Multitime reaction-diffusion solitons

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Abstract. In this paper, we introduce and explore some properties of two types of multitime partial differential equations, one as geometrical prolongation of the reaction-diffusion Kolmogorov-Petrovskii-Piskunov PDE and another as geometrical prolongation of the reaction-diffusion PDE of ultra-parabolic-hyperbolic type. The original ideas include: geometric ingredients to build first order and second order partial derivative operators, two multitime reaction-diffusion PDEs, techniques to obtain multitime solitons defined by a multitime reaction-diffusion PDE in a given direction and multitime solitons defined by a reaction-diffusion PDE of ultra-parabolic-hyperbolic type.

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1 Reaction-diffusion PDE

The modern theory of the nonlinear waves is an important field of today science. The nonlinear waves and coherent structures represent an interdisciplinary area with many applications in physics (nonlinear optics, nonlinear electric circuits, hydrodynamics, plasmas and states of solid), general relativity, chemistry (chemical reactions), biology (atmosphere and oceans, animal dispersal), random media and modern telecommunications.

A great variety of phenomena in physics, chemistry or biology can be described by nonlinear PDEs and particularly by reaction-diffusion equations. For these reasons, the theory of the soliton-solutions of the reaction-diffusion equations is considered one of the fastest development area in modern mathematics. The nonlinear reaction-diffusion (R-D) PDE

$$u_t(x,t) = au_{xx}(x,t) + f(u(x,t)),\ a > 0, \tag{1.1}$$

describes density fluctuations in a material undergoing reaction-diffusion. It appears in population dynamics, combustion theory and chemical kinetics and its solutions are called *reaction-diffusion waves* since 1930. In the mathematical and physical

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literature, the PDE (1.1) is wellknown as Kolmogorov-Petrovskii-Piskunov (KPP). If the reaction term $f(u)$ vanishes, then the equation represents a pure diffusion process. The choice $f(u) = u(1-u)$ yields Fisher's PDE, that was originally used to describe the spreading of the biological populations. For $f(u) = u(1-u^2)$, the Newell-Whitehead-Segel PDE describes Rayleigh-Benard convection, if $f(u) = u(1-u)(u-\alpha)$, $0 < \alpha < 1$, then the more general Zeldovich PDE arises in combustion theory (its particular degenerate case with $f(u) = u^2 - u^3$ is sometimes referred to as the Zeldovich equation as well) and the case $f(u) = u^3$ appears in Differential Geometry in connection to the motion of the curves (curve-shortening flow [7]).

The paper consists in six Sections. Section 2 underlines the history of the multitime solitons. Section 3 introduces the multitime R-D PDE in a given direction and shows that this new PDE is a prolongation of the R-D PDE (1.1). Section 4 analyzes the multitime solitons defined by a multitime R-D PDE in a given direction. Section 5 introduces the multitime R-D PDE of ultra-parabolic-hyperbolic type and proves that such an equation is a prolongation of the nonlinear damped hyperbolic PDE (5.1). Section 6 proves the existence of the multitime solitons defined by a multitime R-D PDE of ultra-parabolic-hyperbolic type.

2 History of multitime solitons

The theory of the multitime solitons starts with the papers [8]-[10], where the original ideas about the multitime sine-Gordon solitons, the multitime Rayleigh solitons and the multitime Boussinesq solitons are formulated, based on some geometrical ingredients similar to those in the present paper.

The evolution PDEs can have multi-temporal behavior if the dynamical systems that they describe permit linear or nonlinear perturbations or if the PDEs contains nonlinearities due to the friction, the deterioration, the flaw or to the presence of the constituents consisting of intelligent materials. In this case, the dynamics extends to many temporal scales evolving from slow to fast and conversely, being able to be described by more temporal variables. The multi-temporal modeling is particularly important in engineering, since it permits the evaluation of the properties of certain materials having as basis the knowledge of the associated geometry.

There are different beautiful connections between Multitime Solitons and Differential Geometry, which we will now shortly describe. First, the three steps to attend the idea of the multitime solitons PDEs are: (1) the geometrical thinking of the deformation with respect to a multitime variable $t = (t^1, ..., t^m)$; (2) the recognition of the geometrical ingredients able to produce differential operators; (3) the possibility of introducing multi-spatial and multi-temporal PDEs with solutions of the type $u(x,t) = \Phi(\langle a, x\rangle - \langle c, t\rangle)$, where $\Phi : \mathbb{R} \to \mathbb{R}$ is a function defining the *soliton shape*, and the *wave-vector* a and the *propagation velocity* c are required to be non-zero for generating non-trivial solutions. Second, in our mind appeared the idea of the multitime solitons as present in many physical, chemical, biological problems having a multidimensional temporal character. Third, though the multitime theory was proposed by Dirac in Physics (1932), the multi-temporal wave equations appeared recently in the context of harmonic analysis on Riemannian symmetric spaces [5].

Multitime soliton profile Let us define the *profile* of a soliton $u(x,t) = \Phi(\langle a, x\rangle - \langle c, t\rangle)$ at multitime $t = (t^1, ..., t^m) \in \mathbb{R}^m$ to be the graph of the function $x \to$

$u(x,t)$, $x \in \mathbb{R}^n$. Then the initial profile (at multitime $t = 0$) is just the graph of $\Phi(\langle a, x\rangle)$, and at any later multitime t (according the *product order*), the profile at multitime t, after a direction $u \in \mathbb{R}^n$, $||u|| = 1$, is obtained by translating each point $(x, \Phi(\langle a, x\rangle))$ of the initial profile to the point $(x + \langle c, t\rangle\, u, \Phi(\langle a, x\rangle))$. In other words, the profile of a multitime soliton propagates by rigid translation.

3 Multitime reaction-diffusion PDE in a given direction

Let us introduce a first multitime version of the nonlinear reaction-diffusion PDE (1.1). Since our temporal variable, $t = (t^\alpha)$, $\alpha = 1, \dots, m$, is m-dimensional, the new PDE will be called *multi-temporal.* Furthermore, since the new PDE admits *multitime soliton solutions*, this will be called *multitime soliton PDE.*

Our multitime version of the reaction-diffusion PDE is based on notions of Differential Geometry. First, the evolution parameter $t = (t^1, \dots, t^m)$, called *multitime*, is a point in $\mathbb{R}^m$, endowed with the *product order.* Second, we need a distinguished vector field $H = (h^\alpha)$, $h^\alpha = h^\alpha(x,t)$, $\alpha = 1, \dots, m$, on $\mathbb{R}^m$, borrowed from the geometry of the jet bundle of order one $J^1(\mathbb{R} \times \mathbb{R}^m, \mathbb{R})$ associated to C^2 function $u : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}$. The distinguished vector field $H = (h^\alpha)$ defines a *multitime derivation operator (along the direction H)*

$$D_H u = h^\alpha\, u_{t^\alpha},\ \alpha = 1, \dots, m. \tag{3.1}$$

This operator defines the multitime PDE

$$h^\alpha(x,t)\, u_{t^\alpha}(x,t) = a\, u_{x^2}(x,t) + f(u(x,t)), \tag{3.2}$$

which will be called *a multitime reaction-diffusion PDE in the direction H.*

Theorem 3.1. *There exists an infinity of distinguished vector fields $H = (h^\alpha)$ on $\mathbb{R}^m$ such that a solution of the reaction-diffusion PDE is also a solution of the multitime reaction-diffusion PDE (3.2).*

Proof Let $t^1 = t$ and $u = u(x,t^1)$. Suppose $u = u(x,t^1)$ is a solution of reaction-diffusion PDE. The function $v(x,t^1,...,t^m) = u(x,t^1)$ is a solution of the multitime reaction-diffusion PDE (3.2) if the family of distinguished vector fields $H = (h^\alpha)$ is fixed by $h^1 = 1$. It is obvious that we have an infinity of distinguished vector fields that satisfy this algebraic equation.

The foregoing theorem justifies the term *multitime geometrical prolongation of the reaction-diffusion PDE.* The reasons and the relations of fixing a vector field $H = (h^\alpha)$ depend on the problem that we want to solve. Many of the core ideas in PDEs theory can be reformulated using an appropriate vector field H.

Remark The theory in this Section can be extended to include the nonlinear Schrödinger equation

$$iu_t + u_{xx} + u|u|^2 = 0,\ u(x,t) = \text{complex unknown function},$$

which recently has been intensively studied because it describes the propagation of pulses of laser light in optical fibers.

4 Multitime solitons defined by a multitime reaction-diffusion PDE in a given direction

The first aim is to find multitime soliton solutions of the multitime PDE

$$h^\alpha(x,t)\, u_{t^\alpha}(x,t) = u_{x^2}(x,t) + u^3(x,t), \tag{4.1}$$

obtained from multitime PDE (3.2) for the diffusion constant $a = 1$ and the reaction term $f(u(x,t)) = u^3(x,t)$. Let $\Phi : I \subset \mathbb{R} \to \mathbb{R}$ be a function of class C^2. We are looking for the multitime solutions in the form

$$u(x,t) = \Phi(x - c_\alpha t^\alpha) = \Phi(z), \tag{4.2}$$

where (c_α) is a constant *index vector* and $z = x - c_\alpha t^\alpha$. Then

$$u_x = \Phi'(z), \quad u_{x^2} = \Phi''(z), \quad u_{t^\alpha} = \Phi'(z)(-c_\alpha).$$

Replacing these derivatives, the PDE (4.1) reduces to

$$-h^\alpha c_\alpha \Phi'(z) = \Phi''(z) + \Phi^3(z).$$

If we impose

$$h^\alpha(x,t) c_\alpha = a(z), \tag{4.3}$$

then the foregoing ODE is reduced to a *second order Riccati ODE*

$$\Phi''(z) + a(z)\Phi'(z) + \Phi^3(z) = 0. \tag{4.4}$$

4.1 Case of coefficients depending on z

We shall prove the existence of the solutions of the ODE (4.4) in two particular cases: $a(z) = -\dfrac{\Phi(z)}{\Phi'(z)}$ and, respectively, $a(z) = 3\Phi(z)$.

Case 1. In the particular case

$$a(z) = -\frac{\Phi(z)}{\Phi'(z)}, \tag{4.5}$$

we shall find solutions by the direct integration method and by the cosine method.

Direct integration. For the particular choice (4.5), the ODE (4.4) becomes

$$\Phi'' - \Phi + \Phi^3 = 0, \tag{4.6}$$

which, after the multiplication of both members by Φ' and integration, leads to the ODE

$${\Phi'}^2 = \Phi^2 - \frac{\Phi^4}{2}\,. \tag{4.7}$$

The ODE (4.7) has solutions for $|\Phi| < \sqrt{2}$. Transforming it in two ODEs with separable variables,

$$\Phi' = \pm\sqrt{\Phi^2 - \frac{\Phi^4}{2}}\,,$$

the solutions are found using primitives. Therefore

$$\pm\sqrt{2}\int \frac{d\Phi}{\sqrt{\Phi^2(2-\Phi^2)}} = z + C, \quad C \in \mathbb{R}. \tag{4.8}$$

Making the change of variable $\sqrt{2-\Phi^2} = \Psi,\ \Psi \in (0, \sqrt{2})$, we obtain the solution

$$\Phi(z) = \sqrt{8K}\,\frac{e^z}{1+Ke^{2z}}, \quad K > 0, \tag{4.9}$$

for both ODEs (4.8).

Cosine method. Using the same choice (4.5) of the function $a(z)$, we shall apply the cosine method to find solutions for the ODE (4.6) in the form

$$\Phi(z) = \lambda \cos^\beta(\mu z), \quad |z| < \frac{\pi}{2\mu},$$

where λ, β, μ are unknown parameters which will be determined further.

Computing the derivatives of the function Φ, we find

$$\Phi'(z) = -\lambda\beta\mu \cos^{\beta-1}(\mu z)\sin(\mu z)$$

and

$$\Phi''(z) = \lambda\beta(\beta-1)\mu^2 \cos^{\beta-2}(\mu z) - \lambda\beta^2\mu^2 \cos^\beta(\mu z).$$

Since $\lambda \neq 0$, the substitution into the nonlinear ODE (4.6) leads to the trigonometric relation

$$\beta(\beta-1)\mu^2\cos^{\beta-2}(\mu z) - (\beta^2\mu^2+1)\cos^\beta(\mu z) + \lambda^2\cos^{3\beta}(\mu z) = 0. \tag{4.10}$$

The single possible case, in which λ, β, μ to be non-zero and all the coefficients of the powers of the function $\cos(\mu z)$ vanish and which yields the solution, is $\beta = -1$. This value is obtained by equating the powers of $\cos(\mu z)$ from the first term and from the last term in the relation (4.10), i.e., $\beta - 2 = 3\beta$.

For $\beta = -1$, the relation (4.10) becomes

$$(\lambda^2 + 2\mu^2)\cos^{-3}(\mu z) - (\mu^2+1)\cos^{-1}(\mu z) = 0.$$

To be an identity, we impose $\lambda^2 + 2\mu^2 = 0$, $mu^2 + 1 = 0$, which yields the solution $\mu = \pm i$ and $\lambda = \pm\sqrt{2}$. Then the functions

$$\Phi(z) = \frac{\pm\sqrt{2}}{\cos(iz)} = \frac{\pm\sqrt{2}}{\cosh(z)} \tag{4.11}$$

are solutions of the ODE (4.6).

Remark. The solution $\Phi(z) = \dfrac{\sqrt{2}}{\cosh(z)}$ coincides to the solution (4.9), previously obtained by direct integration, for the selection $K = 1$.

Theorem 4.1. *If $\Phi(z)$ defined by (4.9) or (4.11) are solutions of the ODE (4.4), in the hypothesis (4.5), then $u(x,t) = \Phi(z)$, where $z = x - c_\alpha t^\alpha$ are multitime soliton solutions of the multitime PDE (4.1), under the assumptions (4.3) and (4.5).*

Case 2. In the sequel we shall consider the second particular choice, namely

$$a(z) = 3\Phi(z). \tag{4.12}$$

According to this assumption, the ODE (4.4) becomes

$$\Phi'' + 3\Phi\Phi' + \Phi^3 = 0. \tag{4.13}$$

If we denote $\Phi = \Psi'$, then $\Phi' = \Psi''$, $\Phi'' = \Psi'''$ and the ODE (4.13) is rewritten as

$$\Psi''' + 3\Psi'\Psi'' + {\Psi'}^3 = 0.$$

By multiplication with e^Ψ we obtain the ODE $\left(e^\Psi\Psi''\right)' + \left(e^\Psi{\Psi'}^2\right)' = 0$ and by integration we get $e^\Psi\Psi'' + e^\Psi{\Psi'}^2 = C_1$. Hence $\left(e^\Psi\Psi'\right)' = C_1$ and a new integration leads to $e^\Psi\Psi' = C_1 z + C_2$. In this way, we find $e^\Psi = \frac{C_1}{2}z^2 + C_2 z + C_3$, on the set where $\frac{C_1}{2}z^2 + C_2 z + C_3 > 0$. It follows $\Psi = \ln\left(\frac{C_1}{2}z^2 + C_2 z + C_3\right)$, hence

$$\Psi' = \frac{C_1 z + C_2}{\dfrac{C_1}{2}z^2 + C_2 + C_3}. \tag{4.14}$$

Coming back to the foregoing notation, we find the solution

$$\Phi(z) = \frac{C_1 z + C_2}{\dfrac{C_1}{2}z^2 + C_2 + C_3}, \tag{4.15}$$

where C_1, C_2, C_3 are real constants.

Theorem 4.2. *If $\Phi(z)$ defined by (4.15) is a solution of the ODE (4.4), in the hypothesis (4.12), then $u(x,t) = \Phi(z)$, where $z = x - c_\alpha t^\alpha$ is a multitime soliton solution of the multitime PDE (4.1), in the hypothesis (4.3) and (4.12).*

4.2 Case of constant coefficients

If we assume $a(z) = constant = l$, then there are two different cases: $l \neq 0$ and, respectively, $l = 0$.

For $l \neq 0$, the equation (4.4) becomes a second order ODE with constant coefficients (second order Riccati type)

$$\Phi''(z) + l\Phi'(z) + \Phi^3(z) = 0, \tag{4.16}$$

and for this equation we shall prove the existence of the solutions using the following method presented in [6]. We suppose that $\Phi(z) = \sum_{i=0}^{M} a_i G^i$, where the function $G = G(z)$ is the solution of the Riccati type ODE

$$G'(z) = \epsilon(1 - G^2), \quad \epsilon = \pm 1.$$

Because the functions $1, G, G^2, \ldots, G^m (m \in \mathbb{N})$ are linearly independent, we can take $M = 1$ and without loss of generality, we may choose $\Phi(z) = a_0 + a_1 G$, where a_0, a_1 are real constants which will be determined later, with $a_1 \neq 0$, for nontrivial solutions.

Then $\Phi'(z) = \epsilon a_1 - \epsilon a_1 G^2$ and $\Phi''(z) = -2a_1 + 2a_1 G^3$. Thus from (4.16) we get a null polynomial

$$a_0^3 + l\epsilon a_1 + (3a_0^2 a_1 - 2a_1)G + (3a_0 a_1^2 - l\epsilon a_1)G^2 + (a_1^3 + 2a_1)G^3 = 0,$$

in the unknown G. It follows a system of algebraic equations

$$a_0^3 + l\epsilon a_1 = 0,\ 3a_0^2 a_1 - 2a_1 = 0,\ 3a_0 a_1^2 - l\epsilon a_1 = 0,\ a_1^3 + 2a_1 = 0,$$

with the unknowns a_0, a_1. Since $a_1 \neq 0$, the last equation has no real solutions. Thus the ODE (4.16), and consequently the equation (4.4), has no solution of the foregoing form if $l \neq 0$.

If we consider now the case in which the function $G = G(z)$ would be a solution for the Riccati type ODEs

$$G'(z) = \epsilon(1 + G^2), \quad \epsilon = \pm 1$$

and we continue in the same way like before, we'll be able to say that the equation (4.4) does not admit solutions of the foregoing form if $l \neq 0$.

In the second case, $l = 0$, the equation (4.4) becomes $\Phi''(z) + \Phi^3(z) = 0$, and it has a solution given by elliptic functions,

$$\Phi(z) = \frac{\sqrt{2}}{2}\, sd\left(z; \frac{\sqrt{2}}{2}\right),$$

where $sd(z; k)$ is the *Jacobi elliptic function* satisfying

$$\left(\frac{d\eta}{dz}\right)^2 = 1 + (2k^2 - 1)\eta^2 + k^2(k^2 - 1)\eta^4.$$

Remark: Using the nonclassical symmetry reduction method [1], if $w(z)$ satisfies the equation $w'' + w^3 = 0$, then PDE (1.1), in the case $a = 1$ and $f(u(x,t)) = u^3(x,t)$, allows a solution of the form $u(x,t) = (x + C)w(z)$, $z = \frac{1}{2}x^2 + Cx + 3t$, where $C \in \mathbb{R}$. In this way, an exact solution of PDE (1.1), in the case $a = 1$ and $f(u(x,t)) = u^3(x,t)$, is given by

$$u(x,t) = \frac{\sqrt{2}}{2}\,(x + C)\ sd\left(\frac{1}{2}x^2 + Cx + 3t; \frac{\sqrt{2}}{2}\right),\ C \in \mathbb{R}.$$

4.3 Maclaurin series soliton of multitime reaction-diffusion PDE in a given direction

The basic idea for finding Maclaurin series-solutions of the multitime PDE (4.1) is that we assume this equation has a solution Φ which is analytic on an interval around the origin $z = 0$. Then we can express Φ as a power series in the form

$$\Phi(z) = \sum_{n=0}^{\infty} \alpha_n z^n \tag{4.17}$$

and we try to determine the unknown coefficients α_n. On an interval around the origin $z = 0$, the resulting power series should converge. We shall say that $\Phi(z)$ is a *series-solution around the origin.*

Computing $\Phi'(z), \Phi''(z)$ and $\Phi^3(z)$, we get

$$\Phi'(z) = \sum_{n=1}^{\infty} n\alpha_n z^{n-1} = \sum_{n=0}^{\infty}(n+1)\alpha_{n+1}z^n,$$

$$\Phi''(z) = \sum_{n=1}^{\infty}(n+1)n\alpha_{n+1}z^{n-1} = \sum_{n=0}^{\infty}(n+2)(n+1)\alpha_{n+2}z^n$$

and

$$\Phi^3(z) = \sum_{n=0}^{\infty}\left(\sum_{i=0}^{n}\sum_{k=0}^{i}\alpha_k\alpha_{i-k}\alpha_{n-i}\right)z^n.$$

If $a(z) = Mz + A$, $M, A \in \mathbb{R}$, is an affine function in z, then the ODE (4.4) becomes

$$\sum_{n=0}^{\infty}(n+2)(n+1)\alpha_{n+2}z^n + (Mz+A)\sum_{n=0}^{\infty}(n+1)\alpha_{n+1}z^n +$$

$$+\sum_{n=0}^{\infty}\left(\sum_{i=0}^{n}\sum_{k=0}^{i}\alpha_k\alpha_{i-k}\alpha_{n-i}\right)z^n = 0.$$

This identity can be written in the form

$$\sum_{n=0}^{\infty}(n+2)(n+1)\alpha_{n+2}z^n + M\sum_{n=0}^{\infty}(n+1)\alpha_{n+1}z^{n+1} + A\sum_{n=0}^{\infty}(n+1)\alpha_{n+1}z^n +$$

$$+\sum_{n=0}^{\infty}\left(\sum_{i=0}^{n}\sum_{k=0}^{i}\alpha_k\alpha_{i-k}\alpha_{n-i}\right)z^n = 0,$$

whence we get

$$\sum_{n=0}^{\infty}(n+2)(n+1)\alpha_{n+2}z^n + M\sum_{n=1}^{\infty}n\alpha_n z^n + A\sum_{n=0}^{\infty}(n+1)\alpha_{n+1}z^n +$$

$$+\sum_{n=0}^{\infty}\left(\sum_{i=0}^{n}\sum_{k=0}^{i}\alpha_k\alpha_{i-k}\alpha_{n-i}\right)z^n = 0$$

and then

$$2\alpha_2 + \sum_{n=1}^{\infty}(n+2)(n+1)\alpha_{n+2}z^n + M\sum_{n=1}^{\infty}n\alpha_n z^n + A\alpha_1 + A\sum_{n=1}^{\infty}(n+1)\alpha_{n+1}z^n$$

$$+\alpha_0^3 + \sum_{n=1}^{\infty}\left(\sum_{i=0}^{n}\sum_{k=0}^{i}\alpha_k\alpha_{i-k}\alpha_{n-i}\right)z^n = 0.$$

Thus we obtain the identity

$$2\alpha_2 + A\alpha_1 + \alpha_0^3 + + \sum_{n=1}^{\infty}\big[(n+2)(n+1)\alpha_{n+2} + A(n+1)\alpha_{n+1}$$

$$+Mn\alpha_n + \sum_{i=0}^{n}\sum_{k=0}^{i}\alpha_k\alpha_{i-k}\alpha_{n-i}\big]z^n = 0.$$

By identifying the coefficients of the powers of z with 0, we find the condition

$$2\alpha_2 + A\alpha_1 + \alpha_0^3 = 0$$

and the recurrence

$$\begin{aligned} &(n+2)(n+1)\alpha_{n+2} + A(n+1)\alpha_{n+1} \\ &+Mn\alpha_n + \sum_{i=0}^{n}\sum_{k=0}^{i}\alpha_k\alpha_{i-k}\alpha_{n-i} = 0, \quad n \geq 1. \end{aligned} \tag{4.18}$$

By the initial conditions $\Phi(0) = \alpha_0$, $\Phi'(0) = \alpha_1$ and $2\alpha_2 + A\alpha_1 + \alpha_0^3 = 0$, this recurrence gives all the coefficients of the power series (4.17), but the difficult part is just solving this recurrence for the unknown α_n or at least showing the convergence of the series, which remain open problems, for the moment.

Theorem 4.3. *In the foregoing hypothesis, the multitime series soliton solution of the multitime PDE (4.1) is*

$$u(x,t) = \sum_{n=0}^{\infty}\alpha_n(x - c_\alpha t^\alpha)^n,$$

with α_0, α_1 *fixed,* $2\alpha_2 + A\alpha_1 + \alpha_0^3 = 0$ *and* $\alpha_n,\ n \geq 3$, *given by the recurrence (4.18).*

5 Multitime reaction-diffusion PDE of ultra-parabolic-hyperbolic type

We consider the *nonlinear damped hyperbolic PDE*

$$\epsilon u_{tt}(x,t) + u_t(x,t) = a\Delta_x u(x,t) + f(u(x,t)), \quad x \in \mathbb{R}^n,\ t \in \mathbb{R}, \tag{5.1}$$

depending on the positive parameter ϵ, not necessarily small, and the nonlinear function $f : \mathbb{R} \to \mathbb{R}$. If we impose $n = 1$, $\epsilon = 0$, then the PDE (5.1) reduces to reaction-diffusion PDE (1.1).

We mention that in the case $n = 1$, the PDE of type (5.1) appeared as mathematical models to describe some natural phenomena, like the propagation of the voltage along a nonlinear transmission line or the random motion of one-celled organisms.

Let us create a second multitime version extending both the ultra-parabolic-hyperbolic PDE (5.1) and the reaction-diffusion PDE (1.1). For this aim, we use ingredients from jet bundle of order one $J^1(\mathbb{R}^n \times \mathbb{R}^m, \mathbb{R})$ induced on $\mathbb{R}^m$: (1) a distinguished symmetric linear connection $\Gamma^\gamma_{\alpha\beta} = \Gamma^\gamma_{\alpha\beta}(x,t)$ and (2) a distinguished contravariant symmetric fundamental tensor field $g^{\alpha\beta} = g^{\alpha\beta}(x,t)$, of constant signature (p,q,r), $p+q+r=m$. Using the function $u : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$, of class C^2, we build the *Hessian operator*

$$(Hess_\Gamma u)_{\alpha\beta} = u_{t^\alpha t^\beta} - \Gamma^\gamma_{\alpha\beta} u_{t^\gamma}, \quad \alpha,\beta = 1,\dots,m \tag{5.2}$$

and its trace, called *ultra-parabolic-hyperbolic operator*,

$$\square_{\Gamma,g} u = g^{\alpha\beta}(Hess_\Gamma u)_{\alpha\beta}. \tag{5.3}$$

This operator defines another multitime PDE of ultra-parabolic-hyperbolic type

$$\square_{\Gamma,g} u(x,t) = a\Delta_x u(x,t) + f(u(x,t)), \tag{5.4}$$

where $x \in \mathbb{R}^n$ and $t = (t^1,\dots,t^m) \in \mathbb{R}^m$. Explicitly,

$$g^{\alpha\beta}(x,t)u_{t^\alpha t^\beta}(x,t) - g^{\alpha\beta}(x,t)\Gamma^\gamma_{\alpha\beta}(x,t)u_{t^\gamma}(x,t) = a\Delta_x u(x,t) + f(u(x,t)). \tag{5.5}$$

Theorem 5.1. *There exists an infinity of distinguished geometrical structures* $\Gamma^\gamma_{\alpha\beta}$, $g^{\alpha\beta}$ *on* $\mathbb{R}^m$ *such that a solution of the nonlinear damped hyperbolic PDE (5.1) is also a solution of the multitime PDE of ultra-parabolic-hyperbolic type (5.5).*

Proof Let $t^1 = t$ and $u = u(x,t^1)$. Suppose $u = u(x,t^1)$ is a solution of the nonlinear damped hyperbolic PDE (5.1). The function $v(x,t^1,...,t^m) = u(x,t^1)$ is a solution of the multitime PDE of ultra-parabolic-hyperbolic type (5.5) if the families of geometrical structures $\Gamma^\gamma_{\alpha\beta}$, $g^{\alpha\beta}$ are fixed by $g^{11} = 1$, $g^{\alpha\beta}\Gamma^1_{\alpha\beta} = 1$. It is obvious that we have an infinity of geometrical structures that satisfy these algebraic equations.

The foregoing theorem justifies the term *multitime geometrical prolongation of the nonlinear damped hyperbolic PDE*. The fundamental tensor field $g^{\alpha\beta}$ and the connection $\Gamma^\gamma_{\alpha\beta}$ are fixed by conditions depending on the problem that we want to solve. A fundamental tensor field and a connection are important in mathematics because they provide a concise mathematical framework for formulating PDE problems in areas such as elasticity, fluid mechanics, general relativity etc.

6 Multitime solitons defined by a reaction-diffusion PDE of ultra-parabolic-hyperbolic type

Let $\Phi : I \subset \mathbb{R} \to \mathbb{R}$ be a function of class C^2. Let $a = (a_i), c = (c_\alpha)$ be two constant index vectors and $z = a_i x^i - c_\alpha t^\alpha = \langle a,x\rangle - \langle c,t\rangle$. We want to find the multitime soliton solutions

$$u(x,t) = \Phi(a_i x^i - c_\alpha t^\alpha) = \Phi(z),$$

of the PDE (5.5). To simplify, we suppose $n=1$, $a_1 = 1$, $z = x - c_\alpha t^\alpha$ and for $a=1$ and $f(u(x,t)) = u^3(x,t)$, we get the multitime PDE

$$g^{\alpha\beta}(x,t)u_{t^\alpha t^\beta}(x,t) - g^{\alpha\beta}(x,t)\Gamma^\gamma_{\alpha\beta}(x,t)u_{t^\gamma}(x,t) = u_{x^2}(x,t) + u^3(x,t), \tag{6.1}$$

where $x \in \mathbb{R}^n$ and $t = (t^1, \ldots, t^m) \in \mathbb{R}^m$. The partial derivatives

$$u_x = \Phi'(z), \quad u_{x^2} = \Phi''(z), \quad u_{t^\alpha} = \Phi'(z)(-c_\alpha), \quad u_{t^\alpha t^\beta} = \Phi''(z) c_\alpha c_\beta,$$

impose the ODE

$$g^{\alpha\beta} c_\alpha c_\beta \Phi''(z) + g^{\alpha\beta} \Gamma^\gamma_{\alpha\beta} c_\gamma \Phi'(z) = \Phi''(z) + \Phi^3(z). \tag{6.2}$$

If we assume

$$1 - g^{\alpha\beta} c_\alpha c_\beta = a(z), \quad -g^{\alpha\beta} \Gamma^\gamma_{\alpha\beta} c_\gamma = b(z), \tag{6.3}$$

we obtain a second order ODE (higher order Riccati type)

$$a(z)\Phi''(z) + b(z)\Phi'(z) + \Phi^3(z) = 0. \tag{6.4}$$

6.1 Case of coefficients depending on z

We shall prove the existence of the solutions for ODE (6.4) in the particular case

$$a(z) = b(z) = -\frac{1}{4\Phi'(z)}. \tag{6.5}$$

For this selection, the ODE (6.4) becomes

$$\Phi''(z) + \Phi'(z) - 4\Phi^3(z)\Phi'(z) = 0.$$

By integration on the interval $[0, z]$, we obtain the ODE $\Phi' + \Phi - \Phi^4 = 0$, with separable variables, which has the solutions

$$\Phi(z) = \frac{1}{\sqrt[3]{1 - Ke^{3z}}}, \quad K > 0, \tag{6.6}$$

respectively

$$\Phi(z) = \frac{1}{\sqrt[3]{1 + Ke^{3z}}}, \quad K > 0. \tag{6.7}$$

Theorem 6.1. *If $\Phi(z)$ given by (6.6) and (6.7) are solutions of the ODE (6.4), under the assumption (6.5), then $u(x,t) = \Phi(z)$,where $z = x - c_\alpha t^\alpha$ are multitime soliton solutions of the multitime PDE (6.1), under the assumptions (6.3) and (6.5).*

6.2 Case of constant coefficients

If we consider $a(z) = constant = k \neq 0$ and $b(z) = constant = l$, then the ODE (6.2) becomes a second order ODE with constant coefficients (higher order Riccati type)

$$k\Phi''(z) + l\Phi'(z) + \Phi^3(z) = 0, \tag{6.8}$$

and for this equation we shall prove the existence of the solutions, using the $\frac{G'}{G}$ expansion method [12]. We suppose that the solution Φ of the equation (6.8) can be expressed by a polynomial in $\frac{G'}{G}$ in the form

$$\Phi(z) = \sum_{i=0}^{n} a_i \left(\frac{G'}{G}\right)^i, \ a_n \neq 0,$$

where $G = G(z)$ is satisfying the second order linear ODE $G'' + \lambda G' + \mu G = 0$ and a_i, λ, μ are real constants which will be determined later. The positive integer n is found by homogeneous balance between the highest order derivatives and the nonlinear terms appearing in equation. Thus, $\deg(\Phi'') = \deg(\Phi^3)$ yields the identity $n + 2 = 3n$, whence $n = 1$. Therefore $\Phi(z) = a_0 + a_1 \frac{G'}{G}$, the constants a_0, a_1 being determined further, with $a_1 \neq 0$. Because $G'' = -\lambda G' - \mu G$, it follows that $\left(\frac{G'}{G}\right)' = -\mu - \lambda \frac{G'}{G} - \left(\frac{G'}{G}\right)^2$ and then, computing Φ', Φ'' and Φ^3, we get

$$\Phi'(z) = -a_1\mu - a_1\lambda \frac{G'}{G} - a_1 \left(\frac{G'}{G}\right)^2,$$

$$\Phi''(z) = a_1\lambda\mu + (a_1\lambda^2 + 2a_1\mu)\frac{G'}{G} + 3a_1\lambda \left(\frac{G'}{G}\right)^2 + 2a_1 \left(\frac{G'}{G}\right)^3$$

and, respectively,

$$\Phi^3(z) = a_0^3 + 3a_0^2 a_1 \frac{G'}{G} + 3a_0 a_1^2 \left(\frac{G'}{G}\right)^2 + a_1^3 \left(\frac{G'}{G}\right)^3.$$

The substitution of these expressions into the equation (6.8) yields the following polynomial identity

$$\begin{aligned} &a_0^3 + k\lambda\mu a_1 - l\mu a_1 + (3a_0^2 a_1 + k\lambda^2 a_1 + 2k\mu a_1 - l\lambda a_1)\frac{G'}{G} + \\ &+ (3a_0 a_1^2 + 3k\lambda a_1 - la_1)\left(\frac{G'}{G}\right)^2 + (a_1^3 + 2ka_1)\left(\frac{G'}{G}\right)^3 = 0, \end{aligned}$$

in the unknown $\frac{G'}{G}$. Equating with 0 all its coefficients, we get the following system of algebraic equations

$$a_0^3 + k\lambda\mu a_1 - l\mu a_1 = 0,\ 3a_0^2 a_1 + k\lambda^2 a_1 + 2k\mu a_1 - l\lambda a_1 = 0,$$
$$3a_0 a_1^2 + 3k\lambda a_1 - la_1 = 0,\ a_1^3 + 2ka_1 = 0,$$

which has solutions only in the case $l = 0$ and $k < 0$; the solutions are $a_0 = \frac{\sqrt{-2k}\lambda}{2}$, $a_1 = \sqrt{-2k}$, $\mu = \frac{\lambda^2}{4}$, $\lambda \in \mathbb{R}$ and $a_0 = -\frac{\sqrt{-2k}\lambda}{2}$, $a_1 = -\sqrt{-2k}$, $\mu = \frac{\lambda^2}{4}$, $\lambda \in \mathbb{R}$. Since $\lambda^2 - 4\mu = 0$, we find $\frac{G'}{G} = \frac{C_2}{C_1 + C_2 z} - \frac{\lambda}{2}$, $C_1, C_2 \in \mathbb{R}$, so we get two different solutions

$$\Phi(z) = \pm \frac{\sqrt{-2k}\, C_2}{C_1 + C_2 z},\ C_1, C_2 \in \mathbb{R}.$$

of equation (6.8), in the additional assumption $k < 0$, $l = 0$.

Theorem 6.2. *If we fix the fundamental tensor $g^{\alpha\beta}$ and the connection $\Gamma^{\gamma}_{\alpha\beta}$ by the conditions*

$$1 - g^{\alpha\beta} c_\alpha c_\beta = constant < 0,\ -g^{\alpha\beta}\Gamma^{\gamma}_{\alpha\beta} c_\gamma = 0,$$

then we get two different soliton-solutions

$$u(x,t) = \pm \frac{\sqrt{2(g^{\alpha\beta} c_\alpha c_\beta - 1)}\, C_2}{C_1 + C_2(x - c_\alpha t^\alpha)},\ C_1, C_2 \in \mathbb{R}$$

of the multitime PDE (6.1).

6.3 Maclaurin series soliton of multitime ultra-parabolic-hyperbolic type PDE

In the same conditions like in the beginning of Subsection 4.3, we shall continue by considering the affine functions $a(z) = Mz + A,\ b(z) = Nz + B,\ M, N, A, B \in \mathbb{R}$ and then equation (6.2) becomes

$$2A\alpha_2 + B\alpha_1 + \alpha_0^3 + \sum_{n=1}^{\infty} \big[A(n+2)(n+1)\alpha_{n+2} + (Mn+B)(n+1)\alpha_{n+1}$$

$$+Nn\alpha_n + \sum_{i=0}^{n}\sum_{k=0}^{i} \alpha_k\alpha_{i-k}\alpha_{n-i}\big] z^n = 0.$$

By identifying the coefficients of the powers of z with 0, we find the condition

$$2A\alpha_2 + B\alpha_1 + \alpha_0^3 = 0$$

and the recurrence

$$A(n+2)(n+1)\alpha_{n+2} + (Mn+B)(n+1)\alpha_{n+1}$$

$$+Nn\alpha_n + \sum_{i=0}^{n}\sum_{k=0}^{i} \alpha_k\alpha_{i-k}\alpha_{n-i} = 0, \quad n \geq 1. \tag{6.9}$$

By the initial conditions $\Phi(0) = \alpha_0, \Phi'(0) = \alpha_1$ and $2A\alpha_2 + B\alpha_1 + \alpha_0^3 = 0$, this recurrence gives all the coefficients of the power series (4.17), but, like we have said before, the difficult part is just solving this recurrence for the unknown α_n, an open problem also.

Theorem 6.2. *In the foregoing hypothesis, the multitime series soliton solution of multitime PDE (5.4) is*

$$u(x,t) = \sum_{n=0}^{\infty} \alpha_n (x - c_\alpha t^\alpha)^n,$$

with α_0, α_1 *fixed,* $2A\alpha_2 + B\alpha_1 + \alpha_0^3 = 0$ *and* $\alpha_n,\ n \geq 3$ *given by the recurrence (6.9).*

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