochastic Legendrian duality).

Theorem 4.1. (Simplified stochastic maximum principle) *We assume (***H1***)-(***H3***). Suppose that the problem of maximizing the functional (4.1) constrained by (4.3) over $\mathcal{A}_{ad}$ has an interior optimal solution $u^*(t)$, which determines the stochastic optimal evolution $x(t)$. Let $\mathcal{H}$ be the Hamiltonian stochastic $1-$form (4.6). Then there exists an adapted processes $(p(t,\omega))_{t\in\Omega_{0T}}$ (adjoint process) satisfying:*

(i) the initial stochastic differential system,

$$dx^i(t)=\frac{\partial\mathcal{H}}{\partial p_i}(t,x_t,u_t^*,p_t)+\sigma_{ax^j}^i(t,x_t,u_t^*)\,\sigma_b^j(t,x_t,u_t^*)\,\delta^{ab}dt+\sigma_a^i(t,x_t,u_t^*)\,dW_t^a;$$

(ii) the adjoint linear stochastic differential system,

$$\begin{aligned} dp_i(t,\omega) &= -\mathcal{H}_{x^i}(t,x_t,u_t^*,p_t)-p_j(t)\,\sigma_{ax^i}^j(t,x_t,u_t^*)\,dW_t^a,\\ p_i(T,\omega) &= \Psi_{x^i}(x_T),\ i\in\overline{1,n}; \end{aligned}$$

(iii) the critical point condition,

$$\mathcal{H}_{u^c}(t,x_t,u_t^*,p_t)=0,\ c=\overline{1,k}.$$

Proof. In the whole this proof, we will omit ω as argument of processes. Suppose that there exists a continuous control $u^*(t)$ over the admissible controls in $\mathcal{A}_{ad}$, which is an optimum point in the previous problem. Consider a variation

$$u(t,\varepsilon)=u^*(t)+\varepsilon h(t),$$

where, by hypothesis, $h(t)$ is an arbitrary continuous function. Since $u^*(t) \in \mathcal{A}_{ad}$ and a continuous function over a compact set Ω_{0T} is bounded, there exists a number $\varepsilon_h > 0$ such that

$$u(t,\varepsilon) = u^*(t) + \varepsilon h(t) \in \mathcal{A}_{ad},\ \forall |\varepsilon| < \varepsilon_h.$$

This ε is used in our variational arguments.

Now, let us define the contact distribution with stochastic perturbations, corresponding to the control variable $u(t,\varepsilon)$, i.e.,

$$dx^i(t,\varepsilon) = \mu^i(t, x(t,\varepsilon), u(t,\varepsilon))\,dt + \sigma_a^i(t, x(t,\varepsilon), u(t,\varepsilon))\,dW_t^a,$$

for $i \in \overline{1,n}$, or,

$$x^i(t,\varepsilon) = x^i(0,\varepsilon) + \int_{\Omega_{0t}} \mu^i(s, x(s,\varepsilon), u(s,\varepsilon))\,ds +$$

$$+ \int_{\Omega_{0t}} \sigma_a^i(s, x(s,\varepsilon), u(s,\varepsilon))\,dW_s^a,\ \forall t \in \Omega_{0T},\ \forall i \in \overline{1,n}$$

and $x(0,\varepsilon) = x_0 \in \mathbb{R}^n$. For $|\varepsilon| < \varepsilon_h$, we define the function

$$J(\varepsilon) = \mathbb{E}\left[\int_{\Omega_{0T}} f(t, x(t,\varepsilon), u(t,\varepsilon))\,dt + \Psi(x(T,\varepsilon))\right].$$

For any adapted process $(p_t)_{t\in\Omega_{0T}}$ we have

$$\int_{\Omega_{0T}} p_i(t)\ \left[\mu^i(t, x(t,\varepsilon), u(t,\varepsilon))\,dt - dx^i(t,\varepsilon)\right] +$$

$$+ \int_{\Omega_{0T}} p_i(t)\ \sigma_a^i(t, x(t,\varepsilon), u(t,\varepsilon))\,dW_t^a = 0,\ i = \overline{1,n},$$

To solve the foregoing constrained optimization problem, we transform it into a free optimization problem ([17]). For this, we use the Lagrange stochastic $1-$form which includes the variations

$$\mathcal{L}(t, x(t,\varepsilon), u(t,\varepsilon), p_t) = f(t, x(t,\varepsilon), u(t,\varepsilon))\,dt +$$

$$+ p_i(t)\left[\mu^i(t, x(t,\varepsilon), u(t,\varepsilon))\,dt + \sigma_a^i(t, x(t,\varepsilon), u(t,\varepsilon))\,dW_t^a - dx^i(t,\varepsilon)\right],$$

where $i = \overline{1,n}$. We have to optimize now the function

$$\widetilde{J}(\varepsilon) = \mathbb{E}\left[\int_{\Omega_{0T}} \mathcal{L}(t, x(t,\varepsilon), u(t,\varepsilon), p_t) + \Psi(x(T,\varepsilon))\right],$$

with doubt any constraints. If the control $u^*(t)$ is optimal, then

$$\widetilde{J}(\varepsilon) \le \widetilde{J}(0), \forall |\varepsilon| < \varepsilon_h.$$

Explicitly,

$$\widetilde{J}(\varepsilon) = \mathbb{E}\int_{\Omega_{0T}} \left[f(t, x(t,\varepsilon), u(t,\varepsilon)) + p_i(t)\,\mu^i(t, x(t,\varepsilon), u(t,\varepsilon))\right] dt -$$

$$-\mathbb{E}\left[\int_{\Omega_{0T}} p_i(t)\, dx^i(t,\varepsilon) + \Psi(x(T,\varepsilon))\right],\ i=\overline{1,n}.$$

To evaluate the integral

$$\int_{\Omega_{0T}} p_i(t)\, dx^i(t,\varepsilon),$$

we integrate by stochastic parts, via Lemma (3.1). Taking into account that $(p_t)_{t\in\Omega_{0T}}$ is an Itô process, we obtain

$$\widetilde{J}(\varepsilon) = \mathbb{E}\int_{\Omega_{0T}} \left[f(t,x(t,\varepsilon),u(t,\varepsilon)) + p_i(t)\,\mu^i(t,x(t,\varepsilon),u(t,\varepsilon))\right] dt-$$

$$-\mathbb{E}\left[p_i(t)\ \ x^i(t,\varepsilon)\big|_0^T - \int_{\Omega_{0T}} x^i(t,\varepsilon)\ \ dp_i(t)\right]+$$

$$+\mathbb{E}\int_{\Omega_{0T}} p_j(t)\,\sigma^j_{ax^i}(t,x(t,\varepsilon),u(t,\varepsilon))\,\sigma^i_b(t,x(t,\varepsilon),u(t,\varepsilon))\,\delta^{ab}\ dt + \mathbb{E}\Psi(x(T,\varepsilon)),$$

with $\sigma^j_{ax^i}(t,x(t,\varepsilon),u(t,\varepsilon))\,|_{\varepsilon=0}\equiv \sigma^j_{ax^i}(t,x_t,u_t)$. Then,

$$\widetilde{J}(\varepsilon) = \mathbb{E}\left[\int_{\Omega_{0T}} f(t,x(t,\varepsilon),u(t,\varepsilon))\, dt\right]+$$

$$+\mathbb{E}\int_{\Omega_{0T}} [p_i(t)\,\mu^i(t,x(t,\varepsilon),u(t,\varepsilon))-$$

$$-p_j(t)\,\sigma^j_{ax^i}(t,x(t,\varepsilon),u(t,\varepsilon))\,\sigma^i_b(t,x(t,\varepsilon),u(t,\varepsilon))\,\delta^{ab}]\ dt+$$

$$+\mathbb{E}\ \left[\int_{\Omega_{0T}} x^i(t,\varepsilon)\ \ dp_i(t) - p_i(t)\ \ x^i(t,\varepsilon)\big|_0^T\right] + \mathbb{E}\Psi(x(T,\varepsilon)).$$

Differentiating with respect to ε (and this is possible, because the derivative with respect to ε, for $\varepsilon=0$, exists in mean square; see, for example, [6] or [19]) it follows

$$\widetilde{J}'(\varepsilon) = \mathbb{E}\int_{\Omega_{0T}} \{[f_{x^k}(t,x(t,\varepsilon),u(t,\varepsilon)) + p_i(t)\,\mu^i_{x^k}(t,x(t,\varepsilon),u(t,\varepsilon))-$$

$$-p_j(t)\,\sigma^j_{ax^ix^k}(t,x(t,\varepsilon),u(t,\varepsilon))\,\sigma^i_b(t,x(t,\varepsilon),u(t,\varepsilon))\,\delta^{ab}-$$

$$-p_j(t)\,\sigma^j_{ax^i}(t,x(t,\varepsilon),u(t,\varepsilon))\,\sigma^i_{bx^k}(t,x(t,\varepsilon),u(t,\varepsilon))\,\delta^{ab}]\ dt + dp_k(t)\}\ x^k_\varepsilon(t,\varepsilon)+$$

$$+\mathbb{E}\int_{\Omega_{0T}} [f_{u^c}(t,x(t,\varepsilon),u(t,\varepsilon)) + p_i(t)\,\mu^i_{u^c}(t,x(t,\varepsilon),u(t,\varepsilon))-$$

$$-p_j(t)\,\sigma^j_{ax^iu^c}(t,x(t,\varepsilon),u(t,\varepsilon))\,\sigma^i_b(t,x(t,\varepsilon),u(t,\varepsilon))\,\delta^{ab}-$$

$$-p_j(t)\,\sigma^j_{ax^i}(t,x(t,\varepsilon),u(t,\varepsilon))\,\sigma^i_{bu^c}(t,x(t,\varepsilon),u(t,\varepsilon))\,\delta^{ab}]\ h^c(t)\ \ dt+$$

$$+\mathbb{E}\Psi_{x^j}(x(T,\varepsilon))\,x^j_\varepsilon(T,\varepsilon),\ c=\overline{1,d}.$$

Evaluating at $\varepsilon=0$, we find

$$\widetilde{J}'(0) = \mathbb{E}\int_{\Omega_{0T}} \{[(f_{x^k}(t,x_t,u_t) + p_i(t)\,\mu^i_{x^k}(t,x_t,u_t)-$$

$$-p_j(t)\,\sigma^j_{ax^ix^k}(t,x_t,u_t)\,\sigma^i_b(t,x_t,u_t)\,\delta^{ab}-$$

$$-p_j(t)\,\sigma^j_{ax^i}(t,x_t,u_t)\,\sigma^i_{bx^k}(t,x_t,u_t)\,\delta^{ab}]dt+dp_k(t)\}\ x^k_\varepsilon(t,0)+$$

$$+\mathbb{E}\Psi_{x^k}(x_T)\,x^k_\varepsilon(T)+\mathbb{E}\int_{\Omega_{0T}}[f_{u^c}(t,x_t,u^*_t)+p_i(t)\,\mu^i_{u^c}(t,x_t,u^*_t)-$$

$$-p_j(t)\,\sigma^j_{ax^iu^c}(t,x_t,u^*_t)\,\sigma^i_b(t,x_t,u^*_t)\,\delta^{ab}-$$

$$-p_j(t)\,\sigma^j_{ax^i}(t,x_t,u^*_t)\,\sigma^i_{bu^c}(t,x_t,u^*_t)\,\delta^{ab}]\ h^c(t)\ \ dt,$$

where $x(t)$ is the state variable corresponding to the optimal control $u^*(t)$.

We need $\widetilde{J}'(0)=0$ for all $h(t)=(h^c(t))_{c=\overline{1,k}}$. On the other hand, the functions $x^i_\varepsilon(t,0)$ are involved in the Cauchy problem

$$dx^i_\varepsilon(t,0)=\left(\mu^i_{x^j}(t,x(t,0),u(t,0))\,dt+\sigma^i_{ax^j}(t,x_t,u_t)\,dW^a_t\right)x^j_\varepsilon(t,0)$$

$$+\left(\mu^i_{u^a}(t,x(t,0),u(t,0))\,dt+\sigma^i_{bu^a}(t,x_t,u_t)\,dW^b_t\right)h^a(t)\,,\ t\in\Omega_{0T},\ \ x_\varepsilon(0,0)=0\in\mathbb{R}^n$$

and hence they depend on $h(t)$. The functions $x^i_\varepsilon(t,0)$ are eliminated by selecting $\mathcal{P}$ as the adjoint contact distribution

$$dp_k(t)=-[f_{x^k}(t,x_t,u_t)+p_i(t)\,\mu^i_{x^k}(t,x_t,u_t)-p_j(t)\,\sigma^j_{ax^ix^k}(t,x_t,u_t)\,\sigma^i_b(t,x_t,u_t)\,\delta^{ab}$$

$$-p_j(t)\,\sigma^j_{ax^i}(t,x_t,u_t)\,\sigma^i_{bx^k}(t,x_t,u_t)\,\delta^{ab}]dt-\sigma^j_{ax^k}(t,x_t,u_t)\,p_j(t)\,dW^a_t, \tag{4.7}$$

for any $\forall t\in\Omega_{0T}$, with stochastic perturbations terminal value problem [13]

$$p_k(T)=\Psi_{x^k}(x_T)\,,\ k=\overline{1,n}.$$

The relation (4.7) shows that

$$a_k(t,x_t,u_t)=-f_{x^k}(t,x_t,u_t)-p_i(t)\,\mu^i_{x^k}(t,x_t,u_t)+$$

$$+\left[p_j(t)\,\sigma^j_{ax^ix^k}(t,x_t,u_t)\,\sigma^i_b(t,x_t,u_t)+p_j(t)\,\sigma^j_{ax^i}(t,x_t,u_t)\,\sigma^i_{bx^k}(t,x_t,u_t)\right]\delta^{ab}.$$

It follows

$$dp_k(t)=-\mathcal{H}_{x^k}(t,x_t,u^*_t,p_t)-p_j(t)\,\sigma^j_{ax^k}(t,x_t,u_t)\,dW^a_t,\ \ k=\overline{1,n}. \tag{4.8}$$

and

$$\mathcal{H}_{u^b}(t,x_t,u^*_t,p_t)=0,\ \forall t\in\Omega_{0T},\ \text{ for all } b=\overline{1,k}. \tag{4.9}$$

Moreover,

$$dx^i(t)=\frac{\partial\mathcal{H}}{\partial p_i}(t,x_t,u^*_t,p_t,q_t)+\sigma^i_{ax^j}(t,x_t,u^*_t)\,\sigma^j_b(t,x_t,u^*_t)\,\delta^{ab}dt+\sigma^i_a(t,x_t,u_t)\,dW^a_t, \tag{4.10}$$

$$\forall t\in\Omega_{0T},x(0)=x_0,\ \forall i=\overline{1,n}.$$

□

Remark 4.2. *1) The relations (4.8), (4.9) and (4.10) suggest Itô (stochastic Euler-Lagrange) equations associated to the Hamiltonian stochastic* 1*-form* $\mathcal{H}$*. The Itô system (4.9) describes the critical points of the Hamiltonian stochastic* $1-$*form* $\mathcal{H}$ *with respect to the control variable.*

2) If the control variables enter the Hamiltonian stochastic $1-$*form* $\mathcal{H}$ *linearly affine, either via the objective function or the stochastic evolution or both, then the problem is called linear stochastic optimal control problem. In this case the control must be bounded, the coefficients of* $u(t)$ *determines the switching function, and it appear the idea of stochastic bang-bang optimal control.*

Example Let $t \in \Omega_{01}$ and the standard Wiener process $W = (W_t)_{t\in\Omega_{01}}$. We consider the following controlled system

$$dx_t = u_t dt + dW_t,\ t \in \Omega_{01},\ x_0 = x_0 = 0,\ x(1) = x_1, \text{free},$$

with the control domain being $\mathcal{A}_{ad} \subset \mathbb{R}$. Denote by $J(u(\cdot)) = -\mathbb{E}\left(\int_{\Omega_{01}} \left(x_t + u_t^2\right) dt\right)$. the cost functional. This problem means to find an optimal control u^* to bring the Itô dynamical system from the origin x_0 at time t to a terminal point x_1, which is unspecified, time $t = 1$ such as to maximize the objective functional J.

The control Hamiltonian $1-$form is

$$\mathcal{H}(t, x_t, u_t, p_t, q_t) = \left(-\left(x_t + u_t^2\right) + p_t u_t\right) dt.$$

Since

$$\frac{\partial \mathcal{H}}{\partial u_t} = \left(-2u_t + p_t\right) dt = 0,\ \frac{\partial^2 \mathcal{H}}{\partial u_t^2} = -2dt,$$

the critical point $u_t = \frac{p_t}{2}$, for the coefficient function of dt, is a maximum point. On the other hand, the adjoint system $dp_t = -\frac{\partial \mathcal{H}}{\partial x_t} + p_t dW_t$ becomes $dp_t = -dt + p_t dW_t$. Also, since the point x_1 is unspecified, the transversality condition implies $p(1) = 0$. It follows the costate $p_t = e^{-\frac{t}{2}+W_t}\left(p_0 - \int_0^t e^{\frac{s}{2}-W_s} ds\right)$, the optimal control $u_t^* = \frac{p_t}{2}$ with the corresponding evolution $dx_s = \frac{1}{2}p_s ds + dW_s,\ s \in \Omega_{01}$. Integrating on Ω_{0t}, we find $x_t = \int_0^t \left(e^{-\frac{\tau}{2}+W_\tau}\left(p_0 - \int_0^\tau e^{\frac{s}{2}-W_s} ds\right)\right) d\tau + W_t$.

The numerical simulation can be done using the Euler method for the following equation of evolution $dx(t) = (p(t)/2)dt + dW(t)$ and the adjoint equation $dp(t) = -dt + p(t)dW(t)$. We obtain the orbits: $(t, p(t,\omega))$, $(t, x(t,\omega))$, $(t, u(t,\omega))$.

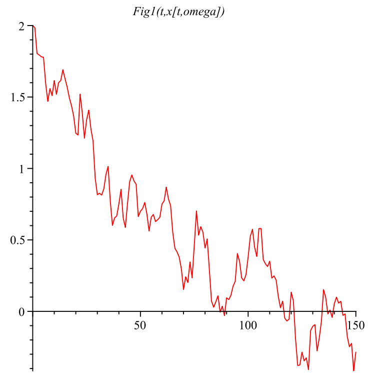
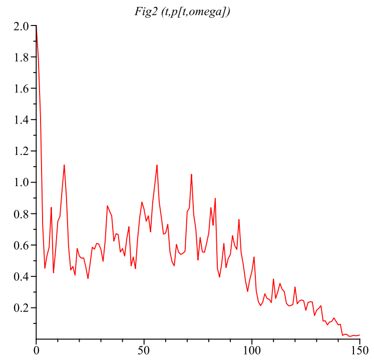
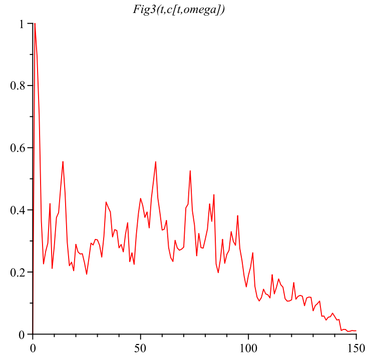
Fig. 1 $(t, p(t,\omega))$	Fig. 2 $(t, x(t,\omega))$
	

Fig. 3 $(t, u(t, \omega))$



5 Optimal feedback control of a continuously monitored spin

The dynamics of the *continuously monitored spin system* is described by the SDEs [7]

$$dr_t = \frac{1}{2}(\frac{\eta}{r_t} - r_t)\sin^2\theta_t\, dt + \sqrt{\eta}\,(1 - r_t^2)\cos\theta_t\, dW_t$$

$$d\theta_t = \left(-u_t + (\frac{\eta}{r_t^2} - \eta - \frac{1}{2})\sin^2\theta_t\cos\theta_t\right) dt + \sqrt{\eta}\,\frac{\sin\theta_t}{r_t}\, dW_t,$$

where the *polar coordinates* (r, θ) represent the *state space*, the parameter $\eta \in [0, 1]$ is the *detection efficiency of the photodetectors* and u_t is the *amplitude of the magnetic field* applied in the y-direction. The special case $\eta = 1$ means *perfect detector efficiency.*

Remark. Letting the initial value of r equal to 1 ($r_0 = 1$) we see that $dr_0 = 0$ and hence $r_t = 1$, $\forall t \geq 0$. Hence, $r = 1$ is a *forward invariant set* of the foregoing stochastic dynamics. Physically, this means that if the detection is perfect, a pure state of the system will remain pure for all time.

We pose now the following optimization problem: Suppose that at time t the state of the system is (r, θ). Let $u_s, s \in [0, T]$ (T is the time at which the experiment terminates) be a square integrable function. We define the following expected cost-to-go

$$J(u(\cdot)) = \mathbb{E}\left[\int_{\Omega_{0T}}\left(-\frac{1}{2}u_t^2 - U(r_t, \theta_t)\right) dt\right],$$

where the expectation value is taken with respect to every possible sample path that starts at (r, θ) at time t. The function U is a measure of the distance of the state from the desired target state $(r, \theta) = (1, 0)$. For well-posedness we require that $U(1, 0) = 0$ and $U(r, \theta) > 0, \forall (r, \theta) \neq (1, 0)$. For example, $U = 1 - r\cos\theta$. We seek the control law u that maximizes J.

The control Hamiltonian $1-$form is

$$\mathcal{H} = \left(-\frac{1}{2}u_t^2 - U(r_t, \theta_t)\right) dt$$

$$+p_1 \left(\frac{1}{2}(\frac{\eta}{r_t} - r_t)\sin^2\theta_t + 2\sqrt{\eta}\ (1-r_t^2)r_t\cos^2\theta_t + \eta\frac{1-r_t^2}{r_t}\sin^2\theta_t \right) dt$$

$$+p_2 \left(-u_t + (\frac{\eta}{r_t^2} - \eta - \frac{1}{2})\sin^2\theta_t\cos\theta_t + \eta\ \frac{1-r_t^2}{r_t^2}\sin\theta_t\cos\theta_t - \eta\frac{1}{r_t^2}\sin\theta_t\cos\theta_t \right) dt.$$

6 Ramsey and Uzawa-Lucas stochastic models

Ramsey stochastic models We suppose that an investor is able to invest his wealth to do some products and he can get profit from this activity. Let $x(t)$ be the capital of investor, $a(t)$ the labor and $c(t) > 0$ the rate of consumption, at time t. Since there must be some risk in the investment, the Ramsey deterministic model [11] $dx(t) = (f(x(t), a(t)) - c(t))\, dt$ can be changed into a stochastic model

$$dx(t) = (f(x(t), a(t)) - c(t))\, dt + \sigma(x(t))dW(t),$$

where $W(t)$ is a 1-dimensional standard Wiener process and σ is a C^1 function. For simplicity, we introduce the following assumptions: (1) the function f is given by the Cobb-Douglas formula $f(x, a) = kx^\alpha a^\beta$, where k, α, β are some constants; (2) during a relatively short period we can take $a(t) = a =$ constant. If $\beta = 1$, then the stochastic evolution is

$$dx(t) = (kax^\alpha(t) - c(t))\, dt + \sigma(x(t))dW(t),\ x(0) = x_0.$$

We propose to maximize the performance function

$$J(c(\cdot)) = \mathbb{E}\left(\int_{\Omega_{0T}} e^{-rt}\ \frac{c^\gamma(t)}{\gamma}\, dt + x(T) \right),$$

where r is the bond rate, and $\gamma \in (0, 1)$ determines the relative risk aversion $1-\gamma$ of the investor.

In this case, the control Hamiltonian stochastic 1-form is

$$\mathcal{H}(t, x_t, c_t, p_t) = e^{-rt}\ \frac{c^\gamma(t)}{\gamma}\, dt + p_t\,(kax^\alpha(t) - c(t) - \sigma'(x(t))\sigma(x(t)))\, dt.$$

Applying the Theorem 4.1, it follows

$$dx(t) = (kax^\alpha(t) - c(t))\, dt + \sigma(x(t))dW(t),\ x(0) = x_0;$$

$$dp(t) = -p(t)\left(\alpha kax(t)^{\alpha-1} - \sigma''(x(t))\sigma(x(t)) - \sigma'(x(t))^2\right) dt$$

$$-p(t)\sigma'(x(t))dW(t),$$

$$p(T) = 1;$$

$$\mathcal{H}_{c(t)} = \left(e^{-rt}\ c^{\gamma-1}(t) - p(t)\right)\, dt = 0.$$

The optimal control

$$c^*(t) = e^{\frac{-rt}{\gamma-1}}\ p(t)^{\frac{1}{\gamma-1}}$$

fixes the optimal stochastic state evolution

$$dx(t) = \left(kax^{\alpha}(t) - e^{\frac{-rt}{\gamma-1}}\ p(t)^{\frac{1}{\gamma-1}}\right) dt + \sigma(x(t))dW(t)$$

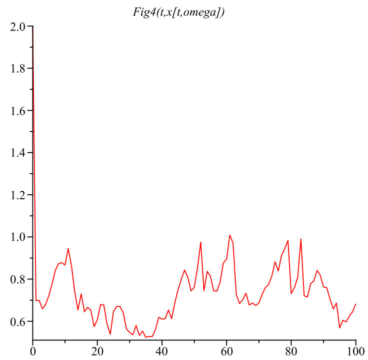
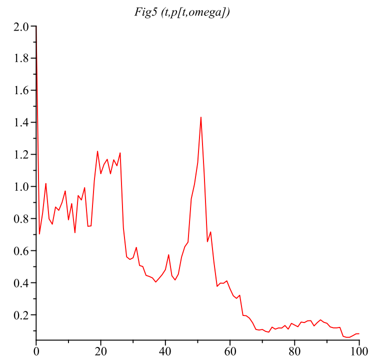
and the optimal stochastic costate evolution

$$dp(t) = -p(t)\left(\alpha kax(t)^{\alpha-1} - \sigma''(x(t))\sigma(x(t)) - \sigma'(x(t))^2\right) dt$$
$$-p(t)\sigma'(x(t))dW(t).$$

The simulations refer to

$$\sigma(x) = \frac{1}{2}x^2; k = 0.005; a = 1000; \gamma = 0.8; r = 0.3; \alpha = 0.5.$$

The figures 4 and 5 represent the orbits: $(t, x(t,\omega))$ and $(t, p(t,\omega))$, respectively.

Fig. 4 $(t, x(t,\omega))$	Fig. 5 $(t, p(t,\omega))$
	

Uzawa-Lucas stochastic models The Uzawa-Lucas deterministic model [5] (first published in 1965 by Uzawa [18] and reconsidered in 1988 by Lucas [9]), refers to a two sector economy: (1) a good sector that produces consumable and gross investment in physical capital; (2) an education sector that produces human capital. Both are subject to maunder conditions of constant returns to scale. All the variables are also accepted as *per capita* quantities. Let introduce the following data: k is the physical capital, h is a human capital, c is the real *per capita* consumption, u is the fraction of labor allocated to the production of physical capital, β is the elasticity of output with respect to physical capital, $\rho > 0$ is a positive discount factor, $\gamma > 0$ is the constant technological level in the good sector, $\delta > 0$ is the constant technological level in the education sector, n is the exogenous growth rate of labor, σ^{-1} is the constant of the elasticity of intertemporal substitution, and $\sigma \neq \beta$. Given the importance of the sustainable development process, a modern economic approach regarding the role of human capital in economic growth in the case of the Romanian economy, using deterministic Uzawa-Lucas model, is done in [1].

The Uzawa-Lucas deterministic problem:

$$\max_{u,c}\ J(u(\cdot), c(\cdot)) = \int_{\Omega_{0T}} e^{-\rho t}\ \frac{c(t)^{1-\sigma} - 1}{1-\sigma}\, dt + k(T),$$

subject to

$$dk(t) = \left(\gamma k(t)^{\beta} u(t)^{1-\beta} - nk(t) - c(t)\right) dt,\ k(0) = k_0,$$
$$dh(t) = \delta(1 - u(t))h(t)dt,\ h(0) = h_0$$

can be extended to the stochastic problem:

$$\max_{u,c}\ J(u(\cdot), c(\cdot)) = \mathbb{E}\left(\int_{\Omega_{0T}} e^{-\rho t}\ \frac{c(t)^{1-\sigma} - 1}{1-\sigma}\, dt + k(T)\right),$$

subject to

$$dk(t) = \left(\gamma k(t)^{\beta} u(t)^{1-\beta} - nk(t) - c(t)\right) dt + \sigma_1(k(t))dW(t),\ k(0) = k_0,$$
$$dh(t) = \delta(1 - u(t))h(t)dt + \sigma_2(h(t))dW(t),\ h(0) = h_0.$$

In this case, the control Hamiltonian stochastic 1-form is

$$\mathcal{H}(t, x_t, c_t, p_t, q_t) = e^{-\rho t}\ \frac{c(t)^{1-\sigma} - 1}{1-\sigma}\, dt$$
$$+p_t\left(\gamma k(t)^{\beta} u(t)^{1-\beta} - nk(t) - c(t) - \sigma_1'(k(t))\sigma_1(k(t))\right) dt$$
$$+q_t\left(\delta(1 - u(t))h(t)) - \sigma_2'(h(t))\sigma_2(h(t))\right) dt$$

Applying the Theorem 4.1, it follows

$$dk(t) = \left(\gamma k(t)^{\beta} u(t)^{1-\beta} h(t)^{1-\beta} - nk(t) - c(t)\right) dt + \sigma_1(k(t))dW(t)$$
$$dh(t) = \delta(1 - u(t))h(t)dt + \sigma_2(h(t))dW(t)$$
$$dp(t) = -p(t)\left(\gamma^{\beta} - k(t)^{1-\beta} h(t)^{1-\beta} - n - \sigma_1''(k(t))\sigma_1(k(t)) - \sigma_1'(k(t))^2\right) dt$$
$$-p(t)\sigma_1'(k(t))dW(t)$$
$$dq(t) = -q(t)\left(\left(\delta(1 - u(t)) - \sigma_2''(h(t))\sigma_2(h(t)) - \sigma_2'(h(t))^2\right) dt + \sigma_2'(h(t))dW(t)\right)$$
$$\mathcal{H}_{u(t)} = \left(p(t)\gamma(1-\beta)k(t)^{\beta} u(t)^{-\beta} h(t)^{1-\beta} - q(t)\delta h(t)\right)\, dt = 0$$
$$\mathcal{H}_{c(t)} = \left(-p(t) + e^{-\rho t}\ c(t)^{-\sigma}\right)\, dt = 0.$$

It follows the optimal control

$$c^*(t) = e^{\frac{-\rho t}{\sigma}} p(t)^{\frac{-1}{\sigma}},\ \ u^*(t) = \left(\frac{\gamma(1-\beta)}{\delta}\frac{p(t)}{q(t)}\right)^{\frac{1}{\beta}}\ \frac{k(t)}{h(t)}$$

which fixes the optimal stochastic state evolution

$$dk(t) = \left(\gamma k(t)\left(\frac{\gamma(1-\beta)}{\delta}\frac{p(t)}{q(t)}\right)^{\frac{1-\beta}{\beta}} - nk(t) - e^{\frac{-\rho t}{\sigma}}\ p(t)^{\frac{-1}{\sigma}}\right) dt + \sigma_1(k(t))dW(t)$$

$$dh(t) = \delta\left(h(t)\left(\frac{\gamma(1-\beta)}{\delta}\frac{p(t)}{q(t)}\right)^{\frac{1}{\beta}}\ k(t)\right) dt + \sigma_2(h(t))dW(t)$$

and the optimal stochastic costate evolution

$$dp(t) = -p(t)\left(\gamma^{\beta} - k(t)^{1-\beta}h(t)^{1-\beta} - n - \sigma_1''(k(t))\sigma_1(k(t)) - \sigma_1'(k(t))^2\right)dt$$

$$-p(t)\sigma_1'(k(t))dW(t)$$

$$dq(t) = -q(t)\left(\left(\delta(1-u(t)) - \sigma_2''(h(t))\sigma_2(h(t)) - \sigma_2'(h(t))^2\right)dt + \sigma_2'(h(t))dW(t)\right)$$

The simulations refer to

$$\sigma_1(x) = \sigma_2(x) = x; \beta = 0.25; \gamma = 1.05;$$

$$\delta = 0.005; n = 0.01; \rho = 0.04; \sigma = 1.5, h_0 = 10; k_0 = 80.$$

The figures 6, 7, 8, 9, 10 and 11 represent the orbits: $(t, k(t,\omega))$, $(t, h(t,\omega))$, $(t, p(t,\omega))$, $(t, q(t,\omega))$, $(t, c(t,\omega))$, $(t, u(t,\omega))$.

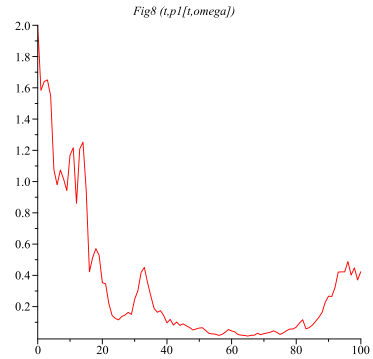
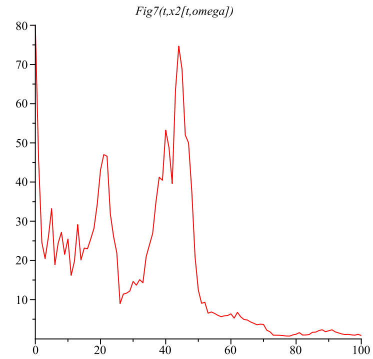
Fig. 6 $(t, k(t,\omega))$	Fig. 7 $(t, h(t,\omega))$
	

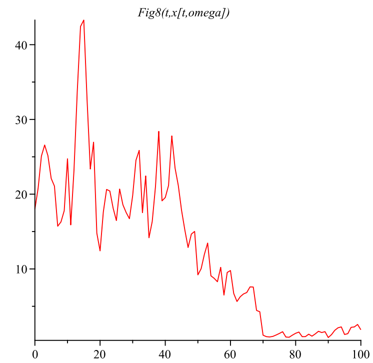
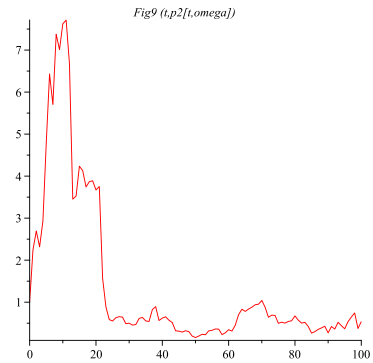
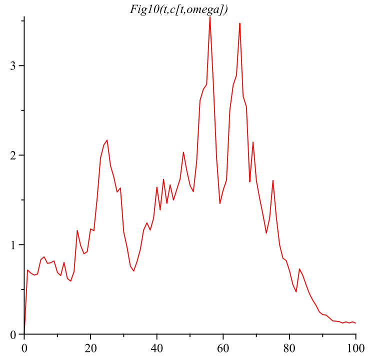
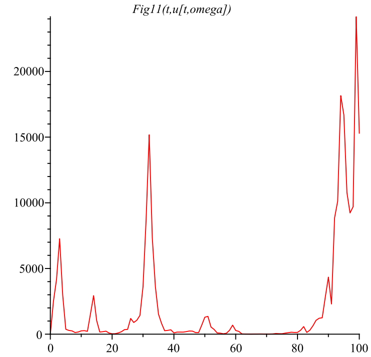
Fig. 8 $(t, p(t,\omega))$	Fig. 9 $(t, q(t,\omega))$
	

Fig. 10 $(t, c(t, \omega))$	Fig. 11 $(t, u(t, \omega))$
	

7 Conclusions

Alternatively to the classical stochastic control literature (e.g., [2], [10]), we start from a non-linear Itô differential system and we build a stochastic variational system and also an adjoint stochastic system. Both systems (variational and adjoint) are created considering a fixed control function. In order to prove the Theorem 4.1, regarding the single-time simplified stochastic maximum principle, we variate the control function. Our approach is different from those existing in the classical literature. We also apply a geometrical formalism, using the integrand 1-form, in order to offer a way of changing the single-time problems into multitime problems (objective functionals given by curvilinear integrals or multiple integrals, constrained by multitime Itô evolution equations).

Acknowledgement Partially supported by Grant POS DRU/6/1.5/S/16, University Politehnica of Bucharest. The authors would like to thank all those who have provided encouraging comments about the contents of this paper: Prof. Dr. Ionel Ţevy (University Politehnica of Bucharest), Vasile Drăgan, Toader Morozan and Constantin Vârsan (Research Workgroup on Optimal Control from Institute of Mathematics of the Romanian Academy), Prof. Dr. Gheorghiţă Zbăganu (University of Bucharest), Prof. Dr. Dumitru Opriş (West University of Timişoara) and Dr. Marius Rădulescu (Institute of Mathematical Statistics and Applied Mathematics, Bucharest).

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