On concircularly recurrent Finsler manifolds

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Abstract. Two special Finsler spaces have been introduced and investigated, namely R^h-recurrent Finsler space and concircularly recurrent Finsler space. The defining properties of these spaces are formulated in terms of the first curvature tensor of Cartan connection. The following three results constitute the main object of the present paper: (i) a concircularly flat Finsler manifold is necessarily of constant curvature (Theorem A); (ii) every R^h-recurrent Finsler manifold is concircularly recurrent with the same recurrence form (Theorem B); (iii) every horizontally integrable concircularly recurrent Finsler manifold is R^h-recurrent with the same recurrence form (Theorem C). The whole work is formulated in a coordinate-free form.

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1 Introduction

In this paper, we present an intrinsic investigation of concircularly recurrent Finsler manifolds. The paper is organized in the following manner.

In section 1, following the introduction, we give a brief account of the basic concepts, definitions and results that will be needed in the sequel.

In section 2, an important tensor field associated to a Finsler manifold, called the concircular curvature tensor, is defined. A necessary and sufficient condition for the vanishing of the concircular curvature tensor is found (Proposition 2.4). We also prove that a concircularly flat Finsler manifold is necessarily of constant curvature (Theorem A).

In section 3, two special Finsler spaces have been introduced and investigated, namely R^h-recurrent Finsler space and concircularly recurrent Finsler space. The defining properties of these spaces are formulated in terms of the first curvature tensor of Cartan connection. Then, we prove that every R^h-recurrent Finsler manifold is concircularly recurrent with the same recurrence form (Theorems B). The converse of

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the above result is not true in general. However, it has been recently proved to be true in Riemannian geometry [4]. For the converse of Theorem B to be true in the Finslerian context, an additional condition is needed, namely the horizontal integrability condition. We thus prove that every horizontally integrable concircularly recurrent Finsler manifold is R^h-recurrent with the same recurrence form (Theorem C). This is the third and most important result of the paper.

Finally, it should be pointed out that the present work is formulated in a coordinate-free form.

2 Notation and preliminaries

In this section, we give a brief account of the basic concepts of the pullback approach to intrinsic Finsler geometry necessary for this work. For more details, we refer to [1], [6] and [7]. We shall use the same notations of [6].

In what follows, we denote by $\pi : \mathcal{T}M \longrightarrow M$ the subbundle of nonzero vectors tangent to M, $\mathfrak{F}(TM)$ the algebra of C^∞ functions on TM, $\mathfrak{X}(\pi(M))$ the $\mathfrak{F}(TM)$-module of differentiable sections of the pullback bundle $\pi^{-1}(TM)$. The elements of $\mathfrak{X}(\pi(M))$ will be called π-vector fields and will be denoted by barred letters $\overline{X}$. The tensor fields on $\pi^{-1}(TM)$ will be called π-tensor fields. The fundamental π-vector field is the π-vector field $\overline{\eta}$ defined by $\overline{\eta}(u) = (u, u)$ for all $u \in TM$.

We have the following short exact sequence of vector bundles

$$0 \longrightarrow \pi^{-1}(TM) \xrightarrow{\gamma} T(\mathcal{T}M) \xrightarrow{\rho} \pi^{-1}(TM) \longrightarrow 0,$$

with the well known definitions of the bundle morphisms ρ and γ. The vector space $V_u(\mathcal{T}M) = \{X \in T_u(\mathcal{T}M) : d\pi(X) = 0\}$ is the vertical space to M at u.

Let D be a linear connection on the pullback bundle $\pi^{-1}(TM)$. We associate with D the map $K : T\mathcal{T}M \longrightarrow \pi^{-1}(TM) : X \longmapsto D_X\overline{\eta}$, called the connection map of D. The vector space $H_u(\mathcal{T}M) = \{X \in T_u(\mathcal{T}M) : K(X) = 0\}$ is called the horizontal space to M at u . The connection D is said to be regular if

$$T_u(\mathcal{T}M) = V_u(\mathcal{T}M) \oplus H_u(\mathcal{T}M) \ \ \forall\, u \in \mathcal{T}M.$$

If M is endowed with a regular connection, then the vector bundle maps γ, $\rho|_{H(\mathcal{T}M)}$ and $K|_{V(\mathcal{T}M)}$ are vector bundle isomorphisms. The map $\beta := (\rho|_{H(\mathcal{T}M)})^{-1}$ will be called the horizontal map of the connection D.

The horizontal ((h)h-) and mixed ((h)hv-) torsion tensors of D, denoted by Q and T respectively, are defined by

$$Q(\overline{X}, \overline{Y}) = \mathbf{T}(\beta\overline{X}\beta\overline{Y}), \quad T(\overline{X}, \overline{Y}) = \mathbf{T}(\gamma\overline{X}, \beta\overline{Y}) \quad \forall\, \overline{X}, \overline{Y} \in \mathfrak{X}(\pi(M)),$$

where $\mathbf{T}$ is the (classical) torsion tensor field associated with D.

The horizontal (h-), mixed (hv-) and vertical (v-) curvature tensors of D, denoted by R, P and S respectively, are defined by

$$R(\overline{X}, \overline{Y})\overline{Z} = \mathbf{K}(\beta\overline{X}\beta\overline{Y})\overline{Z}, \quad P(\overline{X}, \overline{Y})\overline{Z} = \mathbf{K}(\beta\overline{X}, \gamma\overline{Y})\overline{Z}, \quad S(\overline{X}, \overline{Y})\overline{Z} = \mathbf{K}(\gamma\overline{X}, \gamma\overline{Y})\overline{Z},$$

where $\mathbf{K}$ is the (classical) curvature tensor field associated with D.

The contracted curvature tensors of D, denoted by $\widehat{R}$, $\widehat{P}$ and $\widehat{S}$ respectively, known also as the (v)h-, (v)hv- and (v)v-torsion tensors, are defined by

$$\widehat{R}(\overline{X},\overline{Y}) = R(\overline{X},\overline{Y})\overline{\eta}, \quad \widehat{P}(\overline{X},\overline{Y}) = P(\overline{X},\overline{Y})\overline{\eta}, \quad \widehat{S}(\overline{X},\overline{Y}) = S(\overline{X},\overline{Y})\overline{\eta}.$$

If M is endowed with a metric g on $\pi^{-1}(TM)$, we write

$$\mathbf{R}(\overline{X},\overline{Y},\overline{Z},\overline{W}) := g(R(\overline{X},\overline{Y})\overline{Z},\overline{W}), \cdots, \mathbf{S}(\overline{X},\overline{Y},\overline{Z},\overline{W}) := g(S(\overline{X},\overline{Y})\overline{Z},\overline{W}) \tag{2.1}$$

The following result is of extreme importance.

Theorem 2.1. [7] *Let (M, L) be a Finsler manifold and g the Finsler metric defined by L. There exists a unique regular connection ∇ on $\pi^{-1}(TM)$ such that*

(a) ∇ *is metric*: $\nabla g = 0$,

(b) *The (h)h-torsion of* ∇ *vanishes*: $Q = 0$,

(c) *The (h)hv-torsion T of* ∇ *satisfies*: $g(T(\overline{X},\overline{Y}),\overline{Z}) = g(T(\overline{X},\overline{Z}),\overline{Y})$.

Such a connection is called the Cartan connection associated with the Finsler manifold (M, L).

On a Finsler manifold there are other important linear connections. However, the only linear connection we treat in this paper is the Cartan connection. For a Finsler manifold (M, L), we define the following geometric objects:

$$\begin{array}{rcl}
\ell & := & L^{-1} i_{\overline{\eta}}\, g, \\
\hbar & := & g - \ell \otimes \ell : \text{the angular metric tensor}, \\
\phi & : & \text{the vector } \pi\text{-form associated with } \hbar;\ i_{\phi(\overline{X})}\, g := i_{\overline{X}}\, \hbar, \\
\overset{h}{\nabla} & : & \text{the } h\text{-covariant derivatives associated with the Cartan connection}, \\
\overset{v}{\nabla} & : & \text{the } v\text{-covariant derivatives associated with the Cartan connection}, \\
T & : & \text{the Cartan tensor};\ T(\overline{X},\overline{Y},\overline{Z}) := g(T(\overline{X},\overline{Y}),\overline{Z}), \\
H & := & i_{\overline{\eta}}\, \widehat{R} : \text{the deviation tensor}, \\
Ric & : & \text{the horizontal Ricci tensor of Cartan connection}, \\
r & : & \text{the horizontal scalar curvature of Cartan connection.}
\end{array}$$

3 Concircularly flat Finsler manifold

Definition 3.1. [6], [2] A Finsler manifold (M, L) of dimension $n \geq 3$ is said to be h-isotropic if there exists a scalar function $k_o \neq 0$ such that the horizontal curvature tensor R has the form:

$$R = k_o\, G,$$

where G is the π-tensor field defined by

$$G(\overline{X},\overline{Y})\overline{Z} := g(\overline{X},\overline{Z})\overline{Y} - g(\overline{Y},\overline{Z})\overline{X}. \tag{3.1}$$

Definition 3.2. [6], [3] A Finsler manifold (M, L) of dimension $n \geq 3$ is said to be of scalar curvature if the deviation tensor $H := i_{\overline{\eta}}\,\widehat{R}$ satisfies

$$H = \varepsilon L^2 \phi,$$

where ε is a scalar function on TM, positively homogenous of degree zero in y.

In particular, if the scalar function ε is constant, then (M, L) is said to be of constant curvature.

Let us now introduce the notion of concircular curvature.

Definition 3.3. Let (M, L) be a Finsler manifold of dimension $n \geq 3$. The π-tensor field C defined by

$$C := R - \frac{r}{n(n-1)}\,G$$

will be called the concircular curvature tensor, G being the π-tensor field defined by (3.1).

If the concircular curvature tensor C vanishes, then (M, L) is said to be concircularly flat.

It should be noted that the concircular curvature tensor in *Riemannian geometry* has been thoroughly investigated by many authors. The above definition is a generalization to Finsler geometry of that tensor field.

Proposition 3.1. *A Finsler manifold (M, L) is concircularly flat if, and only if, (M, L) is h-isotropic.*

Proof. It is clear that if (M, L) is concircularly flat, then it is h-isotropic (with $k_o = \frac{r}{n(n-1)}$ in Definition 3.1).

Conversely, suppose that (M, L) be h-isotropic. Then, by Definition 3.1, we have

$$R(\overline{X}, \overline{Y})\overline{Z} = k_o \left\{ g(\overline{X}, \overline{Z})\overline{Y} - g(\overline{Y}, \overline{Z})\overline{X} \right\} \tag{3.2}$$

Taking the trace with respect to $\overline{Y}$ of the above relation, we get

$$Ric(\overline{X}, \overline{Z}) = k_o \left\{ n g(\overline{X}, \overline{Z}) - g(\overline{X}, \overline{Z}) \right\}.$$

This equation, again, by taking the trace with respect to the pair of arguments $\overline{X}$ and $\overline{Z}$, reduces to

$$k_o = \frac{r}{n(n-1)}.$$

From which, taking into account (3.2) and Definition 3.3, (M, L) is therefore concircularly flat. □

The following theorem is one of the main results of the present paper.

Theorem A. *A concircularly flat Finsler manifold is necessarily of constant curvature.*

To prove this theorem we need the following three lemmas.

Lemma 3.2. *For a Finsler manifold (M, L), we have:*

(a) $\overset{h}{\nabla} L = 0, \quad \overset{v}{\nabla} L = \ell.$

(b) $\overset{h}{\nabla} \ell = 0, \quad \overset{v}{\nabla} \ell = L^{-1}\hbar.$

(c) $i_{\overline{\eta}}\,\ell = L, \quad i_{\overline{\eta}}\hbar = 0.$

(d) $\phi = I - L^{-1}\ell \otimes \overline{\eta}.$

Lemma 3.3. *For a concircularly flat Finsler manifold (M, L), we have*

$$\mathfrak{S}_{\overline{X},\overline{Y},\overline{Z}} R(\overline{X}, \overline{Y})\overline{Z} = 0.$$

Proof. Let (M, L) be concircularly flat. Then, by Definition 3.3 and the fact that $i_{\overline{\eta}}\, g = L\ell$, we have

$$\widehat{R}(\overline{X}, \overline{Y}) = kL(\ell(\overline{X})\overline{Y} - \ell(\overline{Y})\overline{X}), \tag{3.3}$$

where $k(x, y) := \frac{r}{n(n-1)}$, necessarily homogenous of degree 0 in y.
From (3.3), taking into account the fact that the $(h)hv$-torsion T is symmetric, we obtain

$$\begin{aligned}
\mathfrak{S}_{\overline{X},\overline{Y},\overline{Z}} T(\widehat{R}(\overline{X}, \overline{Y}), \overline{Z}) &= kL\left\{\ell(\overline{X})T(\overline{Y}, \overline{Z}) - \ell(\overline{Y})T(\overline{X}, \overline{Z})\right\}\\
&\quad + kL\left\{\ell(\overline{Y})T(\overline{Z}, \overline{X}) - \ell(\overline{Z})T(\overline{Y}, \overline{X})\right\}\\
&\quad + kL\left\{\ell(\overline{Z})T(\overline{X}, \overline{Y}) - \ell(\overline{X})T(\overline{Z}, \overline{Y})\right\}\\
&= 0. \qquad (3.4)
\end{aligned}$$

On the other hand, we have [8]

$$\mathfrak{S}_{\overline{X},\overline{Y},\overline{Z}} R(\overline{X}, \overline{Y})\overline{Z} = \mathfrak{S}_{\overline{X},\overline{Y},\overline{Z}} T(\widehat{R}(\overline{X}, \overline{Y}), \overline{Z}). \tag{3.5}$$

Hence, the result follows from (3.5) and (3.4). □

Lemma 3.4. *For a π-tensor field ω of type $(1, 1)$ on a Finsler manifold (M, L), we have*

$$(\overset{v}{\nabla}\overset{h}{\nabla} \omega)(\overline{X}, \overline{Y}, \overline{Z}) - (\overset{h}{\nabla}\overset{v}{\nabla} \omega)(\overline{Y}, \overline{X}, \overline{Z}) = -(P(\overline{X}, \overline{Y})\omega)(\overline{Z}) \tag{3.6}$$

$$+(\overset{v}{\nabla} \omega)(\widehat{P}(\overline{X}, \overline{Y}), \overline{Z}) + (\overset{h}{\nabla} \omega)(T(\overline{Y}, \overline{X}), \overline{Z}). \tag{3.7}$$

In particular, for a scalar function $f(x, y)$, we have

$$\overset{v}{\nabla}\overset{h}{\nabla} f = \overset{h}{\nabla}\overset{v}{\nabla} f.$$

Proof of Theorem A: Let (M, L) be a concircularly flat Finsler manifold, then the $(v)h$-torsion tensor $\widehat{R}$ satisfies Equation (3.3). As a consequence of Lemma 3.2, (3.3) reduces to

$$H = kL^2\phi \tag{3.8}$$

If k is constant, then the result follows from (3.8) and Definition 3.2. Now, we will show that $\overset{v}{\nabla} k = \overset{h}{\nabla} k = 0$.

We have [8]

$$\begin{aligned}&(\nabla_{\gamma\overline{X}}R)(\overline{Y},\overline{Z},\overline{W})+(\nabla_{\beta\overline{Y}}P)(\overline{Z},\overline{X},\overline{W})-(\nabla_{\beta\overline{Z}}P)(\overline{Y},\overline{X},\overline{W})\\&-P(\overline{Z},\widehat{P}(\overline{Y},\overline{X}))\overline{W}+R(T(\overline{X},\overline{Y}),\overline{Z})\overline{W}-S(\widehat{R}(\overline{Y},\overline{Z}),\overline{X})\overline{W}\\&+P(\overline{Y},\widehat{P}(\overline{Z},\overline{X}))\overline{W}-R(T(\overline{X},\overline{Z}),\overline{Y})\overline{W}=0.\end{aligned}$$

Setting $\overline{W}=\overline{\eta}$ into the above relation, noting that $K\circ\gamma=id_{\mathfrak{X}(\pi(M))}$, $K\circ\beta=0$ and $\widehat{S}=0$, it follows that

$$\begin{aligned}&(\nabla_{\gamma\overline{X}}\widehat{R})(\overline{Y},\overline{Z})-R(\overline{Y},\overline{Z})\overline{X}+(\nabla_{\beta\overline{Y}}\widehat{P})(\overline{Z},\overline{X})-(\nabla_{\beta\overline{Z}}\widehat{P})(\overline{Y},\overline{X})\\&-\widehat{P}(\overline{Z},\widehat{P}(\overline{Y},\overline{X}))+\widehat{P}(\overline{Y},\widehat{P}(\overline{Z},\overline{X}))+\widehat{R}(T(\overline{X},\overline{Y}),\overline{Z})-\widehat{R}(T(\overline{X},\overline{Z}),\overline{Y})=0.\end{aligned}$$

Applying the cyclic sum $\mathfrak{S}_{\overline{X},\overline{Y},\overline{Z}}$ on the above equation, taking Lemma 3.3 into account, we get

$$\mathfrak{S}_{\overline{X},\overline{Y},\overline{Z}}(\nabla_{\gamma\overline{X}}\widehat{R})(\overline{Y},\overline{Z})=0. \tag{3.9}$$

Substituting (3.3) into (3.9), using $(\nabla_{\gamma\overline{X}}\ell)(\overline{Y})=L^{-1}\hbar(\overline{X},\overline{Y})$ (Lemma 3.2(**b**)), we have

$$\begin{aligned}&L(\nabla_{\gamma\overline{Z}}k)\left\{(\ell(\overline{X})\overline{Y}-\ell(\overline{Y})\overline{X})\right\}+L(\nabla_{\gamma\overline{Y}}k)\left\{(\ell(\overline{Z})\overline{X}-\ell(\overline{X})\overline{Z})\right\}\\&+L(\nabla_{\gamma\overline{X}}k)\left\{(\ell(\overline{Y})\overline{Z}-\ell(\overline{Z})\overline{Y})\right\}+k\ell(\overline{Z})\left\{(\ell(\overline{X})\overline{Y}-\ell(\overline{Y})\overline{X})\right\}\\&+k\ell(\overline{Y})\left\{(\ell(\overline{Z})\overline{X}-\ell(\overline{X})\overline{Z})\right\}+k\ell(\overline{X})\left\{(\ell(\overline{Y})\overline{Z}-\ell(\overline{Z})\overline{Y})\right\}\\&+kL\left\{(\hbar(\overline{X},\overline{Z})\overline{Y}-\hbar(\overline{Y},\overline{Z})\overline{X})\right\}+kL\left\{(\hbar(\overline{Z},\overline{Y})\overline{X}-\hbar(\overline{X},\overline{Y})\overline{Z})\right\}\\&+kL\left\{(\hbar(\overline{Y},\overline{X})\overline{Z}-\hbar(\overline{Z},\overline{X})\overline{Y})\right\}=0.\end{aligned}$$

Setting $\overline{Z}=\overline{\eta}$ into the above relation, noting that $i_{\overline{\eta}}\ell=L$, $i_{\overline{\eta}}\hbar=0$ (Lemma 3.2(**c**)) and $\nabla_{\gamma\overline{\eta}}k=0$, we conclude that

$$L^2\left\{\phi(\overline{Y})\,\nabla_{\gamma\overline{X}}k-\phi(\overline{X})\,\nabla_{\gamma\overline{Y}}k\right\}=0. \tag{3.10}$$

Taking the trace of both sides of (3.10) with respect to $\overline{Y}$, noting that $Tr(\phi)=n-1$ [6], it follows that

$$(n-2)\nabla_{\gamma\overline{X}}k=0.$$

Consequently, as $n\geq 3$,

$$\overset{v}{\nabla} k=0. \tag{3.11}$$

Now, from (3.3) and the fact that the $(v)hv$-torsion $\widehat{P}$ is symmetric [8], we get

$$\mathfrak{S}_{\overline{X},\overline{Y},\overline{Z}}\widehat{P}(\widehat{R}(\overline{X},\overline{Y}),\overline{Z})\quad=\quad 0. \tag{3.12}$$

On the other hand, we have [8]

$$\mathfrak{S}_{\overline{X},\overline{Y},\overline{Z}}\left\{(\nabla_{\beta\overline{X}}R)(\overline{Y},\overline{Z},\overline{W})+P(\widehat{R}(\overline{X},\overline{Y}),\overline{Z})\overline{W}\right\}=0.$$

From which, together with (3.12), it follows that

$$\mathfrak{S}_{\overline{X},\overline{Y},\overline{Z}}\,(\nabla_{\beta\overline{X}}\widehat{R})(\overline{Y},\overline{Z})=0. \tag{3.13}$$

Again from (3.3), noting that $\nabla_{\beta\overline{X}}\ell=0$ (Lemma 3.2(**b**)), (3.13) reads

$$\begin{aligned}&L(\nabla_{\beta\overline{X}}k)\left\{\ell(\overline{Y})\overline{Z}-\ell(\overline{Z})\overline{Y}\right\}+L(\nabla_{\beta\overline{Y}}k)\left\{\ell(\overline{Z})\overline{X}-\ell(\overline{X})\overline{Z}\right\}\\&+L(\nabla_{\beta\overline{Z}}k)\left\{\ell(\overline{X})\overline{Y}-\ell(\overline{Y})\overline{X}\right\}=0.\end{aligned}$$

Setting $\overline{Z}=\overline{\eta}$ into the above equation, noting that $\ell(\overline{\eta})=L$ (Lemma 3.2(**c**)), we obtain

$$\begin{aligned}&L(\nabla_{\beta\overline{X}}k)\left\{\ell(\overline{Y})\overline{\eta}-L\overline{Y}\right\}+L(\nabla_{\beta\overline{Y}}k)\left\{L\overline{X}-\ell(\overline{X})\overline{\eta}\right\}\\&+L(\nabla_{\beta\overline{\eta}}k)\left\{\ell(\overline{X})\overline{Y}-\ell(\overline{Y})\overline{X}\right\}=0.\end{aligned}$$

Taking the trace of both sides with respect to $\overline{Y}$, it follows that

$$\nabla_{\beta\overline{X}}k=L^{-1}(\nabla_{\beta\overline{\eta}}k)\ell(\overline{X}). \tag{3.14}$$

Applying the v-covariant derivative with respect to $\overline{Y}$ on both sides of (3.14), yields

$$\ell(\overline{Y})\nabla_{\beta\overline{X}}k+L(\overset{v}{\nabla}\overset{h}{\nabla}k)(\overline{X},\overline{Y})=L^{-1}\hbar(\overline{X},\overline{Y})(\nabla_{\beta\overline{\eta}}k)+\ell(\overline{X})(\overset{v}{\nabla}\overset{h}{\nabla}k)(\overline{\eta},\overline{Y}).$$

Since, $\overset{v}{\nabla}\overset{h}{\nabla}k=\overset{h}{\nabla}\overset{v}{\nabla}k=0$, by Lemma 3.4 and (3.11), the above relation reduces to

$$\ell(\overline{Y})\nabla_{\beta\overline{X}}k=L^{-1}\hbar(\overline{X},\overline{Y})(\nabla_{\beta\overline{\eta}}k).$$

Setting $\overline{Y}=\overline{\eta}$ into the above equation, taking Lemma 3.2 into account, it follows that $\nabla_{\beta\overline{X}}k=0$. Consequently,

$$\overset{h}{\nabla}k=0. \tag{3.15}$$

Now, (3.11) and (3.15) imply that $\nabla k=0$. Hence, k is constant and the theorem is proved. □

4 Concircularly recurrent Finsler manifold

We first introduce the following two special Finsler spaces which will be the object of our study in this section.

Definition 4.1. A Finsler manifold (M,L) of dimension $n\geq 3$ is called R^h-recurrent if its h-curvature tensor R is horizontally recurrent:

$$\overset{h}{\nabla}R=\lambda\otimes R\quad\text{and}\quad R\neq 0, \tag{4.1}$$

where λ is a scalar π-form, positively homogenous of degree zero in y, called the recurrence form.

In particular, if $\overset{h}{\nabla}R=0$, then (M,L) is called R^h-symmetric.

Definition 4.2. A Finsler manifold (M, L) of dimension $n \geq 3$ is called concircularly recurrent if its concircular curvature tensor C is horizontally recurrent:

$$\overset{h}{\nabla} C = \alpha \otimes C \quad \text{and} \quad C \neq 0, \tag{4.2}$$

where α is a scalar π-form, positively homogenous of degree zero in y, called the recurrence form.

In particular, if $\overset{h}{\nabla} C = 0$, then (M, L) is called concircularly symmetric.

The following theorem is the second main result of the present paper.

Theorem B. *Every R^h-recurrent Finsler manifold is concircularly recurrent with the same recurrence form.*

Proof. Let (M, L) be an R^h-recurrent Finsler manifold with recurrence form λ. Then (4.1) is satisfied. Consequently,

$$\overset{h}{\nabla} r = \lambda \otimes r. \tag{4.3}$$

Now, from Definition 4.2, (4.1) and (4.3), we get

$$\begin{aligned} \overset{h}{\nabla} C &= \overset{h}{\nabla} \left\{ R - \frac{r}{n(n-1)} G \right\} = \overset{h}{\nabla} R - \frac{1}{n(n-1)} \overset{h}{\nabla} r \otimes G \\ &= \lambda \otimes \left\{ R - \frac{r}{n(n-1)} G \right\} = \lambda \otimes C. \end{aligned}$$

Therefore, (M, L) is concircularly recurrent with the same recurrence form λ. $\square$

Remark 4.3. *The converse of the above theorem is not true in general. However, it has been recently proved to be true in Riemannian geometry [4].*

For the converse of Theorem B to be true in the Finslerian context, an additional condition is needed, namely the horizontal integrability condition. A Finsler manifold is said to be horizontally integrable if its horizonal distribution is completely integrable (or, equivalently, $\widehat{R} = 0$).

Now, we are in a position to announce our third main and most important result.

Theorem C. *Every horizontally integrable concircularly recurrent Finsler manifold is R^h-recurrent with the same recurrence form.*

To prove this theorem we need the following three lemmas.

Lemma 4.1. *For a concircularly recurrent Finsler manifold with recurrence form α, we have*

$$\overset{h}{\nabla} R = \alpha \otimes R + \mu \otimes G,$$

where μ is a π-scalar form defined by

$$\mu := \frac{1}{n(n-1)} \left\{ \overset{h}{\nabla} r - r\alpha \right\}.$$

Proof. Let (M, L) be a concircularly recurrent Finsler manifold with recurrence form α. Then, by Definitions 4.2 and 3.3, we have

$$\overset{h}{\nabla}\left\{R - \frac{r}{n(n-1)}\,G\right\} = \alpha \otimes \left\{R - \frac{r}{n(n-1)}\,G\right\}.$$

From which, together with the the fact that $\overset{h}{\nabla}\,G = 0$, we get

$$\overset{h}{\nabla}\,R - \frac{\overset{h}{\nabla}\,r}{n(n-1)} \otimes G \quad = \quad \alpha \otimes \left\{R - \frac{r}{n(n-1)}\,G\right\}.$$

Hence, the result follows. □

Lemma 4.2. *For a horizontally integrable Finsler manifold, we have*

(a) $\mathfrak{S}_{\overline{X},\overline{Y},\overline{Z}} R(\overline{X},\overline{Y})\overline{Z} = 0.$

(b) $\mathbf{R}(\overline{X},\overline{Y},\overline{Z},\overline{W}) = \mathbf{R}(\overline{Z},\overline{W},\overline{X},\overline{Y}).$

(c) *The horizontal Ricci tensors Ric is symmetric.*

(d) $\mathfrak{S}_{\overline{U},\overline{V};\ \overline{W},\overline{X};\ \overline{Y},\overline{Z}}\left\{(R(\overline{U},\overline{V})\mathbf{R})(\overline{W},\overline{X},\overline{Y},\overline{Z})\right\} = 0$ [1]

(e) $(\overset{h}{\nabla}\overset{h}{\nabla}\,\omega)(\overline{Y},\overline{X},\overline{Z}) - (\overset{h}{\nabla}\overset{h}{\nabla}\,\omega)(\overline{X},\overline{Y},\overline{Z}) = (R(\overline{X},\overline{Y})\omega)(\overline{Z}),$
where ω is a π-tensor field of type $(1,1)$.

Proof. **(a)** Follows from (3.5) and the horizontal integrability condition ($\widehat{R} = 0$).
(b) Follows from **(a)** and the two identities [8]:

$$\mathbf{R}(\overline{X},\overline{Y},\overline{Z},\overline{W}) = -\mathbf{R}(\overline{Y},\overline{X},\overline{Z},\overline{W}), \tag{4.4}$$

$$\mathbf{R}(\overline{X},\overline{Y},\overline{Z},\overline{W}) = -\mathbf{R}(\overline{X},\overline{Y},\overline{W},\overline{Z}). \tag{4.5}$$

(c) Follows from **(b)**.
(d) We have:

$$\begin{aligned}
(R(\overline{U},\overline{V})\mathbf{R})(\overline{W},\overline{X},\overline{Y},\overline{Z}) \;&=\; -\mathbf{R}(R(\overline{U},\overline{V})\overline{W},\overline{X},\overline{Y},\overline{Z}) - \mathbf{R}(\overline{W},R(\overline{U},\overline{V})\overline{X},\overline{Y},\overline{Z})\\
&\quad\; -\mathbf{R}(\overline{W},\overline{X},R(\overline{U},\overline{V})\overline{Y},\overline{Z}) - \mathbf{R}(\overline{W},\overline{X},\overline{Y},R(\overline{U},\overline{V})\overline{Z})\\
(R(\overline{W},\overline{X})\mathbf{R})(\overline{Y},\overline{Z},\overline{U},\overline{V}) \;&=\; -\mathbf{R}(R(\overline{W},\overline{X})\overline{Y},\overline{Z},\overline{U},\overline{V}) - \mathbf{R}(\overline{Y},R(\overline{W},\overline{X})\overline{Z},\overline{U},\overline{V})\\
&\quad\; -\mathbf{R}(\overline{Y},\overline{Z},R(\overline{W},\overline{X})\overline{U},\overline{V}) - \mathbf{R}(\overline{Y},\overline{Z},\overline{U},R(\overline{W},\overline{X})\overline{V})\\
(R(\overline{Y},\overline{Z})\mathbf{R})(\overline{U},\overline{V},\overline{W},\overline{X}) \;&=\; -\mathbf{R}(R(\overline{Y},\overline{Z})\overline{U},\overline{V},\overline{W},\overline{X}) - \mathbf{R}(\overline{U},R(\overline{Y},\overline{Z})\overline{V},\overline{W},\overline{X})\\
&\quad\; -\mathbf{R}(\overline{U},\overline{V},R(\overline{Y},\overline{Z})\overline{W},\overline{X}) - \mathbf{R}(\overline{U},\overline{V},\overline{W},R(\overline{Y},\overline{Z})\overline{X}).
\end{aligned}$$

Adding the above three equations, making use of (4.4), (4.5) and **(b)**, the result follows.
(e) One can show that for the Cartan connection, we have:

$$\begin{aligned}
(\overset{h}{\nabla}\overset{h}{\nabla}\,\omega)(\overline{X},\overline{Y},\overline{Z}) - (\overset{h}{\nabla}\overset{h}{\nabla}\,\omega)(\overline{Y},\overline{X},\overline{Z}) \;&=\; \omega(R(\overline{X},\overline{Y})\overline{Z}) - R(\overline{X},\overline{Y})\omega(\overline{Z})\\
&\quad\; +(\overset{v}{\nabla}\,\omega)(\widehat{R}(\overline{X},\overline{Y}),\overline{Z}).
\end{aligned}$$

[1] $\mathfrak{S}_{\overline{U},\overline{V};\ \overline{W},\overline{X};\ \overline{Y},\overline{Z}}$ denotes the cyclic sum over the three pairs of arguments $\overline{U},\overline{V};\ \overline{W},\overline{X}$ and $\overline{Y},\overline{Z}$.

From which, together with the assumption of horizontal integrability, the result follows. □

Lemma 4.2(**d**) and the next lemma are the global Finslerian versions of Walker's lemmas [5], proved locally in Riemannian geometry.

Lemma 4.3. *Let A be a symmetric scalar π-form and B a scalar π-form. If for all $\overline{X}, \overline{Y}, \overline{Z} \in \mathfrak{X}(\pi(M))$,*

$$\mathfrak{S}_{\overline{X},\overline{Y},\overline{Z}} \left\{A(\overline{X},\overline{Y})B(\overline{Z})\right\} = 0, \tag{4.6}$$

then $A = 0$ or $B = 0$.

In particular, for a horizontally integrable non-flat (non-concircularly flat) Finsler manifold, if one of the following relations holds

$$\begin{aligned} \mathfrak{S}_{\overline{U},\overline{V};\,\overline{W},\overline{X};\,\overline{Y},\overline{Z}} \left\{\omega(\overline{U},\overline{V})\mathbf{R}(\overline{W},\overline{X},\overline{Y},\overline{Z})\right\} &= 0,\\ \mathfrak{S}_{\overline{U},\overline{V};\,\overline{W},\overline{X};\,\overline{Y},\overline{Z}} \left\{\omega(\overline{U},\overline{V})\mathbf{C}(\overline{W},\overline{X},\overline{Y},\overline{Z})\right\} &= 0, \end{aligned}$$

then the scalar π-form ω vanishes, where

$$\mathbf{C}(\overline{X},\overline{Y},\overline{Z},\overline{W}) := g(C(\overline{X},\overline{Y})\overline{Z},\overline{W}).$$

Proof. Let A be a symmetric scalar π-form and B a scalar π-form which satisfy Relation (4.6). If B vanishes, the result follows. If B dose not vanish, then from (4.6), we have

$$\begin{aligned} 3A(\overline{X},\overline{X})B(X) &= 0 \quad \forall\, \overline{X} \in \mathfrak{X}(\pi(M))\\ \Longrightarrow \qquad A(\overline{X},\overline{X}) &= 0, \quad \forall\, \overline{X} \in \mathfrak{X}(\pi(M))\\ \Longrightarrow \; A(\overline{X}+\overline{Y},\overline{X}+\overline{Y}) &= 0, \quad \forall\, \overline{X},\overline{Y} \in \mathfrak{X}(\pi(M))\\ \Longrightarrow \qquad 2A(\overline{X},\overline{Y}) &= 0, \quad \forall\, \overline{X},\overline{Y} \in \mathfrak{X}(\pi(M)). \end{aligned}$$

Hence, the scalr π-form A vanishes.

The second part of this lemma follows from the first part, taking into account the assumption that $\mathbf{R} \neq 0\,(\mathbf{C} \neq 0)$, together with Lemma 4.2(**b**). □

Proof of Theorem C: Let (M, L) be a horizontally integrable concircularly recurrent Finsler manifold with recurrence form α. The proof is achieved in three steps:

First step: *The h-covariant derivative of the recurrence form α is symmetric*:

The concircular recurrence condition (4.2) gives

$$\begin{aligned} \overset{h}{\nabla}\mathbf{C} &= \alpha \otimes \mathbf{C}\\ \overset{h}{\nabla}\overset{h}{\nabla}\mathbf{C} &= (\overset{h}{\nabla}\alpha)\otimes\mathbf{C} + \alpha\otimes\overset{h}{\nabla}\mathbf{C}\\ &= (\overset{h}{\nabla}\alpha + \alpha\otimes\alpha)\otimes\mathbf{C}. \end{aligned}$$

From this, taking into account Lemma 4.2, we obtain

$$\begin{aligned}(R(\overline{U},\overline{V})\mathbf{C})(\overline{W},\overline{X},\overline{Y},\overline{Z}) &= (\overset{h}{\nabla}\overset{h}{\nabla}\mathbf{C})(\overline{V},\overline{U},\overline{W},\overline{X},\overline{Y},\overline{Z}) - (\overset{h}{\nabla}\overset{h}{\nabla}\mathbf{C})(\overline{U},\overline{V},\overline{W},\overline{X},\overline{Y},\overline{Z})\\ &= \left\{(\overset{h}{\nabla}\alpha)(\overline{V},\overline{U}) - (\overset{h}{\nabla}\alpha)(\overline{U},\overline{V})\right\}\mathbf{C}(\overline{W},\overline{X},\overline{Y},\overline{Z})\\ &= -(\bar{d}\alpha)(\overline{U},\overline{V})\mathbf{C}(\overline{W},\overline{X},\overline{Y},\overline{Z}),\end{aligned}\tag{4.7}$$

where

$$(\bar{d}\alpha)(\overline{U},\overline{V}) := (\overset{h}{\nabla}\alpha)(\overline{U},\overline{V}) - (\overset{h}{\nabla}\alpha)(\overline{V},\overline{U}).\tag{4.8}$$

On the other hand, in view of Definition 3.3, we have

$$\mathbf{C} := \mathbf{R} - \frac{r}{n(n-1)}\,\mathbf{G},\tag{4.9}$$

where $\mathbf{G}$ is the π-tensor field defined by

$$\mathbf{G}(\overline{X},\overline{Y},\overline{Z},\overline{W}) := g(G(\overline{X},\overline{Y})\overline{Z},\overline{W}).$$

Using (4.9) and the identities $R(\overline{U},\overline{V})r = 0 = R(\overline{U},\overline{V})\mathbf{G}$, we get

$$(R(\overline{U},\overline{V})\mathbf{C})(\overline{W},\overline{X},\overline{Y},\overline{Z}) = (R(\overline{U},\overline{V})\mathbf{R})(\overline{W},\overline{X},\overline{Y},\overline{Z}.)\tag{4.10}$$

Now, from (4.7) and (4.10), taking Lemma 4.2(**d**) into account, it follows that

$$\bar{d}\alpha(\overline{U},\overline{V})\mathbf{C}(\overline{W},\overline{X},\overline{Y},\overline{Z}) + \bar{d}\alpha(\overline{W},\overline{X})\mathbf{C}(\overline{Y},\overline{Z},\overline{U},\overline{V}) + \bar{d}\alpha(\overline{Y},\overline{Z})\mathbf{C}(\overline{U},\overline{V},\overline{W},\overline{X}) = 0.$$

From this, together with Lemma 4.3, we conclude that $\bar{d}\alpha = 0$. Hence the result follows from (4.8).

<u>Second step</u>: (M,L) *has the property that* $R(\overline{X},\overline{Y})\mathbf{R} = 0$:

We have, by Lemma 4.1,

$$\begin{aligned}\overset{h}{\nabla}\mathbf{R} &= \alpha\otimes\mathbf{R} + \mu\otimes\mathbf{G}\\ \overset{h}{\nabla}\overset{h}{\nabla}\mathbf{R} &= (\overset{h}{\nabla}\alpha)\otimes\mathbf{R} + \alpha\otimes\overset{h}{\nabla}\mathbf{R} + (\overset{h}{\nabla}\mu)\otimes\mathbf{G}\\ &= (\overset{h}{\nabla}\alpha)\otimes\mathbf{R} + \alpha\otimes(\alpha\otimes\mathbf{R} + \mu\otimes\mathbf{G}) + (\overset{h}{\nabla}\mu)\otimes\mathbf{G}\\ &= (\overset{h}{\nabla}\alpha + \alpha\otimes\alpha)\otimes\mathbf{R} + (\overset{h}{\nabla}\mu + \alpha\otimes\mu)\otimes\mathbf{G}.\end{aligned}$$

The above equation together with Lemma 4.2(**e**) and (4.8) imply that

$$\begin{aligned}(R(\overline{U},\overline{V})\mathbf{R})(\overline{W},\overline{X},\overline{Y},\overline{Z}) &= -(\bar{d}\alpha)(\overline{U},\overline{V})\mathbf{R}(\overline{W},\overline{X},\overline{Y},\overline{Z})\\ &\quad -(\bar{d}\mu + \alpha\wedge\mu)(\overline{U},\overline{V})\mathbf{G}(\overline{W},\overline{X},\overline{Y},\overline{Z}).\end{aligned}$$

Now, taking into account the fact that $\bar{d}\alpha = 0$ (*First step*), the above equation reduces to

$$(R(\overline{U},\overline{V})\mathbf{R})(\overline{W},\overline{X},\overline{Y},\overline{Z}) = -(\bar{d}\mu + \alpha\wedge\mu)(\overline{U},\overline{V})\mathbf{G}(\overline{W},\overline{X},\overline{Y},\overline{Z}).\tag{4.11}$$

From which, taking Lemma 4.2 into account, we obtain

$$(\bar{d}\mu + \alpha \wedge \mu)(\overline{U}, \overline{V})\mathbf{G}(\overline{W}, \overline{X}, \overline{Y}, \overline{Z}) + (\bar{d}\mu + \alpha \wedge \mu)(\overline{W}, \overline{X})\mathbf{G}(\overline{Y}, \overline{Z}, \overline{U}, \overline{V})$$
$$+(\bar{d}\mu + \alpha \wedge \mu)(\overline{Y}, \overline{Z})\mathbf{G}(\overline{U}, \overline{V}, \overline{W}, \overline{X}) = 0.$$

Applying Lemma 4.3, the above relation implies that

$$\bar{d}\mu + \alpha \wedge \mu = 0.$$

Consequently, in view of (4.11), we conclude that $R(\overline{U}, \overline{V})\mathbf{R} = 0$.
Third step: *(M, L) is R^h-recurrent with the same recurrence form α*:
We have, from the second step,

$$\begin{aligned} &\mathbf{R}(R(\overline{U}, \overline{V})\overline{W}, \overline{X}, \overline{Y}, \overline{Z}) + \mathbf{R}(\overline{W}, R(\overline{U}, \overline{V})\overline{X}, \overline{Y}, \overline{Z}) \\ &+\mathbf{R}(\overline{W}, \overline{X}, R(\overline{U}, \overline{V})\overline{Y}, \overline{Z}) + \mathbf{R}(\overline{W}, \overline{X}, \overline{Y}, R(\overline{U}, \overline{V})\overline{Z}) = 0. \end{aligned} \tag{4.12}$$

Differentiating h-covariantly both sides of the above relation with respect to $\overline{\xi}$, we get

$$\begin{aligned} &(\nabla_{\beta\overline{\xi}}\mathbf{R})(R(\overline{U}, \overline{V})\overline{W}, \overline{X}, \overline{Y}, \overline{Z}) + \mathbf{R}((\nabla_{\beta\overline{\xi}}R)(\overline{U}, \overline{V})\overline{W}, \overline{X}, \overline{Y}, \overline{Z}) \\ &+(\nabla_{\beta\overline{\xi}}\mathbf{R})(\overline{W}, R(\overline{U}, \overline{V})\overline{X}, \overline{Y}, \overline{Z}) + \mathbf{R}(\overline{W}, (\nabla_{\beta\overline{\xi}}R)(\overline{U}, \overline{V})\overline{X}, \overline{Y}, \overline{Z}) \\ &+(\nabla_{\beta\overline{\xi}}\mathbf{R})(\overline{W}, \overline{X}, R(\overline{U}, \overline{V})\overline{Y}, \overline{Z}) + \mathbf{R}(\overline{W}, \overline{X}, (\nabla_{\beta\overline{\xi}}R)(\overline{U}, \overline{V})\overline{Y}, \overline{Z}) \\ &+(\nabla_{\beta\overline{\xi}}\mathbf{R})(\overline{W}, \overline{X}, \overline{Y}, R(\overline{U}, \overline{V})\overline{Z}) + \mathbf{R}(\overline{W}, \overline{X}, \overline{Y}, (\nabla_{\beta\overline{\xi}}R)(\overline{U}, \overline{V})\overline{Z}) = 0. \end{aligned}$$

Applying Lemma 4.1, we find

$$\begin{aligned} &(\alpha(\overline{\xi})\mathbf{R} + \mu(\overline{\xi})\mathbf{G})(R(\overline{U}, \overline{V})\overline{W}, \overline{X}, \overline{Y}, \overline{Z}) + \mathbf{R}((\alpha(\overline{\xi})R + \mu(\overline{\xi})G)(\overline{U}, \overline{V})\overline{W}, \overline{X}, \overline{Y}, \overline{Z}) \\ &+(\alpha(\overline{\xi})\mathbf{R} + \mu(\overline{\xi})\mathbf{G})(\overline{W}, R(\overline{U}, \overline{V})\overline{X}, \overline{Y}, \overline{Z}) + \mathbf{R}(\overline{W}, (\alpha(\overline{\xi})R + \mu(\overline{\xi})G)(\overline{U}, \overline{V})\overline{X}, \overline{Y}, \overline{Z}) \\ &+(\alpha(\overline{\xi})\mathbf{R} + \mu(\overline{\xi})\mathbf{G})(\overline{W}, \overline{X}, R(\overline{U}, \overline{V})\overline{Y}, \overline{Z}) + \mathbf{R}(\overline{W}, \overline{X}, (\alpha(\overline{\xi})R + \mu(\overline{\xi})G)(\overline{U}, \overline{V})\overline{Y}, \overline{Z}) \\ &+(\alpha(\overline{\xi})\mathbf{R} + \mu(\overline{\xi})\mathbf{G})(\overline{W}, \overline{X}, \overline{Y}, R(\overline{U}, \overline{V})\overline{Z}) + \mathbf{R}(\overline{W}, \overline{X}, \overline{Y}, (\alpha(\overline{\xi})R + \mu(\overline{\xi})G)(\overline{U}, \overline{V})\overline{Z}) = 0. \end{aligned}$$

Now, let us assume that $\mu \neq 0$ at a certain point of TM. At this point, using (4.12), the above equation reduces to

$$\begin{aligned} &\mathbf{G}(R(\overline{U}, \overline{V})\overline{W}, \overline{X}, \overline{Y}, \overline{Z}) + \mathbf{R}(G(\overline{U}, \overline{V})\overline{W}, \overline{X}, \overline{Y}, \overline{Z}) \\ &+\mathbf{G}(\overline{W}, R(\overline{U}, \overline{V})\overline{X}, \overline{Y}, \overline{Z}) + \mathbf{R}(\overline{W}, G(\overline{U}, \overline{V})\overline{X}, \overline{Y}, \overline{Z}) \\ &+\mathbf{G}(\overline{W}, \overline{X}, R(\overline{U}, \overline{V})\overline{Y}, \overline{Z}) + \mathbf{R}(\overline{W}, \overline{X}, G(\overline{U}, \overline{V})\overline{Y}, \overline{Z}) \\ &+\mathbf{G}(\overline{W}, \overline{X}, \overline{Y}, R(\overline{U}, \overline{V})\overline{Z}) + \mathbf{R}(\overline{W}, \overline{X}, \overline{Y}, G(\overline{U}, \overline{V})\overline{Z}) = 0. \end{aligned}$$

Using the definition of $\mathbf{G}$ and G, the last equality takes the form

$$\begin{aligned} &g(\overline{V}, \overline{W})\mathbf{R}(\overline{U}, \overline{X}, \overline{Y}, \overline{Z}) - g(\overline{U}, \overline{W})\mathbf{R}(\overline{V}, \overline{X}, \overline{Y}, \overline{Z}) \\ &\quad+g(\overline{V}, \overline{X})\mathbf{R}(\overline{W}, \overline{U}, \overline{Y}, \overline{Z}) - g(\overline{U}, \overline{X})\mathbf{R}(\overline{W}, \overline{V}, \overline{Y}, \overline{Z}) \\ &\quad+g(\overline{V}, \overline{Y})\mathbf{R}(\overline{W}, \overline{X}, \overline{U}, \overline{Z}) - g(\overline{U}, \overline{Y})\mathbf{R}(\overline{W}, \overline{X}, \overline{V}, \overline{Z}) \\ &\quad+g(\overline{V}, \overline{Z})\mathbf{R}(\overline{W}, \overline{X}, \overline{Y}, \overline{U}) - g(\overline{U}, \overline{Z})\mathbf{R}(\overline{W}, \overline{X}, \overline{Y}, \overline{V}) = 0. \end{aligned}$$

Taking the trace of the above equation with respect to the pair of arguments $(\overline{V}, \overline{W})$, we obtain

$$(n-2)\mathbf{R}(\overline{U}, \overline{X}, \overline{Y}, \overline{Z}) + \mathbf{R}(\overline{Y}, \overline{X}, \overline{U}, \overline{Z}) + \mathbf{R}(\overline{Z}, \overline{X}, \overline{Y}, \overline{U})$$
$$-g(\overline{U}, \overline{Y})Ric(\overline{X}, \overline{Z}) + g(\overline{U}, \overline{Z})Ric(\overline{X}, \overline{Y}) = 0.$$

This equation, using Lemma 4.2**(a)**, reduces to

$$(n-1)\mathbf{R}(\overline{U}, \overline{X}, \overline{Y}, \overline{Z}) - g(\overline{U}, \overline{Y})Ric(\overline{X}, \overline{Z}) + g(\overline{U}, \overline{Z})Ric(\overline{X}, \overline{Y}) = 0. \tag{4.13}$$

Again, taking the trace of the above equation with respect to the pair of arguments $\overline{X}$ and $\overline{Z}$, we get $Ric = \frac{r}{n}\, g$, which when inserted to (4.13), gives

$$R = \frac{r}{n(n-1)}\, G.$$

Hence, the concircular curvature C vanishes, which contradicts our assumption. Therefore, $\mu = 0$ at every point on TM. Consequently, by lemma 4.1, (M, L) is R^h-recurrent with the same recurrence form α. □

References

[1] H. Akbar-Zadeh, *Initiation to global Finsler geometry*, Elsevier, 2006.

[2] M. Matsumoto, *On h-isotropic and* C^h*-recurrent Finsler spaces*, J. Math. Kyoto Univ., 11, (1971), 1-9.

[3] S. Numata, *On Landesberg spaces of scalar curvature*, J. Korean Math. Soc., 12, 2(1975), 97-100.

[4] K. Olszak and Z. Olszak, *On pseudo-Riemannian manifolds with recurrent concircular curvature tensor*, Acta Math. Hungar. DOI: 10.1007/s 10474-012-0216-5, 8pp. ArXiv: 1108.0018v4 [math.DG].

[5] A. G. Walker, *On Ruses's spaces of recurrent curvature*, Proc. London Math. Soc., 52, (1950), 36-64.

[6] Nabil L. Youssef, S. H. Abed and A. Soleiman, *A global approach to the theory of special Finsler manifolds*, J. Math. Kyoto Univ., 48, 4(2008), 857-893. ArXiv: 0704.0053 [math. DG].

[7] ———, *A global approach to the theory of connections in Finsler geometry*, Tensor, N. S., 71, 3(2009), 187-208. ArXiv: 0801.3220 [math.DG].

[8] ———, *Geometric objects associated with the fundumental connections in Finsler geometry*, J. Egypt. Math. Soc., 18, 1(2010), 67-90. ArXiv: 0805.2489 [math.DG].

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