The characterization of eigenfunctions for Laplacian operators

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Abstract. In this paper, we consider the characterization of eigenfunctions for Laplacian operators on some Riemannian manifolds. Firstly we prove that for the space form (M_K^n, g_K) with the constant sectional curvature K, the first eigenvalue of Laplacian operator $\lambda_1(M_K^n)$ is greater than the limit of the first Dirichlet eigenvalue of Laplacian operator $\lambda_1^D(B_K(p,r))$. Based on this, we then present a characterization of the Ricci soliton being an n-dim space form by the eigenfunctions corresponding to the first eigenvalue of Laplacian operator, which gives a generalization of an interesting result by Cheng in [3] from 2-dim to n-dim. Moreover, this result also gives a partly proof of a conjecture by Hamilton that a compact gradient shrinking Ricci soliton with positive curvature operator must be Einstein.

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1 Introduction and main results

Suppose that (M^n, g) is an n-dim C^∞ complete Riemannian manifold, and let Δ denote the Laplacian operator. If the manifold is compact, it is well known that the eigenvalue problem $-\Delta\varphi = \lambda\varphi$ has discrete eigenvalues, and we list them as

$$0 = \lambda_0(M^n) < \lambda_1(M^n) \le \lambda_2(M^n) \le \cdots .$$

We call $\lambda_i(M^n)$ the ith eigenvalue and call a function satisfying $\Delta\varphi = -\lambda_i\varphi$ an ith eigenfunction.

Recall that the first eigenvalue $\lambda_1(M^n)$ for the closed Riemannian manifold M^n is defined as follows:

$$\lambda_1(M^n) = \inf_{f\in\Omega} \frac{\int_{M^n} |\nabla f|^2 d\mu}{\int_{M^n} f^2 d\mu}, \tag{1.1}$$

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where Ω is the completing Hilbert space of

$$\Omega_0 = \left\{ f \in C^\infty (M^n) \middle| \int_{M^n} f d\mu = 0 \right\}$$

under the norm

$$\|f\|_1^2 = \int_{M^n} f^2 d\mu + \int_{M^n} |\nabla f|^2 d\mu.$$

For the first eigenvalue $\lambda_1(M^n)$ for closed Riemannian manifolds, we have the following famous theorem For the first eigenvalue $\lambda_1(M^n)$ for closed Riemannian manifolds, we have the following famous theorem in [4]:

Theorem 1.1 (Lichnerowicz-Obata). *Let (M^n, g) be a closed Riemannian manifold satisfying $Rc \geq (n-1) K > 0$. Then*

$$\lambda_1(M^n) \geq nK,$$

and the equality holds iff (M^n, g) is isometric to the space form (M_K^n, g_K) with constant sectional curvature K and the eigenfunction

$$f(x) = A \cos\left(\sqrt{K} r\right) + B \sin\left(\sqrt{K} r\right),$$

where $r = d(p, x)$.

On the other hand, we denote the open geodesic ball with center p and radius r by $B(p, r)$, and let $B_K(p, r)$ denote the geodesic ball with radius r in the n-dim simply connected space form (M_K^n, g_K) with constant sectional curvature K. Then the first Dirichlet eigenvalue $\lambda_1^D(B(p, r))$ of $B(p, r)$ can be denoted as:

$$\lambda_1^D(B(p, r)) = \inf_{f \in H_0^2(B(p,r))} \frac{\int_{B(p,r)} |\nabla f|^2 d\mu}{\int_{B(p,r)} f^2 d\mu}, \tag{1.2}$$

where $H_0^2(B(p, r))$ is the completing Hilbert space of $C_0^\infty(B(p, r))$ under the norm

$$\|f\|_1^2 = \int_{B(p,r)} f^2 d\mu + \int_{B(p,r)} |\nabla f|^2 d\mu.$$

For $\lambda_1^D(B(p, r))$ we have the following famous theorem in [2]:

Theorem 1.2 (Cheng). *Let (M^n, g) be a complete Riemannian manifold satisfying $Rc \geq (n-1) K$. Then*

$$\lambda_1^D(B(p, r)) \leq \lambda_1^D(B_K(p, r))$$

and the equality holds iff $B(p, r)$ is isometric to $B_K(p, r)$.

Moreover by using Theorem 1.2 we have the following corollary:

Corollary 1.3 (Cheng). *Let (M^n, g) be a compact Riemannian manifold satisfying $Rc \geq 0$. Then*

$$\lambda_1(M^n) \leq \lambda_1^D\left(B\left(p, \frac{d_{M^n}}{2}\right)\right) \leq \frac{C_n}{d_{M^n}^2},$$

where $C_n = 2n(n+4)$ and d_{M^n} is the diameter of M^n.

In this paper, by using Theorem 1.1 and 1.2 we firstly prove a useful result, which is one of the main results as follows:

Theorem 1.4. *Let (M_K^n, g_K) be a space form with constant sectional curvature K, and $\lambda_1^D(M_K^n)$ denote the first Dirichlet eigenvalue of M_K^n which is defined by*

$$\lambda_1^D(M_K^n) = \lim_{r\to\frac{\pi}{\sqrt{K}}} \lambda_1^D(B_K(p,r)),$$

where $\lambda_1^D(B_K(p,r))$ is the first Dirichlet eigenvalue of Laplacian operator of $B_K(p,r)$, then

$$\lambda_1^D(M_K^n) = \lim_{r\to\frac{\pi}{\sqrt{K}}} \lambda_1^D(B_K(p,r)) \le \lambda_1(M_K^n) \le \lambda_1^D\left(B_K\left(p, \frac{d_{M_K^n}}{2}\right)\right). \tag{1.3}$$

One of the basic problems in Riemannian geometry is to relate curvature and topology, in [1] Böhm and Wilking prove that n-dimensional closed Riemannian manifolds with 2-positive curvature operators are diffeomorphic to spherical space forms, i.e., they admit metrics with constant positive sectional curvature. Moreover it is a well-known theorem of Tachibana in [7] that any compact Einstein manifold with positive sectional curvature must be of constant curvature. Since Einstein manifolds are special Ricci solitons with constant potential functions, hence, inspired by his own work in [5] and [6], Hamilton made the following conjecture:

Conjecture 1.5 (Hamilton). *A compact gradient shrinking Ricci soliton with positive curvature operator must be Einstein.*

Recall that the definition of the gradient Ricci soliton is as follows:

Definition 1.1. A complete Riemannian manifold (M^n, g) is called a gradient Ricci soliton if there is a smooth function $f : M^n \to \mathbb{R}$, such that

$$Rc + \nabla\nabla f + \frac{\varepsilon}{2} g = 0,$$

where Rc is the Ricci curvature tensor and ε is a real number. Moreover
(i) If $\varepsilon < 0$, it is called a shrinking gradient Ricci soliton;
(ii) If $\varepsilon = 0$, it is called a steady one;
(iii) If $\varepsilon > 0$, it is called an expanding one.

For the compact gradient Ricci soliton, Hamilton proved the following fact:

Theorem 1.6 (Hamilton). *A compact gradient steady or expanding Ricci soliton must be Einstein.*

On the other hand, for the special case S^2, in [3] Cheng derived an interesting result by using the approach of tensor analysis and Gauss-Bonnet Theorem as follows:

Theorem 1.7 (Cheng). *Suppose that M^2 is homeomorphic to S^2 and φ_1, φ_2 and φ_3 are three first eigenfunctions such that their square sum is a constant. Then M^2 is actually isometric to a sphere with constant sectional curvature.*

Thus for the manifolds with $dim \geq 3$, it is natural to ask the following question:

Problem 1.2. Is Theorem 1.7 also true when $dim \geq 3$?

Although the general answer is hard, recall that each manifold with $dim = 2$ is actually an Einstein manifold, which is a special Ricci soliton with the constant potential function. Based on this observation, we will give an affirmative answer for the special case of Ricci solitons. In fact, we present a characterization of a Ricci soliton being an n-dim space form by the first eigenfunctions of Laplacian operator, which also gives a partly proof of Conjecture 1.5 of Hamilton as follows:

Theorem 1.8. *Let (M^n, g) be a compact gradient Ricci soliton with positve Ricci curvature, and suppose that there exists a geodesic ball $B(p, r)$ with center p and radius r such that the eigenfunctions $\{\varphi_i\}_{i=1}^m$ corresponding to the first Dirichlet eigenvalue of Laplacian operator $\lambda_1^D(B(p,r))$ satisfy $\sum_i \varphi_i^2 \equiv C$, where C is a nonzero constant. Let*

$$\mu = \inf\{\lambda \in \mathbb{R} \,|\, \nabla\nabla f \leq \lambda g\},$$

then
(i) (M^n, g) *is a shrinking Ricci soliton,*
(ii) (M^n, g) *is locally isometric to the space form M_K^n with constant sectional curvature $K = -\frac{\mu + \frac{\varepsilon}{2}}{n-1}$.*

The paper is organized as follows: In section 2, we prove Theorem 1.4. In section 3, we present the proof of Theorem 1.8 by using the approach of tensor analysis and Theorem 1.1, 1.2 and 1.4.

2 Proof of Theorem 1.4

Proof of Theorem 1.4. Firstly let points $p, q \in M_K^n$ such that $d(p,q) = d_{M_K^n}$, then we consider the geodesic balls $B_K\left(p, \frac{d_{M_K^n}}{2}\right)$ and $B_K\left(q, \frac{d_{M_K^n}}{2}\right)$ in the n-dim simply connected space form M_K^n. We denote u and v as the first Dirichlet eigenfunctions of Laplacian operator corresponding to $B_K\left(p, \frac{d_{M_K^n}}{2}\right)$ and $B_K\left(q, \frac{d_{M_K^n}}{2}\right)$, and define the following two functions:

$$\widetilde{u}(x) = \begin{cases} u(x), & x \in B_K\left(p, \frac{d_{M_K^n}}{2}\right) \\ 0, & x \in M_K^n \backslash B_K\left(p, \frac{d_{M_K^n}}{2}\right) \end{cases}$$

and

$$\widetilde{v}(x) = \begin{cases} v(x), & x \in B_K\left(q, \frac{d_{M_K^n}}{2}\right) \\ 0, & x \in M_K^n \backslash B_K\left(q, \frac{d_{M_K^n}}{2}\right). \end{cases}$$

Thus

$$\frac{\int_{M_K^n} |\nabla \widetilde{u}|^2 d\mu}{\int_{M_K^n} \widetilde{u}^2 d\mu} = \frac{\int_{B_K\left(p, \frac{d_{M_K^n}}{2}\right)} |\nabla u|^2 d\mu}{\int_{B_K\left(p, \frac{d_{M_K^n}}{2}\right)} u^2 d\mu} = \lambda_1^D\left(B_K\left(p, \frac{d_{M_K^n}}{2}\right)\right) \tag{2.1}$$

and

$$\text{(2.2)} \qquad \frac{\int_{M_K^n} |\nabla \widetilde{v}|^2 \, d\mu}{\int_{M_K^n} \widetilde{v}^2 d\mu} = \frac{\int_{B_K\left(q, \frac{d_{M_K^n}}{2}\right)} |\nabla u|^2 \, d\mu}{\int_{B_K\left(q, \frac{d_{M_K^n}}{2}\right)} u^2 d\mu} = \lambda_1^D \left(B_K \left(q, \frac{d_{M_K^n}}{2} \right) \right).$$

Then we choose a constant C such that

$$\int_{M_K^n} (\widetilde{u} + C\widetilde{v}) \, d\mu = 0,$$

and by the definition of the first eigenvalue of Laplacian operator $\lambda_1 (M_K^n)$ we have

$$\lambda_1 (M_K^n) \leq \frac{\int_{M_K^n} |\nabla (\widetilde{u} + C\widetilde{v})|^2 \, d\mu}{\int_{M_K^n} (\widetilde{u} + C\widetilde{v})^2 \, d\mu}.$$

Since M_K^n is a space form with diameter $d_{M_K^n}$, we have

$$\lambda_1^D \left(B_K \left(p, \frac{d_{M_K^n}}{2} \right) \right) = \lambda_1^D \left(B_K \left(q, \frac{d_{M_K^n}}{2} \right) \right)$$

and

$$Vol \left(B_K \left(p, \frac{d_{M_K^n}}{2} \right) \cap B_K \left(q, \frac{d_{M_K^n}}{2} \right) \right) = 0.$$

Thus

$$\begin{aligned} \lambda_1 (M_K^n) &\leq \frac{\int_{M_K^n} |\nabla (\widetilde{u} + C\widetilde{v})|^2 \, d\mu}{\int_{M_K^n} (\widetilde{u} + C\widetilde{v})^2 \, d\mu} \\ &= \frac{\int_{M_K^n} |\nabla \widetilde{u}|^2 \, d\mu + C^2 \int_{M_K^n} |\nabla \widetilde{v}|^2 \, d\mu}{\int_{M_K^n} \widetilde{u}^2 d\mu + C^2 \int_{M_K^n} \widetilde{v}^2 d\mu} \\ &= \lambda_1^D \left(B_K \left(p, \frac{d_{M_K^n}}{2} \right) \right) \\ &= \lambda_1^D \left(B_K \left(q, \frac{d_{M_K^n}}{2} \right) \right), \end{aligned}$$

for the last two equalities we use (2.1) and (2.2).

For the other inequality, since the metric of space form M_K^n has the form

$$g_{M_K^n} = dr^2 + s_K (r) \, g_{S^{n-1}},$$

and by Theorem 1.1 we can choose the function

$$\varphi (x) = A \cos \left(\sqrt{K} r \right) + B \sin \left(\sqrt{K} r \right)$$

where $r = d (p, x)$ such that

$$\begin{aligned}
\lambda_1\left(M_K^n\right) &= \frac{\int_{M_K^n} |\nabla\varphi|^2 \, d\mu}{\int_{M_K^n} \varphi^2 d\mu} \\
&= \frac{\int_0^{\frac{\pi}{\sqrt{K}}} |\nabla\varphi(r)|^2 s_K(r)^{n-1} \, dr \int_{S^{n-1}} d\mu_{S^{n-1}}}{\int_0^{\frac{\pi}{\sqrt{K}}} \varphi(r)^2 s_K(r)^{n-1} \, dr \int_{S^{n-1}} d\mu_{S^{n-1}}} \\
&= \frac{\int_0^{\frac{\pi}{\sqrt{K}}} |\nabla\varphi(r)|^2 s_K(r)^{n-1} \, dr}{\int_0^{\frac{\pi}{\sqrt{K}}} \varphi(r)^2 s_K(r)^{n-1} \, dr} \\
&= \lim_{s \to \frac{\pi}{\sqrt{K}}} \frac{\int_0^s |\nabla\varphi(r)|^2 s_K(r)^{n-1} \, dr}{\int_0^s \varphi(r)^2 s_K(r)^{n-1} \, dr},
\end{aligned}$$

where we use M_K^n is a space form with diameter $d_{M_K^n}$. Then we define a function $\varphi_s \in H_0^2\left(B_K(p,s)\right)$ such that $\varphi_s(x) = \varphi(x)$ for any $x \in B_K(p,s)$. Then by the definition of the first Dirichlet eigenvalue $\lambda_1^D\left(B_K(p,s)\right)$ it follows that

$$\begin{aligned}
\frac{\int_0^s |\nabla\varphi(r)|^2 s_K(r)^{n-1} \, dr}{\int_0^s \varphi(r)^2 s_K(r)^{n-1} \, dr} &= \frac{\int_0^s |\nabla\varphi(r)|^2 s_K(r)^{n-1} \, dr \int_{S^{n-1}} d\mu_{S^{n-1}}}{\int_0^s \varphi(r)^2 s_K(r)^{n-1} \, dr \int_{S^{n-1}} d\mu_{S^{n-1}}} \\
&= \frac{\int_{B_K(p,s)} |\nabla\varphi_s|^2 \, d\mu}{\int_{B_K(p,s)} \varphi_s^2 d\mu} \\
&\geq \lambda_1^D\left(B_K(p,s)\right).
\end{aligned}$$

Consequently

$$\lambda_1\left(M_K^n\right) = \lim_{s \to \frac{\pi}{\sqrt{K}}} \frac{\int_0^s |\nabla\varphi(r)|^2 s_K(r)^{n-1} \, dr}{\int_0^s \varphi(r)^2 s_K(r)^{n-1} \, dr} \geq \lim_{s \to \frac{\pi}{\sqrt{K}}} \lambda_1^D\left(B_K(p,s)\right) = \lambda_1^D\left(M_K^n\right).$$

□

3 Proof of the main result

By using Theorem 1.1, 1.2 and 1.4, we now turn to prove our main result Theorem 1.8.

Proof of Theorem 1.8. (i) If (M^n, g) is steady or expanding Ricci soliton, which satisfies

$$Rc + \nabla\nabla f + \frac{\varepsilon}{2} g = 0$$

such that $\varepsilon \geq 0$, then by using Theorem 1.6 we have (M^n, g) is an Einstein manifold and the Ricci potential function f is a constant. Thus

$$Rc = -\frac{\varepsilon}{2} g \leq 0,$$

which leads a contradiction to the positive Ricci curvature.

(ii) Note that the assumption of Theorem 1.8 says that

$$\begin{cases} \Delta\varphi_i + \lambda_1^D\left(B\left(p,r\right)\right)\varphi_i = 0, i = 1, \cdots, m \\ \sum_i \varphi_i^2 \equiv C, \end{cases}$$

thus

$$\begin{aligned} 0 &= \Delta\left(\sum_i \varphi_i^2\right) \\ &= \sum_i 2\left|\nabla\varphi_i\right|^2 + 2\sum_i \varphi_i\Delta\varphi_i \\ &= \sum_i 2\left|\nabla\varphi_i\right|^2 - 2\lambda_1^D\left(B\left(p,r\right)\right)\sum_i \varphi_i^2, \end{aligned}$$

which implies

$$\sum_i \left|\nabla\varphi_i\right|^2 = C\lambda_1^D\left(B\left(p,r\right)\right). \tag{3.1}$$

Recall that the Bochner formula (see [4]) says that

$$\Delta\left|\nabla f\right|^2 = 2\left|\nabla\nabla f\right|^2 + 2\left\langle\nabla f, \nabla\Delta f\right\rangle + 2Rc\left(\nabla f, \nabla f\right),$$

where Rc denotes the Ricci curvature and ∇f is the vector field. Taking Laplacian of both sides of (3.1) we have

$$\begin{aligned} 0 &= \sum_i \Delta\left|\nabla\varphi_i\right|^2 = 2\sum_i \left|\nabla\nabla\varphi_i\right|^2 + 2\sum_i \left\langle\nabla\varphi_i, \nabla\Delta\varphi_i\right\rangle + 2\sum_i Rc\left(\nabla\varphi_i, \nabla\varphi_i\right) \\ &= 2\sum_i \left|\nabla\nabla\varphi_i\right|^2 - 2\lambda_1^D\left(B\left(p,r\right)\right)^2\sum_i \varphi_i^2 + 2\sum_i Rc\left(\nabla\varphi_i, \nabla\varphi_i\right) \\ &= 2\sum_i \left|\nabla\nabla\varphi_i\right|^2 - 2\lambda_1^D\left(B\left(p,r\right)\right)^2\sum_i \varphi_i^2 - 2\sum_i \nabla\nabla f\left(\nabla\varphi_i, \nabla\varphi_i\right) - \varepsilon\sum_i \left|\nabla\varphi_i\right|^2 \\ &\geq 2\sum_i \left|\nabla\nabla\varphi_i\right|^2 - 2\lambda_1^D\left(B\left(p,r\right)\right)^2\sum_i \varphi_i^2 - 2\mu\sum_i \left|\nabla\varphi_i\right|^2 - \varepsilon\sum_i \left|\nabla\varphi_i\right|^2 \\ &\geq \frac{2}{n}\sum_i \left|\Delta\varphi_i\right|^2 - 2\lambda_1^D\left(B\left(p,r\right)\right)^2\sum_i \varphi_i^2 - 2\mu\sum_i \left|\nabla\varphi_i\right|^2 - \varepsilon\sum_i \left|\nabla\varphi_i\right|^2 \\ &= -2\left(1 - \frac{1}{n}\right)\lambda_1^D\left(B\left(p,r\right)\right)^2 C - 2\left(\mu + \frac{\varepsilon}{2}\right)\lambda_1^D\left(B\left(p,r\right)\right)C. \end{aligned}$$

Since $C > 0$ we have

$$\lambda_1^D\left(B\left(p,r\right)\right) \geq -\frac{1}{\left(1 - \frac{1}{n}\right)}\left(\mu + \frac{\varepsilon}{2}\right).$$

For the space form with constant sectional curvature $K = -\frac{\mu + \frac{\varepsilon}{2}}{n-1}$, we have

$$Rc = -(\mu + \frac{\varepsilon}{2})g.$$

Then by using Theorem 1.1 and 1.4 we have

$$\lambda_1^D\left(B_K(p,r)\right) \leq \lambda_1\left(M_K^n\right) = -\frac{1}{\left(1-\frac{1}{n}\right)}\left(\mu+\frac{\varepsilon}{2}\right),$$

where $\lambda_1\left(M_K^n\right)$ is the first eigenvalue of Laplacian operator for the space form M_K^n. This implies

$$\lambda_1^D\left(B\left(p,r\right)\right) \geq \lambda_1^D\left(B_K(p,r)\right). \tag{3.2}$$

On the other hand, by the definition of μ we have

$$Rc = -\left(\nabla\nabla f + \frac{\varepsilon}{2}g\right) \geq -\left(\mu+\frac{\varepsilon}{2}\right)g = (n-1)\left(-\frac{\mu+\frac{\varepsilon}{2}}{n-1}\right),$$

and by Theorem 1.2, it follows that

$$\lambda_1^D\left(B\left(p,r\right)\right) \leq \lambda_1^D\left(B_K\left(p,r\right)\right), \tag{3.3}$$

where $K = -\frac{\mu+\frac{\varepsilon}{2}}{n-1}$.

Then by (3.2) and (3.3) we derive that

$$\lambda_1^D\left(B\left(p,r\right)\right) = \lambda_1^D\left(B_K\left(p,r\right)\right), \tag{3.4}$$

thus by using the equality condition in Theorem 1.2, we complete the proof. □

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