Optimal control problems with higher order ODEs constraints

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Abstract. In this paper, the analysis is focused on single-time optimal control problems based on simple integral cost functionals from Lagrangians whose order is smaller than the higher order of ODEs constraints. The basic topics of our theory include: variational differential systems, adjoint differential systems, Legendrian duality, single-time maximum principle. The main original results refer to the form of adjoint differential systems and the simplified single-time maximum principle, based on higher order ingredients. For completeness, we added Euler-Lagrange and Hamilton equations of higher order obtained from the maximum principle.

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1 Single-time optimal control problem with second order ODEs constraints

Our paper has three sources of inspiration: (1) the Analytical Mechanics based on second order Lagrangians studied, with remarkable results, by many researchers (see [5]-[7]), (2) some optimization problems via second order Lagrangians solved in the papers [8]-[10], [13], [22] and (3) the optimal control problem governed by the nonlinear elastic beam equation (see [4]).

Here we develop our view-point by introducing some new results regarding higher order Lagrangians and ODEs constraints. Section 1 introduces and studies an optimal control problem involving second order ODEs constraints and, using the notion of adjointness, there are given necessary conditions of optimality. Section 2 takes into account the general case when there are considered higher order ODEs constraints for an optimal control problem. Section 3 is devoted to higher order Euler-Lagrange and Hamilton ODEs via simplified single-time maximum principle, highlighting the main results. Section 4 points out future research.

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Let study an optimal control problem based on a simple integral cost functional with second order ODEs constraints:

$$\max_{u(\cdot),x_{t_0}} \left\{ I\left(u(\cdot)\right) = \int_0^{t_0} X\left(t, x(t), \dot{x}(t), u(t)\right) dt \right\} \tag{1.1}$$

subject to

$$\ddot{x}^i(t) = X^i\left(t, x(t), \dot{x}(t), u(t)\right), \quad i = \overline{1,n} \tag{1.2}$$

$$u(t) \in \mathcal{U},\ \forall t \in [0, t_0]; \quad x(0) = x_0,\ x(t_0) = x_{t_0},\ \dot{x}(0) = \tilde{x}_0,\ \dot{x}(t_0) = \tilde{x}_{t_0}. \tag{1.3}$$

Terminology and notations: $t \in [0, t_0]$ is a parameter of evolution, or *single-time*; $[0, t_0] \subset \mathbb{R}_+$ is the *time interval*; $x(t) = (x^i(t)),\ i = \overline{1,n}$, is a C^3-class function, called *state vector*; $u(t) = (u^\alpha(t)),\ \alpha = \overline{1,k}$, is a *continuous control vector*; the *running cost* $X\left(t, x(t), \dot{x}(t), u(t)\right)$ is a C^1-class function, called *non-autonomous Lagrangian*.

In this section we are looking for necessary conditions of optimality (for a pair (x, u)) in the previous optimal control problem. Further, the summation over the repeated indices is assumed.

We remark that the differential system (1.2) can be rewritten as follows

$$\dot{x}^i(t) := z^i(t),\ \dot{z}^i(t) = X^i\left(t, x(t), z(t), u(t)\right), \quad i = \overline{1,n}. \tag{1.2$'$}$$

Using the Lagrange function (*Lagrangian*),

$$L\left(t, x(t), \dot{x}(t), z(t), \dot{z}(t), u(t), p(t), q(t)\right) = X\left(t, x(t), z(t), u(t)\right)$$

$$+ p_i(t)\left[z^i(t) - \dot{x}^i(t)\right] + q_i(t)\left[X^i\left(t, x(t), z(t), u(t)\right) - \dot{z}^i(t)\right],$$

where $p(t) = (p_i(t))\,,\ q(t) = (q_i(t))\,,\ i = \overline{1,n}$, are called *co-state variables* or *Lagrange multipliers*, we build the *control Hamiltonian*,

$$H\left(t, x(t), z(t), u(t), p(t), q(t)\right) = X\left(t, x(t), z(t), u(t)\right) + p_i(t) z^i(t)$$

$$+ q_i(t) X^i\left(t, x(t), z(t), u(t)\right),$$

or, equivalently, $H = L + p_i \dot{x}^i + q_i \dot{z}^i$ (*modified Legendrian duality*).

1.1 Variational differential system and adjoint differential system

We start with the ODE system (1.2′), for a fixed control $u(t)$ and a corresponding solution $(x(t), z(t))$. Consider the differentiable variations $x(t, \varepsilon),\ z(t, \varepsilon)$, fulfilling

$$\dot{x}^i(t, \varepsilon) = z^i(t, \varepsilon)$$

$$\dot{z}^i(t, \varepsilon) = X^i\left(t, x(t, \varepsilon), z(t, \varepsilon), u(t)\right)$$

$$x(t, 0) = x(t), \quad z(t, 0) = z(t), \quad i = \overline{1,n}.$$

By a derivation with respect to ε, evaluating at $\varepsilon = 0$, we get the ODE system

$$\dot{y}^i(t) = v^i(t), \quad \dot{v}^i(t) = X^i_{x^j}\left(t, x(t), z(t), u(t)\right) y^j(t) + X^i_{z^j}\left(t, x(t), z(t), u(t)\right) v^j(t),$$

called *variational differential system*, where we used the notations $x^i_\varepsilon(t,0) := y^i(t)$, $z^i_\varepsilon(t,0) := v^i(t)$ (see $x^i_\varepsilon(t,0)$ as the derivative of $x^i(t,\varepsilon)$ with respect to ε, evaluated at $\varepsilon = 0$). The *matrix form* of the previous variational differential system is $\dot{W}(t) = A(t)W(t)$, where

$$W(t) := \begin{pmatrix} y^1(t) \\ y^2(t) \\ \vdots \\ y^n(t) \\ v^1(t) \\ v^2(t) \\ \vdots \\ v^n(t) \end{pmatrix}, \quad A(t) := \begin{pmatrix} O_n & I_n \\ \left(X^i_{x^j}\right) & \left(X^i_{\dot{x}^j}\right) \end{pmatrix}.$$

Denote $R(t) := \left[p_1(t)\ p_2(t)\ \cdots\ p_n(t)\ q_1(t)\ q_2(t)\ \cdots\ q_n(t)\right]^T$ (see M^T as the *transposed matrix* of M) the matrix of co-state variables. The following differential system

$$\dot{p}_j(t) = -X^i_{x^j}\left(t, x(t), \dot{x}(t), u(t)\right) q_i(t)$$

$$\dot{q}_j(t) = -p_j(t) - X^i_{\dot{x}^j}\left(t, x(t), \dot{x}(t), u(t)\right) q_i(t)$$

is called the *adjoint differential system* of the previous variational differential system because the scalar product $R^T(t)W(t)$ is a first integral of the two systems, i.e.,

$$\frac{d}{dt}\left[R^T(t)W(t)\right] = 0.$$

The *matrix form* of the previous adjoint differential system is $\dot{R}(t) = -A^T(t)R(t)$.

For another viewpoint regarding this subject, we address the reader to the works [1]-[5].

1.2 The optimal control problem solution: necessary conditions

The main result of Section 1 is represented by the following

Theorem 1.1. *(***Simplified single-time maximum principle based on second order ingredients***) Let $(x, \hat{u})$ be an optimal pair in* (1.1)*, subject to* (1.2) *and* (1.3)*. Then there exist a C^1-class co-state variable $p = (p_i)$, respectively a C^2-class co-state variable $q = (q_i)$, defined over $[0, t_0]$, such that*

$$\dot{x}^j(t) = \frac{\partial H}{\partial p_j}\left(t, x(t), \dot{x}(t), \hat{u}(t), p(t), q(t)\right) \tag{1.4}$$

$$\ddot{x}^j(t) = \frac{\partial H}{\partial q_j}\left(t, x(t), \dot{x}(t), \hat{u}(t), p(t), q(t)\right), \quad \forall t \in [0, t_0], \quad j = \overline{1, n}$$

$$x(0) = x_0, \quad \dot{x}(0) = \tilde{x}_0,$$

the functions $p = (p_i)$, $q = (q_i)$ *satisfy*

$$\dot{p}_j(t) = -H_{x^j}\left(t, x(t), \dot{x}(t), \hat{u}(t), p(t), q(t)\right), \quad p_j(t_0) = 0 \tag{1.5}$$

$$\dot{q}_j(t) = -H_{\dot{x}^j}\left(t, x(t), \dot{x}(t), \hat{u}(t), p(t), q(t)\right), \quad q_j(t_0) = 0,$$

the critical point conditions are

$$H_{u^\alpha}\left(t, x(t), \dot{x}(t), \hat{u}(t), p(t), q(t)\right) = 0, \quad \forall t \in [0, t_0], \quad \alpha = \overline{1, k} \tag{1.6}$$

and

$$\frac{\partial H}{\partial x^j}\left(t, x(t), \dot{x}(t), \hat{u}(t), p(t), q(t)\right)$$

$$-\frac{d}{dt}\left[\frac{\partial H}{\partial \dot{x}^j}\left(t, x(t), \dot{x}(t), \hat{u}(t), p(t), q(t)\right) - p_j(t)\right] + \frac{d^2}{dt^2}\left[-q_j(t)\right] = 0, \quad \forall t \in [0, t_0].$$

Proof. The adjective "simplified" means that the principle is obtained via techniques from Variational Calculus, under simplified hypothesis.

We use the Lagrangian L. The solutions of the foregoing optimization problem are among the solutions of the free maximization problem of the simple integral functional

$$J\left(u(\cdot)\right) = \int_0^{t_0} L\left(t, x(t), \dot{x}(t), z(t), \dot{z}(t), u(t), p(t), q(t)\right) dt,$$

with

$$u(t) \in \mathcal{U}, \quad p(t), q(t) \in \mathcal{P}, \quad \forall t \in [0, t_0]$$

$$x(0) = x_0, \quad x(t_0) = x_{t_0}, \quad \dot{x}(0) = \tilde{x}_0, \quad \dot{x}(t_0) = \tilde{x}_{t_0},$$

where the set $\mathcal{P}$ of co-state variables will be defined later.

Let us suppose that there exists a continuous control $\hat{u}(t)$ defined on the closed interval $[0, t_0]$, with $\hat{u}(t) \in \mathrm{Int}\mathcal{U}$, which is an optimum point of the previous problem. Consider a control variation, $u(t, \varepsilon) = \hat{u}(t) + \varepsilon h(t)$, where h is an arbitrary continuous vector function, and a state variation $x(t, \varepsilon)$, $t \in [0, t_0]$, related by

$$\ddot{x}^i(t, \varepsilon) = X^i\left(t, x(t, \varepsilon), \dot{x}(t, \varepsilon), u(t, \varepsilon)\right), \quad i = \overline{1, n}, \ \forall t \in [0, t_0],$$

with $x(0, \varepsilon) = x_0$, $\dot{x}(0, \varepsilon) = \tilde{x}_0$. Since $\hat{u}(t) \in \mathrm{Int}\mathcal{U}$ and a continuous function on a compact interval $[0, t_0]$ is bounded, there exists a value $\varepsilon_h > 0$ such that $u(t, \varepsilon) = \hat{u}(t) + \varepsilon h(t) \in \mathrm{Int}\mathcal{U}$, $\forall\, |\varepsilon| < \varepsilon_h$. This ε is used in our variational arguments.

For $|\varepsilon| < \varepsilon_h$, let consider the function (integral with parameter)

$$J(\varepsilon) = \int_0^{t_0} L\left(t, x(t, \varepsilon), \dot{x}(t, \varepsilon), z(t, \varepsilon), \dot{z}(t, \varepsilon), u(t, \varepsilon), p(t), q(t)\right) dt$$

$$= \int_0^{t_0} \left[H\left(t, x(t, \varepsilon), z(t, \varepsilon), u(t, \varepsilon), p(t), q(t)\right) - p_i(t)\dot{x}^i(t, \varepsilon) - q_i(t)\dot{z}^i(t, \varepsilon)\right] dt.$$

Assume that the co-state variables $p(t) = (p_i(t))$, $q(t) = (q_i(t))$ are of C^1-class. By derivation with respect to ε, evaluating at $\varepsilon = 0$, we obtain

$$J'(0) = \int_0^{t_0} \left[H_{x^j}\left(t, x(t), z(t), \hat{u}(t), p(t), q(t)\right) + \dot{p}_j(t)\right] x^j_\varepsilon(t, 0) dt$$

$$+\int_0^{t_0}\left[H_{z^j}\left(t,x(t),z(t),\hat{u}(t),p(t),q(t)\right)+\dot{q}_j(t)\right]z^j_\varepsilon(t,0)dt$$

$$+\int_0^{t_0}H_{u^\alpha}\left(t,x(t),z(t),\hat{u}(t),p(t),q(t)\right)h^\alpha(t)dt-\left[p_j(t)x^j_\varepsilon(t,0)+q_j(t)z^j_\varepsilon(t,0)\right]|_0^{t_0},$$

where $x(t)$ is the state variable corresponding to the optimal control $\hat{u}(t)$. We must have $J'(0)=0$ for any continuous vector function $h(t)=(h^\alpha(t))$. On the other hand, the functions $x^i_\varepsilon(t,0)$ and $z^i_\varepsilon(t,0)$ solve the Cauchy problem

$$\nabla_t x_\varepsilon(t,0)=z_\varepsilon(t,0)$$

$$\nabla_t z_\varepsilon(t,0)=X_x\left(t,x(t),z(t),u(t)\right)x_\varepsilon(t,0)+X_z\left(t,x(t),z(t),u(t)\right)z_\varepsilon(t,0)$$
$$+X_u\left(t,x(t),z(t),u(t)\right)h(t)$$
$$t\in[0,t_0],\quad x_\varepsilon(0,0)=0,\quad z_\varepsilon(0,0)=0$$

and consequently they depend on h. To eliminate this dependence, using the adjoint differential system, define the set $\mathcal{P}$ of co-state variables as the set of solutions of the following problem

$$\dot{p}_j(t)=-H_{x^j}\left(t,x(t),\dot{x}(t),\hat{u}(t),p(t),q(t)\right),\quad p_j(t_0)=0$$

$$\dot{q}_j(t)=-H_{\dot{x}^j}\left(t,x(t),\dot{x}(t),\hat{u}(t),p(t),q(t)\right),\quad q_j(t_0)=0.$$

We have

$$H_{u^\alpha}\left(t,x(t),\dot{x}(t),\hat{u}(t),p(t),q(t)\right)=0,\quad \forall t\in[0,t_0]$$

$$\frac{\partial H}{\partial x^j}\left(t,x(t),\dot{x}(t),\hat{u}(t),p(t),q(t)\right)$$

$$-\frac{d}{dt}\left[\frac{\partial H}{\partial \dot{x}^j}\left(t,x(t),\dot{x}(t),\hat{u}(t),p(t),q(t)\right)-p_j(t)\right]+\frac{d^2}{dt^2}\left[-q_j(t)\right]=0,\quad \forall t\in[0,t_0].$$

Moreover,

$$\dot{x}^j(t)=\frac{\partial H}{\partial p_j}\left(t,x(t),\dot{x}(t),\hat{u}(t),p(t),q(t)\right)$$

$$\ddot{x}^j(t)=\frac{\partial H}{\partial q_j}\left(t,x(t),\dot{x}(t),\hat{u}(t),p(t),q(t)\right),\quad \forall t\in[0,t_0]$$

$$x(0)=x_0,\quad \dot{x}(0)=\tilde{x}_0.$$

□

Remark 1.1. (i) The algebraic system (1.6),

$$H_{u^\alpha}\left(t,x(t),\dot{x}(t),u(t),p(t),q(t)\right)=0,\quad \forall t\in[0,t_0],$$

describes the critical points of the control Hamiltonian H with respect to the control vector $u=(u^\alpha)$.

(ii) The differential equations (1.6), (1.5) and (1.4) represent the Euler-Lagrange ODEs

$$\frac{\partial L}{\partial u^\alpha}-\frac{d}{dt}\frac{\partial L}{\partial u^{(1)\alpha}}=0,\quad \alpha=\overline{1,k}$$

$$\frac{\partial L}{\partial x^j} - \frac{d}{dt}\frac{\partial L}{\partial x^{(1)j}} = 0, \quad \frac{\partial L}{\partial z^j} - \frac{d}{dt}\frac{\partial L}{\partial z^{(1)j}} = 0, \quad j = \overline{1,n}$$
$$\frac{\partial L}{\partial p_j} - \frac{d}{dt}\frac{\partial L}{\partial p_j^{(1)}} = 0, \quad \frac{\partial L}{\partial q_j} - \frac{d}{dt}\frac{\partial L}{\partial q_j^{(1)}} = 0,$$

corresponding to the new Lagrangian L.

2 Single-time optimal control problem with higher order ODEs constraints

Next, we shall consider the general case when the constraints are higher order ODEs, that is, the case $k > 2$ with k an arbitrary fixed natural number. Also, we accept the notation: $f^{(k)i}(t) = f^{i(k)}(t)$.

Let be an optimal control problem based on a simple integral cost functional with higher order ODEs constraints:

$$\max_{u(\cdot),x_{t_0}} \left\{ I\left(u(\cdot)\right) = \int_0^{t_0} X\left(t, x(t), x^{(1)}(t), ..., x^{(k-1)}(t), u(t)\right) dt \right\} \tag{2.1}$$

subject to

$$x^{(k)i}(t) = X^i\left(t, x(t), x^{(1)}(t), ..., x^{(k-1)}(t), u(t)\right), \quad i = \overline{1,n} \tag{2.2}$$

$$u(t) \in \mathcal{U},\ \forall t \in [0, t_0]; \quad x^{(\gamma)}(0) = \tilde{x}_{\gamma 0},\ x^{(\gamma)}(t_0) = \tilde{x}_{\gamma t_0},\ \gamma = \overline{0, k-1}. \tag{2.3}$$

As in the previous section, $t \in [0, t_0] \subset \mathbb{R}_+$ is a parameter of evolution, or a *single-time*; $[0, t_0] \subset \mathbb{R}_+$ is the *time interval*; $x(t) = (x^i(t))$, $i = \overline{1,n}$, is a C^{k+1}-class function, called *state vector*; $x^{(\beta)}(t)$, $\beta = \overline{1,k}$, is the derivative of order β of the state variable $x(t)$; $u(t) = (u^\alpha(t))$, $\alpha = \overline{1,m}$, is a *continuous control vector*; the *running cost* $X\left(t, x(t), x^{(1)}(t), ..., x^{(k-1)}(t), u(t)\right)$ is a C^1-class function, called *non-autonomous Lagrangian*.

Rewrite the differential system (2.2) using the following auxiliary variables $y_1^i(t) := x^i(t)$, $y_2^i(t) := x^{(1)i}(t)$, $\dots$, $y_{k-1}^i(t) := x^{(k-2)i}(t)$, $y_k^i(t) := x^{(k-1)i}(t)$, or, equivalently,

$$\begin{aligned} \dot{y}_1^i(t) &:= y_2^i(t) \\ \dot{y}_2^i(t) &:= y_3^i(t) \\ &\vdots \\ \dot{y}_{k-1}^i(t) &:= y_k^i(t) \\ \dot{y}_k^i(t) &:= X^i\left(t, y_1(t), ..., y_k(t), u(t)\right). \end{aligned} \tag{2.2'}$$

The *matrix form* of the previous differential system is $\dot{Y}(t) = AY(t) + W(t)$, where $Y(t) := [y_1(t)\ y_2(t)\ \cdots\ y_k(t)]^T$ (see M^T as the *transposed matrix* of M; also, see $O_{p,q}$ as the $(p \times q)$ *null matrix* and I_p as the *unit (identity) matrix* of order p) and

$$W(t) := \begin{pmatrix} O_{k-1,1} \\ X\left(t, y_1(t), ..., y_k(t), u(t)\right) \end{pmatrix}, \quad A := \begin{pmatrix} O_{k-1,1} & I_{k-1} \\ \\ 0 & O_{1,k-1} \end{pmatrix}.$$

Build the Lagrange function

$$L\left(t, y_1(t), y_2(t), \dots, y_k(t), \dot{y}_1(t), \dot{y}_2(t), \dots, \dot{y}_k(t), u(t), p^1(t), p^2(t), ..., p^k(t)\right)$$
$$= X\left(t, y_1(t), y_2(t), \dots, y_k(t), u(t)\right) + p_i^1(t)\left[y_2^i(t) - \dot{y}_1^i(t)\right]$$
$$+ \dots + \; p_i^k(t)\left[X^i\left(t, y_1(t), y_2(t), \dots, y_k(t), u(t)\right) - \dot{y}_k^i(t)\right]$$

(each expression $k \to p_i^k(t)\dot{y}_k^i(t)$, indexed after k, contains summation only upon i) that changes the initial optimal control problem (with higher order ODEs constraints) into the following problem

$$\max_{u(\cdot), x_{t_0}} \int_0^{t_0} L\left(t, Y^T(t), \dot{Y}^T(t), u(t), p^1(t), p^2(t), ..., p^k(t)\right) dt$$

subject to

$$u(t) \in \mathcal{U}, \quad \{p^1(t), ..., p^k(t)\} \subseteq \mathcal{P}, \quad \forall t \in [0, t_0]$$
$$x^{(\gamma)}(0) = \tilde{x}_{\gamma 0}, \quad x^{(\gamma)}(t_0) = \tilde{x}_{\gamma t_0}, \quad \gamma = \overline{0, k-1},$$

where the set $\mathcal{P}$ of co-state variables will be defined later. Using the *control Hamiltonian*,

$$H\left(t, Y^T(t), u(t), p^1(t), p^2(t), ..., p^k(t)\right)$$
$$= X\left(t, Y^T(t), u(t)\right) + p_i^1(t)y_2^i(t) + p_i^2(t)y_3^i(t)$$
$$+ ... + \; p_i^{k-1}(t)y_k^i(t) + p_i^k(t)X^i\left(t, Y^T(t), u(t)\right),$$

or, equivalently,

$$H = L + p_i^1\dot{y}_1^i + p_i^2\dot{y}_2^i + ... + p_i^k\dot{y}_k^i,$$

(*modified higher order Legendrian duality*) we can rewrite the previous problem as

$$\max_{u(\cdot), x_{t_0}} \left\{ \int_0^{t_0} \left[H\left(t, Y^T(t), u(t), p^1(t), p^2(t), ..., p^k(t)\right)\right] dt\right.$$
$$\left. - \int_0^{t_0} \left[p_i^1(t)\dot{y}_1^i(t) + p_i^2(t)\dot{y}_2^i(t) + ... + p_i^k(t)\dot{y}_k^i(t)\right] dt\right\}$$

subject to

$$u(t) \in \mathcal{U}, \quad \{p^1(t), ..., p^k(t)\} \subseteq \mathcal{P}, \quad \forall t \in [0, t_0]$$
$$x^{(\gamma)}(0) = \tilde{x}_{\gamma 0}, \quad x^{(\gamma)}(t_0) = \tilde{x}_{\gamma t_0}, \quad \gamma = \overline{0, k-1}.$$

2.1 Variational differential system and adjoint differential system

We consider the ODE system (2.2′) with a fixed control $u(t)$ and the corresponding solution $(y_1(t), y_2(t), ..., y_k(t))$. Consider the differentiable variations $\{y_1(t,\varepsilon), y_2(t,\varepsilon), ..., y_k(t,\varepsilon)\}$, fulfilling $\dot{y}_1^i(t,\varepsilon) = y_2^i(t,\varepsilon)$, $\dot{y}_2^i(t,\varepsilon) = y_3^i(t,\varepsilon), ...,$ $\dot{y}_{k-1}^i(t,\varepsilon) = y_k^i(t,\varepsilon)$, $\dot{y}_k^i(t,\varepsilon) = X^i\left(t, y_1(t,\varepsilon), ..., y_k(t,\varepsilon), u(t)\right)$, $y_\beta(t,0) = y_\beta(t)$, $\beta = \overline{1,k}$. Let denote $y^i_{\beta,\varepsilon}(t,0) := v^i_\beta(t)$, $\beta = \overline{1,k}$, that is the derivative of $y^i_\beta(t,\varepsilon)$ with respect to ε, evaluated at $\varepsilon = 0$. By a derivation with respect to ε, evaluating at $\varepsilon = 0$, we get

$$\begin{gathered}
\dot{v}_1^i(t) = v_2^i(t)\\
\dot{v}_2^i(t) = v_3^i(t)\\
\vdots\\
\dot{v}_{k-1}^i(t) = v_k^i(t)\\
\dot{v}_k^i(t) = X^i_{y_1^j}\left(t, y_1(t), ..., y_k(t), u(t)\right) v_1^j(t) + ... + X^i_{y_k^j}\left(t, y_1(t), ..., y_k(t), u(t)\right) v_k^j(t),
\end{gathered}$$

called *variational differential system.*

The *matrix form* of the previous variational differential system is $\dot{V}(t) = B(t)V(t)$ (see $v_\zeta(t) = \left[v_\zeta^1(t)\ v_\zeta^2(t)\ \cdots\ v_\zeta^n(t)\right]^T$, $\zeta = \overline{1,k}$), where

$$V(t) := \begin{pmatrix} v_1(t)\\ v_2(t)\\ \vdots\\ v_k(t)\end{pmatrix}, \quad B(t) := \begin{pmatrix} O_n & I_n & O_n & ... & O_n\\ O_n & O_n & I_n & ... & O_n\\ \vdots & \vdots & \vdots & ... & \vdots\\ O_n & O_n & O_n & ... & I_n\\ X_{y_1} & X_{y_2} & X_{y_3} & ... & X_{y_k}\end{pmatrix}.$$

Denote $R(t) := \left[p^1(t)\ p^2(t)\ \cdots\ p^k(t)\right]^T$ the matrix of co-state variables. The following differential system

$$\begin{gathered}
\dot{p}_j^1(t) = -X^i_{x^j}\left(t, x(t), x^{(1)}(t), ..., x^{(k-1)}(t), u(t)\right) p_i^1(t)\\
\dot{p}_j^2(t) = -p_j^1(t) - X^i_{x^{(1)j}}\left(t, x(t), x^{(1)}(t), ..., x^{(k-1)}(t), u(t)\right) p_i^2(t)\\
\vdots\\
\dot{p}_j^k(t) = -p_j^{k-1}(t) - X^i_{x^{(k-1)j}}\left(t, x(t), x^{(1)}(t), ..., x^{(k-1)}(t), u(t)\right) p_i^k(t)
\end{gathered}$$

is called the *adjoint differential system* of the previous variational differential system because the scalar product $R^T(t)V(t)$ is a first integral of the two systems, i.e.,

$$\frac{d}{dt}\left[R^T(t)V(t)\right] = 0.$$

The *matrix form* of the previous adjoint differential system is $\dot{R}(t) = -B^T(t)R(t)$.

2.2 Necessary conditions of optimality

Assume there exists a continuous control vector $\hat{u}(t)$ defined on the closed interval $[0, t_0]$, with $\hat{u}(t) \in \mathrm{Int}\mathcal{U}$, which is an optimal solution for our problem. Let take a variation of the optimal control vector, $u(t,\varepsilon) = \hat{u}(t) + \varepsilon h(t)$, where $h = (h^\alpha(t))$, $\alpha = \overline{1, m}$, is an arbitrary continuous vector function. Since $\hat{u}(t) \in \mathrm{Int}\mathcal{U}$ and a continuous function on a compact interval $[0, t_0]$ is bounded, there exists $\varepsilon_h > 0$ such that $u(t,\varepsilon) = \hat{u}(t) + \varepsilon h(t) \in \mathrm{Int}\mathcal{U}$, $\forall\, |\varepsilon| < \varepsilon_h$. This ε is used in our variational arguments.

Consider $x(t,\varepsilon)$ as the state vector corresponding to the control vector $u(t,\varepsilon)$, i.e.,

$$x^{(k)i}(t,\varepsilon) = X^i\left(t, x(t,\varepsilon), x^{(1)}(t,\varepsilon), ..., x^{(k-1)}(t,\varepsilon), u(t,\varepsilon)\right)$$

$$i = \overline{1, n}, \quad \forall t \in [0, t_0]$$

and $x^{(\gamma)}(0,\varepsilon) = \tilde{x}_{\gamma 0}$, $\gamma = \overline{0, k-1}$. For $|\varepsilon| < \varepsilon_h$, let define the function (integral with parameter)

$$I(\varepsilon) := \int_0^{t_0} X\left(t, Y^T(t,\varepsilon), u(t,\varepsilon)\right) dt.$$

Also, the continuous control vector $\hat{u}(t)$ must be an optimal control vector. Therefore, we obtain $I(0) \geq I(\varepsilon)$, $\forall\, |\varepsilon| < \varepsilon_h$. We have

$$\int_0^{t_0} p_i^1(t)\left[y_2^i(t,\varepsilon) - \dot{y}_1^i(t,\varepsilon)\right] dt = 0$$

$$\int_0^{t_0} p_i^2(t)\left[y_3^i(t,\varepsilon) - \dot{y}_2^i(t,\varepsilon)\right] dt = 0$$

$$\vdots$$

$$\int_0^{t_0} p_i^k(t)\left[X^i\left(t, Y^T(t,\varepsilon), u(t,\varepsilon)\right) - \dot{y}_k^i(t,\varepsilon)\right] dt = 0,$$

for any continuous vector functions $p^1 = (p_i^1), ..., p^k = (p_i^k) : [0, t_0] \to R^n$. Necessarily, we must use the Lagrange function with variations

$$L\left(t, Y^T(t,\varepsilon), \dot{Y}^T(t,\varepsilon), u(t,\varepsilon), p^1(t), p^2(t), ..., p^k(t)\right)$$

$$= X\left(t, Y^T(t,\varepsilon), u(t,\varepsilon)\right) + p_i^1(t)\left[y_2^i(t,\varepsilon) - \dot{y}_1^i(t,\varepsilon)\right]$$
$$+ ... + \; p_i^k(t)\left[X^i\left(t, Y^T(t,\varepsilon), u(t,\varepsilon)\right) - \dot{y}_k^i(t,\varepsilon)\right]$$

and the associated function (integral with parameter)

$$I(\varepsilon) = \int_0^{t_0} L\left(t, Y^T(t,\varepsilon), \dot{Y}^T(t,\varepsilon), u(t,\varepsilon), p^1(t), ..., p^k(t)\right) dt.$$

Suppose that the co-state variables $\{p^1 = (p_i^1), ..., p^k = (p_i^k)\}$ are of C^1-class. Introduce the corresponding control Hamiltonian with variations

$$H\left(t, Y^T(t,\varepsilon), u(t,\varepsilon), p^1(t), p^2(t), ..., p^k(t)\right)$$

$$= X\left(t, Y^T(t,\varepsilon), u(t,\varepsilon)\right) + p_i^1(t)y_2^i(t,\varepsilon) + p_i^2(t)y_3^i(t,\varepsilon)$$
$$+ ... +\ p_i^{k-1}(t)y_k^i(t,\varepsilon) + p_i^k(t)X^i\left(t, Y^T(t,\varepsilon), u(t,\varepsilon)\right).$$

The previous integral with parameter can be rewritten as follows

$$I(\varepsilon) = \int_0^{t_0} H\left(t, y_1(t,\varepsilon), y_2(t,\varepsilon), ..., y_k(t,\varepsilon), u(t,\varepsilon), p^1(t), ..., p^k(t)\right) dt$$
$$- \int_0^{t_0} \left[p_j^1(t)\dot{y}_1^j(t,\varepsilon) + p_j^2(t)\dot{y}_2^j(t,\varepsilon) + ... + p_j^k(t)\dot{y}_k^j(t,\varepsilon)\right] dt,$$

or (using the formula of integration by parts),

$$I(\varepsilon) = \int_0^{t_0} H\left(t, y_1(t,\varepsilon), y_2(t,\varepsilon), ..., y_k(t,\varepsilon), u(t,\varepsilon), p^1(t), ..., p^k(t)\right) dt$$
$$+ \int_0^{t_0} \left[\dot{p}_j^1(t)y_1^j(t,\varepsilon) + ... + \dot{p}_j^k(t)y_k^j(t,\varepsilon)\right] dt$$
$$- \left[p_j^1(t)y_1^j(t,\varepsilon) + ... + p_j^k(t)y_k^j(t,\varepsilon)\right] |_0^{t_0}.$$

By derivation with respect to ε, evaluating at $\varepsilon = 0$, we find

$$I'(0) = \int_0^{t_0} \left[H_{y_1^j}\left(t, y_1(t), ..., y_k(t), \hat{u}(t), p(t)\right) + \dot{p}_j^1(t)\right] y_{1,\varepsilon}^j(t,0)dt$$
$$+ \int_0^{t_0} \left[H_{y_2^j}\left(t, y_1(t), ..., y_k(t), \hat{u}(t), p(t)\right) + \dot{p}_j^2(t)\right] y_{2,\varepsilon}^j(t,0)dt$$
$$\vdots$$
$$+ \int_0^{t_0} \left[H_{y_k^j}\left(t, y_1(t), ..., y_k(t), \hat{u}(t), p(t)\right) + \dot{p}_j^k(t)\right] y_{k,\varepsilon}^j(t,0)dt$$
$$+ \int_0^{t_0} H_{u^\alpha}\left(t, y_1(t), ..., y_k(t), \hat{u}(t), p(t)\right) h^\alpha(t)dt$$
$$- \left[p_j^1(t)y_{1,\varepsilon}^j(t,0) + ... + p_j^k(t)y_{k,\varepsilon}^j(t,0)\right] |_0^{t_0},$$

where $x(t)$ is the state variable corresponding to the optimal control $\hat{u}(t)$ (see $p(t) := \{p^1(t), ..., p^k(t)\}$). We must have $I'(0) = 0$ for any continuous vector function $h(t) = (h^\alpha(t))$. Also, the functions $\{y_{1,\varepsilon}^i(t,0), ..., y_{k,\varepsilon}^i(t,0)\}$ solve the Cauchy problem

$$\nabla_t y_{\beta,\varepsilon}(t,0) = y_{\beta+1,\varepsilon}(t,0), \quad \beta = \overline{1, k-1}$$

$$\nabla_t y_{k,\varepsilon}(t,0) = X_{y_1}\left(t, y_1(t), y_2(t), ..., y_k(t), u(t)\right) y_{1,\varepsilon}(t,0)$$
$$+ ... +\ X_{y_k}\left(t, y_1(t), y_2(t), ..., y_k(t), u(t)\right) y_{k,\varepsilon}(t,0)$$
$$+ X_u\left(t, y_1(t), y_2(t), ..., y_k(t), u(t)\right) h(t), \quad \beta = k$$
$$t \in [0, t_0], \quad y_{\beta,\varepsilon}(0,0) = 0, \quad \forall \beta = \overline{1, k}.$$

Consequently, they are dependent on h. To eliminate this dependence, we use the adjoint differential system in the previous section, i.e., we consider the set $\mathcal{P}$ of co-state variables as the set of solutions for the following problem

$$\begin{aligned}
&p_j^{(1)1}(t) = -H_{x^j}\left(t, x(t), ..., x^{(k-1)}(t), \hat{u}(t), p(t)\right), \quad p_j^1(t_0) = 0 \\
&p_j^{(1)2}(t) = -H_{x^{(1)j}}\left(t, x(t), ..., x^{(k-1)}(t), \hat{u}(t), p(t)\right), \quad p_j^2(t_0) = 0 \\
&\vdots \\
&p_j^{(1)k}(t) = -H_{x^{(k-1)j}}\left(t, x(t), ..., x^{(k-1)}(t), \hat{u}(t), p(t)\right), \quad p_j^k(t_0) = 0.
\end{aligned} \tag{2.4}$$

We have

$$H_{u^\alpha}\left(t, x(t), ..., x^{(k-1)}(t), \hat{u}(t), p(t)\right) = 0, \quad \forall t \in [0, t_0] \tag{2.5}$$

$$\begin{aligned}
&\frac{\partial H}{\partial x^i}\left(t, x(t), ..., x^{(k-1)}(t), \hat{u}(t), p(t)\right) \\
&- \frac{d}{dt}\left[\frac{\partial H}{\partial x^{(1)i}}\left(t, x(t), ..., x^{(k-1)}(t), \hat{u}(t), p(t)\right) - p_i^1(t)\right] \\
&+ \frac{d^2}{dt^2}\left[\frac{\partial H}{\partial x^{(2)i}}\left(t, x(t), ..., x^{(k-1)}(t), \hat{u}(t), p(t)\right) - p_i^2(t)\right] \\
&- ... + (-1)^{k-1}\frac{d^{k-1}}{dt^{k-1}}\left[\frac{\partial H}{\partial x^{(k-1)i}}\left(t, x(t), ..., x^{(k-1)}(t), \hat{u}(t), p(t)\right) - p_i^{k-1}(t)\right] \\
&+ (-1)^k \frac{d^k}{dt^k}\left[-p_i^k(t)\right] = 0, \quad \forall t \in [0, t_0].
\end{aligned}$$

Moreover,

$$x^{(\beta)j}(t) = \frac{\partial H}{\partial p_j^\beta}\left(t, x(t), ..., x^{(k-1)}(t), \hat{u}(t), p(t)\right), \quad \beta = \overline{1, k}, \ \forall t \in [0, t_0] \tag{2.6}$$

$$x^{(\gamma)}(0) = \tilde{x}_{\gamma 0}, \quad \gamma = \overline{0, k-1}.$$

Remark 2.1. (i) The algebraic system

$$H_{u^\alpha}\left(t, x(t), ..., x^{(k-1)}(t), \hat{u}(t), p(t)\right) = 0, \quad \forall t \in [0, t_0]$$

describes the critical points of the control Hamiltonian H with respect to the control vector $u = (u^\alpha)$.

(ii) We can obtain the result via the Euler-Lagrange ODEs

$$\frac{\partial L}{\partial u^\alpha} - \frac{d}{dt}\frac{\partial L}{\partial u^{(1)\alpha}} + \frac{d^2}{dt^2}\frac{\partial L}{\partial u^{(2)\alpha}} - ... + (-1)^k \frac{d^k}{dt^k}\frac{\partial L}{\partial u^{(k)\alpha}} = 0, \quad \alpha = \overline{1, m}$$

$$\frac{\partial L}{\partial x^i} - \frac{d}{dt}\frac{\partial L}{\partial x^{(1)i}} + \frac{d^2}{dt^2}\frac{\partial L}{\partial x^{(2)i}} - ... + (-1)^k \frac{d^k}{dt^k}\frac{\partial L}{\partial x^{(k)i}} = 0, \quad i = \overline{1, n}$$

$$\frac{\partial L}{\partial p_j^\beta} - \frac{d}{dt}\frac{\partial L}{\partial p_j^{(1)\beta}} + \frac{d^2}{dt^2}\frac{\partial L}{\partial p_j^{(2)\beta}} - ... + (-1)^k \frac{d^k}{dt^k}\frac{\partial L}{\partial p_j^{(k)\beta}} = 0, \quad \beta = \overline{1, k}, \ j = \overline{1, n},$$

where L is a suitable Lagrangian.

In summary, we get the *simplified single-time Pontryagin maximum principle*, a result that gives us only necessary conditions for the optimal point $u = (u^\alpha)$. The adjective "simplified" means that the principle is obtained via techniques from Variational Calculus, under simplified hypothesis.

Theorem 2.1. *(***Simplified single-time maximum principle based on higher order ingredients***) Assume that the problem of maximizing the functional* (2.1)*, subject to the higher order ODEs constraints* (2.2) *and to the conditions* (2.3)*, with* X*,* X^i *of* C^1*-class, has an interior solution* $\hat{u}(t) \in Int\mathcal{U}$ *which determines the optimal state vector* $x(t) = \left(x^i(t)\right)$*. Then there exist the* C^β*-class co-state variables,* $p^\beta = \left(p_j^\beta\right)$*,* $\beta = \overline{1,k}$*, defined over* $[0, t_0]$*, such that the relations* (2.4)*,* (2.5)*,* (2.6) *hold.*

3 Euler-Lagrange and Hamilton ODEs via single-time Pontryagin maximum principle

To get the (higher order) Euler-Lagrange and Hamilton ODEs from the single-time Pontryagin maximum principle, based on higher order ingredients, let consider the following simple integral cost functional

$$\max_{u(\cdot), x_{t_0}} \left\{ I\left(u(\cdot)\right) = \int_0^{t_0} X\left(x(t), x^{(1)}(t), ..., x^{(k-1)}(t), u(t)\right) dt \right\}$$

subject to

$$x^{(k)i}(t) = u_k^i(t), \quad i = \overline{1,n}, \quad k \geq 2 \; (fixed\ natural\ number)$$
$$t \in [0, t_0] \subset \mathbb{R}_+, \quad x^{(\gamma)}(0) = \tilde{x}_{\gamma 0}, \quad \gamma = \overline{0, k-1}.$$

Here, the running cost $X\left(x(t), x^{(1)}(t), ..., x^{(k-1)}(t), u(t)\right)$ is a C^1-class *autonomous Lagrangian* and the control matrix $u(t) = \left(u_k^i(t)\right)$.

For solving the problem we need the control Hamiltonian,

$$H\left(x(t), x^{(1)}(t), ..., x^{(k-1)}(t), u(t), p^1(t), p^2(t), ..., p^k(t)\right)$$
$$= X\left(x(t), x^{(1)}(t), ..., x^{(k-1)}(t), u(t)\right) + p_i^1(t) y_2^i(t) + p_i^2(t) y_3^i(t)$$
$$+ ... + \; p_i^{k-1}(t) y_k^i(t) + p_i^k(t) u_k^i(t),$$

where $\{y_1(t), ..., y_k(t)\}$ are *auxiliary variables* defined as $y_1^i(t) := x^i(t)$, $y_2^i(t) := x^{(1)i}(t)$, ..., $y_{k-1}^i(t) := x^{(k-2)i}(t)$, $y_k^i(t) := x^{(k-1)i}(t)$. Using the relations

$$H_{x^{(\eta)i}}\left(x(t), ..., x^{(k-1)}(t), u(t), p^1(t), ..., p^k(t)\right) = -p_i^{(1)\eta+1}(t), \quad \eta = \overline{0, k-1}$$

$$H_{u_k^i}\left(x(t), ..., x^{(k-1)}(t), u(t), p^1(t), ..., p^k(t)\right) = 0,$$

obtained from (2.4) and (2.5), and

$$H_{x^i}\left(x(t), x^{(1)}(t), ..., x^{(k-1)}(t), u(t), p^1(t), p^2(t), ..., p^k(t)\right) \tag{3.1}$$

$$= X_{x^i}\left(x(t), x^{(1)}(t), ..., x^{(k-1)}(t), u(t)\right);$$

$$H_{x^{(\eta)i}}\left(x(t), x^{(1)}(t), ..., x^{(k-1)}(t), u(t), p^1(t), p^2(t), ..., p^k(t)\right)$$

$$= X_{x^{(\eta)i}}\left(x(t), x^{(1)}(t), ..., x^{(k-1)}(t), u(t)\right) + p_i^\eta(t), \quad \eta = \overline{1, k-1};$$

$$H_{u_k^i}\left(x(t), x^{(1)}(t), ..., x^{(k-1)}(t), u(t), p^1(t), p^2(t), ..., p^k(t)\right)$$

$$= H_{x^{(k)i}}\left(x(t), x^{(1)}(t), ..., x^{(k-1)}(t), u(t), p^1(t), p^2(t), ..., p^k(t)\right)$$

$$= X_{u_k^i}\left(x(t), x^{(1)}(t), ..., x^{(k-1)}(t), u(t)\right) + p_i^k(t)$$

$$= X_{x^{(k)i}}\left(x(t), x^{(1)}(t), ..., x^{(k-1)}(t), u(t)\right) + p_i^k(t),$$

obtained from (3.1), we have the following relations

$$X_{x^i}\left(x(t), x^{(1)}(t), ..., x^{(k-1)}(t), u(t)\right) + p_i^{(1)1}(t) = 0 \tag{3.2}$$

$$X_{x^{(\eta)i}}\left(x(t), x^{(1)}(t), ..., x^{(k-1)}(t), u(t)\right) + p_i^\eta(t) + p_i^{(1)\eta+1}(t) = 0, \quad \eta = \overline{1, k-1}$$

$$X_{x^{(k)i}}\left(x(t), x^{(1)}(t), ..., x^{(k-1)}(t), u(t)\right) + p_i^k(t) = 0.$$

Assume that the running cost $X\left(x(t), x^{(1)}(t), ..., x^{(k-1)}(t), u(t)\right)$ is a C^{k+1}-class function. Consequently, by a direct computation (a simple substitution of terms) at (3.3), we get the *higher order Euler-Lagrange ODEs*

$$\frac{\partial X}{\partial x^i} - \frac{d}{dt}\frac{\partial X}{\partial x^{(1)i}} + \frac{d^2}{dt^2}\frac{\partial X}{\partial x^{(2)i}} - ... + (-1)^k \frac{d^k}{dt^k}\frac{\partial X}{\partial x^{(k)i}} = 0, \quad i = \overline{1, n}.$$

Let $u(t) = \left(u_k^i(t)\right)$ be an optimal control vector, $x(t) = \left(x^i(t)\right)$ the optimal evolution, and $\{p^1 = (p_i^1), ..., p^k = (p_i^k)\}$ the solution for

$$p_i^{(1)\eta+1}(t) = -H_{x^{(\eta)i}}\left(x(t), ..., x^{(k-1)}(t), u(t), p^1(t), ..., p^k(t)\right), \quad \eta = \overline{0, k-1},$$

(see (2.4)) corresponding to $u(t)$ and $x(t)$. The critical point equations,

$$H_{u_k^i}\left(x(t), x^{(1)}(t), ..., x^{(k-1)}(t), u(t), p^1(t), p^2(t), ..., p^k(t)\right) \tag{3.3}$$

$$= X_{u_k^i}\left(x(t), x^{(1)}(t), ..., x^{(k-1)}(t), u(t)\right) + p_i^k(t) = 0, \quad i = \overline{1, n},$$

define the co-state variable $p^k(t) = \left(p_i^k(t)\right)$ as a *non-standard (modified) moment*. Let suppose that (3.4) has a unique solution

$$u_k^i(t) = u_k^i\left(x(t), x^{(1)}(t), ..., x^{(k-1)}(t), p^1(t), p^2(t), ..., p^k(t)\right) = x^{(k)i}(t).$$

By a direct computation (see (3.1) and $u^i_k(t) = x^{(k)i}(t)$), we get *the first part of the higher order single-time Hamilton ODEs*

$$x^{(\beta)i}(t) = \frac{\partial H}{\partial p^\beta_i}\left(x(t), x^{(1)}(t), ..., x^{(k-1)}(t), u(t), p^1(t), ..., p^k(t)\right), \quad \beta = \overline{1,k}. \tag{3.4}$$

We have (see (2.5))

$$\begin{aligned} &\frac{\partial H}{\partial x^i}\left(x(t), ..., x^{(k-1)}(t), u(t), p^1(t), ..., p^k(t)\right) \\ &- \frac{d}{dt}\left[\frac{\partial H}{\partial x^{(1)i}}\left(x(t), ..., x^{(k-1)}(t), u(t), p^1(t), ..., p^k(t)\right) - p^1_i(t)\right] \\ &+ \frac{d^2}{dt^2}\left[\frac{\partial H}{\partial x^{(2)i}}\left(x(t), ..., x^{(k-1)}(t), u(t), p^1(t), ..., p^k(t)\right) - p^2_i(t)\right] \\ -... + \ &(-1)^{k-1}\frac{d^{k-1}}{dt^{k-1}}\left[\frac{\partial H}{\partial x^{(k-1)i}}\left(x(t), ..., x^{(k-1)}(t), u(t), p^1(t), ..., p^k(t)\right) - p^{k-1}_i(t)\right] \\ &+ (-1)^k \frac{d^k}{dt^k}\left[-p^k_i(t)\right] = 0, \quad \forall t \in [0, t_0]. \end{aligned}$$

Knowing that the running cost $X\left(x(t), x^{(1)}(t), ..., x^{(k-1)}(t), u(t)\right)$ satisfies the higher order single-time Euler-Lagrange ODEs in the previous and taking $\tilde{p}^\eta_i := -X_{x^{(\eta)i}} = p^\eta_i - H_{x^{(\eta)i}}$, $\eta = \overline{1, k-1}$, and $\tilde{p}^k_i := p^k_i$, we get *the second part of the higher order single-time Hamilton ODEs*

$$\sum_{\beta=1}^{k}(-1)^{\beta+1}\frac{d^\beta}{dt^\beta}\tilde{p}^\beta_i(t) = -\frac{\partial H}{\partial x^i}\left(x(t), ..., x^{(k-1)}(t), u(t), p^1(t), ..., p^k(t)\right). \tag{3.5}$$

4 Conclusion and further development

In this work we introduced and studied single-time optimal control problems which involve higher order ODEs constraints. Reducing the constraints to first order differential equations, employing variational and adjoint differential systems, we have derived necessary conditions of optimality for our optimization problems (see Theorems 1.1 and 2.1). Of course, we can work directly with a constraint as ODE of order k, but then it just uses a single Lagrange multiplier with its derivatives of order k.

Section 3 is dedicated to Euler-Lagrange and Hamilton ODEs, of superior order, via single-time Pontryagin maximum principle based on higher order ODEs constraints.

The main results of this research paper are original and they complement previously known results. Further, we shall direct our research to the development of the multitime case for similar problems (see [13]-[21]).

For other different but connected viewpoints to this subject, the reader is addressed to the research papers [1], [2] and [12], [13].

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