Characterizations of the Euclidean complex space form

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Abstract. In this paper we are interested in obtaining characterizations of the Euclidean complex space form $(\mathbb{C}^n, J, \langle , \rangle)$ using specific conformal vector fields on a Kaehler manifold. On the Euclidean complex space form there exist a conformal vector field, whose expression for its covariant derivative motivates the definition of a specific vector field on a Kaehler manifold, which we call a special conformal vector field. We show that a complete simply connected complex space form $M(c)$ (a Kaehler manifold of constant holomorphic sectional curvature c) admits a special conformal vector field if and only if it is isometric to the Euclidean complex space form. We also show that a complete simply connected Kaehler manifold (M, J, g) that admits a non-parallel harmonic special conformal vector field, is isometric to the Euclidean complex space form.

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1 Introduction

Characterizations of important spaces such as the Euclidean space $\mathbb{R}^n$, the Euclidean sphere $\mathbb{S}^n$, and the complex projective space $\mathbb{CP}^n$, is an important problem in differential geometry and was taken up by several authors (cf. [1], [8]-[14]). In most of these characterizations conformal vector fields play a notable role. Conformal vector fields are important objects for studying the geometry of several kinds of manifolds. They have been widely studied on Riemannian manifolds (cf. [2, 3, 5, 7, 10, 12, 14]). However, although the Kaehler geometry is quite rich, conformal vector fields have not been studied intensively on a Kaehler manifold. The main reason for this is perhaps the result "on a compact Kaehler manifold a conformal vector field is Killing" (cf. [8, 13]), however, on a non-compact Kaehler manifold the non-Killing conformal vector fields are in abundance. For example, consider the Euclidean space $\mathbb{C}^n$ of real dimension $2n$, which is a Kaehler manifold with natural canonical complex structure

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J. The vector field $\xi = \psi + J\psi$ is a conformal vector field on the Kaehler manifold $(\mathbb{C}^n, J, \langle,\rangle)$ which is not Killing, where ψ is the position vector field and $\langle,\rangle$ is the Euclidean metric on $\mathbb{C}^n$. If ∇ is the Levi-Civita connection on $(\mathbb{C}^n, J, \langle,\rangle)$, then we have

$$\nabla_X \xi = X + JX, \quad X \in \mathfrak{X}(\mathbb{C}^n),$$

where $\mathfrak{X}(\mathbb{C}^n)$ is the Lie algebra of smooth vector fields on $\mathbb{C}^n$. This conformal vector field ξ satisfies $\Delta\xi = 0$, where Δ is the Laplacian operator acting on smooth vector fields, that is, ξ is a harmonic vector field on $(\mathbb{C}^n, J, \langle,\rangle)$. A natural question can be raised "whether the existence of this vector field is a characteristic property of the Euclidean complex space form $(\mathbb{C}^n, J, \langle,\rangle)$?" The answer to this question is affirmative as seen in the following results, which is the aim of this paper. Motivated by this example, we call a vector field ξ on a Kaehler manifold (M, J, g) a special conformal vector field if it satisfies

$$\nabla_X \xi = \rho X + fJX, \quad X \in \mathfrak{X}(M),$$

for some smooth real functions $\rho, f \in C^\infty(M)$. It is interesting to note that on a complete simply connected complex space form $M(c)$ (Kaehler manifold of constant holomorphic sectional curvature c) the presence of a non-Killing special conformal vector field render it to be isometric to $(\mathbb{C}^n, J, \langle,\rangle)$. In fact we prove the following

Theorem 1.1. *Let $M(c)$ be a complete simply connected complex space form with* $\dim M > 2$. *Then $M(c)$ admits a non-Killing special conformal vector field if and only if it is isometric to* $(\mathbb{C}^n, J, \langle,\rangle)$.

For complete simply connected Kaehler manifold that admit a non-parallel harmonic special conformal vector field we prove the following

Theorem 1.2. *Let (M, J, g) be a complete simply connected Kaehler manifold with* $\dim M > 2$. *Then (M, J, g) admits a non-parallel harmonic special conformal vector field if and only if it is isometric to* $(\mathbb{C}^n, J, \langle,\rangle)$.

2 Preliminaries

Let (M, g) be an n-dimensional Riemannian manifold with metric g. We denote by ∇ the Levi-Civita connection and by $\mathfrak{X}(M)$ the Lie algebra of smooth vector fields on M. The curvature tensor field R and the Ricci tensor field Ric of M are defined by

$$\begin{aligned} R(X,Y)Z &= \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z, \\ \mathrm{Ric}(X,Y) &= \mathrm{tr}_g(Z \to R(Z,X)Y), \end{aligned}$$

where $X, Y, Z \in \mathfrak{X}(M)$, and tr_g denotes the trace with respect to g.

Recall that a smooth vector field $\xi \in \mathfrak{X}(M)$ is said to be a conformal vector field on M if

$$\mathcal{L}_\xi(g) = 2\rho g \,, \tag{2.1}$$

for a smooth function $\rho \in C^\infty(M)$, where $\mathcal{L}_\xi$ is the Lie derivative with respect to ξ. We call the smooth function ρ the *potential function* associated to conformal vector field ξ. Let us denote by η the 1-form dual to the conformal vector field ξ, then we can define a skew-symmetric tensor field ϕ of type $(1,1)$ on M by

$$d\eta(X,Y) = 2g(\phi X, Y), \quad X, Y \in \mathfrak{X}(M). \tag{2.2}$$

We will call ϕ the *skew-symmetric tensor* associated to the conformal vector field ξ.

We use the well known irreducible orthogonal decomposition

$$TM \otimes TM = \mathbb{R} \oplus S_0^2(M) \oplus \Lambda^2(M)$$

and the identification of the tangent bundle TM with the cotangent bundle T^*M using the Riemannian metric g, together with the fact that the covariant derivative $\nabla\xi$ is a section of the bundle $TM \otimes T^*M$, to deduce the following expression

$$\nabla_X \xi = \rho X + \phi X, \quad X \in \mathfrak{X}(M) \tag{2.3}$$

for a conformal vector field ξ on a Riemannian manifold (M, g) with potential function ρ and skew-symmetric tensor ϕ.

We recall that if S is an $(1,1)$ tensor then its divergence $\operatorname{div}(S) \in \mathfrak{X}(M)$ is the vector field defined by

$$g(\operatorname{div}(S), X) = \operatorname{tr}_g(Z \to (\nabla_Z S)(X)) = \sum_{i=1}^{n} g((\nabla_{e_i} S)(X), e_i), \quad X \in \mathfrak{X}(M),$$

where $\{e_1, \ldots, e_n\}$ is a local orthonormal frame on M (cf. [11]), and $(\nabla_X S)(Y) = \nabla_X SY - S\nabla_X Y$, for $X, Y \in \mathfrak{X}(M)$.

The Ricci operator Q is a symmetric $(1,1)$ tensor field defined by $\operatorname{Ric}(X,Y) = g(Q(X), Y)$, for $X, Y \in \mathfrak{X}(M)$.

Lemma 2.1. *Let ξ be a conformal vector field on a n-dimensional Riemannian manifold (M, g) with potential function ρ and skew-symmetric tensor ϕ. Then* $\operatorname{div}(\xi) = n\rho$ *and the Ricci operator Q of M satisfies*

$$Q(\xi) = -(n-1)\nabla\rho + \operatorname{div}(\phi).$$

Proof. Let $\{e_1, \ldots, e_n\}$ be a local orthonormal frame on M. Using equation (2.3) we get

$$\operatorname{div}(\xi) = \sum_i g(\nabla_{e_i}\xi, e_i) = \sum_i \rho g(e_i, e_i) + g(\phi e_i, e_i) = n\rho,$$

since ϕ is skew-symmetric. For the second claim we use again equation (2.3) to obtain

$$R(X,Y)\xi = X(\rho)Y - Y(\rho)X + (\nabla_X\phi)(Y) - (\nabla_Y\phi)(X), \quad X, Y \in \mathfrak{X}(M). \tag{2.4}$$

Consequently using (2.4), the definition of $\operatorname{div}(\phi)$, and the fact that $(\nabla_X\phi)$ is an skew-symmetric tensor, we get

$$\begin{aligned}\operatorname{Ric}(X,\xi) &= -(n-1)g(\nabla\rho, X) + \sum_i g((\nabla_{e_i}\phi)(X), e_i) - \sum_i g((\nabla_X\phi)(e_i), e_i) \\ &= -(n-1)g(\nabla\rho, X) + g(\operatorname{div}(\phi), X).\end{aligned}$$

$\square$

On a Riemannian manifold (M,g) the Hessian operator $\mathcal{H}_f$ of a smooth function $f \in C^\infty(M)$ is the $(1,1)$-tensor defined by

$$\mathcal{H}_f(X) = \nabla_X \nabla f, \quad X \in \mathfrak{X}(M).$$

The Hessian operator is symmetric and also can be viewed as the symmetric $(0,2)$-tensor given by $\mathcal{H}_f(X,Y) = g(\mathcal{H}_f(X),Y)$, for $X,Y \in \mathfrak{X}(M)$ (cf. [11, 4]). The Laplacian Δf of the smooth function f is related to the Hessian $\mathcal{H}_f$ by $\Delta f = \mathrm{tr}_g(\mathcal{H}_f)$. Recently García-Río and others [6] have initiated the study of the Laplacian operator $\Delta : \mathfrak{X}(M) \to \mathfrak{X}(M)$, defined on a Riemannian manifold (M,g) by

$$\Delta X = \sum_{i=1}^{n} \left(\nabla_{e_i} \nabla_{e_i} X - \nabla_{\nabla_{e_i} e_i} X \right), \quad X \in \mathfrak{X}(M),$$

where $\{e_1, \dots, e_n\}$ is a local orthonormal frame on M. This operator is a self adjoint elliptic operator with respect to the inner product $\langle , \rangle$ on the set $\mathfrak{X}_c(M)$ of compactly supported vector fields in $\mathfrak{X}(M)$, defined by

$$\langle X, Y \rangle = \int_M g(X,Y), \quad X, Y \in \mathfrak{X}_c(M).$$

A vector field X is said to be *harmonic* if $\Delta X = 0$. Note that a parallel vector field is harmonic, therefore we shall call a harmonic vector field *nontrivial harmonic vector field* if it is not parallel. We shall denote by Δ both the Laplacian operators, the one acting on smooth functions on M as well as that acting on smooth vector fields.

Let (M,J,g) be a $2n$-dimensional Kaehler manifold with complex structure J and Hermitian metric g. We denote by ∇ the Levi-Civita connection and by $\mathfrak{X}(M)$ the Lie algebra of smooth vector fields on M. Then we have

$$\nabla_X JY = J \nabla_X Y, \quad g(JX, JY) = g(X,Y), \quad X, Y \in \mathfrak{X}(M). \tag{2.5}$$

The curvature tensor field R and the Ricci tensor field Ric of a Kaehler manifold (M,J,g) satisfy

$$g(R(JX,JY)JZ,JW) = g(R(X,Y)Z,W), \quad \mathrm{Ric}(JX,JY) = \mathrm{Ric}(X,Y) \tag{2.6}$$

for $X, Y, Z, W \in \mathfrak{X}(M)$.

Lemma 2.2. *Let ξ be a conformal vector field on a $2n$-dimensional Kaehler manifold (M,J,g) with potential ρ and skew-symmetric tensor ϕ. Then*

$$2(n-1)\nabla\rho = \mathrm{div}(\phi) - \mathrm{div}(J\phi J) - J\nabla \mathrm{tr}_g(J\phi)\ .$$

Proof. We will compute $Q(J\xi)$. From (2.5) it follows that $R(Z,X)J\xi = JR(Z,X)\xi$, for $X, Z \in \mathfrak{X}(M)$. Let $\{e_1, \dots, e_{2n}\}$ be an orthonormal frame on M, then

$$\begin{aligned}\mathrm{Ric}(X, J\xi) &= \sum_i g(e_i(\rho)JX - X(\rho)Je_i + (\nabla_{e_i} J\phi)(X) - (\nabla_X J\phi)(e_i), e_i) \\ &= -g(J\nabla\rho, X) + g(\mathrm{div}(J\phi), X) - \sum_i g((\nabla_X J\phi)(e_i), e_i),\end{aligned}$$

where we have used (2.4) jointly with the fact that $J(\nabla_X\phi) = (\nabla_X J\phi)$, for $X \in \mathfrak{X}(M)$, and also that J is skew-symmetric. On the other hand we know that (cf. [11])

$$\sum_i g((\nabla_X J\phi)(e_i), e_i) = \mathrm{tr}_g((\nabla_X J\phi)) = X(\mathrm{tr}_g(J\phi)).$$

Therefore we have proved that

$$Q(J\xi) = -J\nabla\rho + \mathrm{div}(J\phi) - \nabla\mathrm{tr}_g(J\phi). \tag{2.7}$$

On a Kaehler manifold (M, J, g), it is well known that $Q(JX) = J(Q(X))$, for $X \in \mathfrak{X}(M)$, hence by Lemma 2.1 and (2.7) we obtain

$$2(n-1)\nabla\rho = \mathrm{div}(\phi) + J\mathrm{div}(J\phi) - J\nabla\mathrm{tr}_g(J\phi)\ .$$

Since $(\nabla_X\phi)(JY) = (\nabla_X\phi J)(Y)$, $X, Y \in \mathfrak{X}(M)$, we get that $J\mathrm{div}(J\phi) = -\mathrm{div}(J\phi J)$, which finishes the proof. □

Lemma 2.3. *Let $u \in C^\infty(M)$ be a smooth function on a $2n$-dimensional Kaehler manifold (M, J, g). Then* $\mathrm{div}(J\nabla u) = 0$.

Proof. Let $\{e_1, \dots, e_{2n}\}$ be an local orthonormal frame on M. We compute

$$\mathrm{div}(J\nabla u) = \sum_i g(\nabla_{e_i} J\nabla u, e_i) = \sum_i g(J\nabla_{e_i}\nabla u, e_i) = \sum_i g(J\mathcal{H}_u(e_i), e_i).$$

Since $\mathcal{H}_u$ is symmetric we can choose an orthonormal basis that diagonalizes $\mathcal{H}_u$, that is, $\mathcal{H}_u(e_i) = \kappa_i e_i$ for $i = 1, \dots, 2n$. Therefore $g(J\mathcal{H}_u(e_i), e_i) = \kappa_i g(Je_i, e_i) = 0$, which finishes the proof. □

A Kaehler manifold of constant holomorphic sectional curvature c is called a complex space form and it is denoted by $M(c)$. The curvature tensor field of a complex space form $M(c)$ has the expression

$$\begin{aligned} R(X,Y)Z = \frac{c}{4}\{&g(Y,Z)X - g(X,Z)Y + g(JY,Z)JX \\ &-g(JX,Z)JY + 2g(X,JY)JZ\}\,. \end{aligned} \tag{2.8}$$

Now consider the Kaehler manifold $(\mathbb{C}^n, J, \langle,\rangle)$, where $\mathbb{C}^n = \mathbb{R}^{2n}$ is the $2n$-dimensional Euclidean space, J and $\langle,\rangle$ are the canonical complex structure and the Euclidean metric on $\mathbb{C}^n$, respectively. We denote the position vector field of $\mathbb{C}^n$ by ψ. Then on the Kaehler manifold $(\mathbb{C}^n, J, \langle,\rangle)$, the vector field $\xi = \psi + J\psi$ is a conformal vector field, as one can verify that it satisfies

$$\nabla_X\xi = X + JX, \quad X \in \mathfrak{X}(\mathbb{C}^n),$$

where the potential function $\rho = 1$ is a constant, the skew-symmetric tensor field $\phi = J$, and that ξ is not Killing. Motivated by this vector field, and the expression of its covariant derivative, we define what we call a *special conformal vector field* on a Kaehler manifold in the following

Definition 2.1. Let (M, J, g) be a Kaehler manifold. A conformal vector field $\xi \in \mathfrak{X}(M)$ with skew-symmetric tensor ϕ is said to be a special conformal vector field if $\phi = fJ$ for some smooth function $f \in C^\infty(M)$.

We would like to point out that the special conformal vector fields in the above definition are particular case of the vector fields considered in [9]. We also remark that for a special conformal vector field ξ on a Kaehler manifold (M, J, g) it follows from $\phi = fJ$ that $J\phi J = -\phi$ and therefore

$$\operatorname{div}(\phi) = -J\nabla f, \qquad \operatorname{div}(J\phi J) = J\nabla f, \qquad \operatorname{tr}_g(J\phi) = -2nf. \tag{2.9}$$

Hence from Lemma 2.2 we get

$$(n-1)\{\nabla\rho - J\nabla f\} = 0,$$

which proves the following

Lemma 2.4. *Let ξ be a special conformal vector field on a $2n$-dimensional Kaehler manifold (M, J, g). Then either $n = 1$ or $\nabla\rho = J\nabla f$.*

3 Proof of theorems

Initially we will prove in the following proposition a slightly more general result than was enunciated in Theorem 1.1.

Proposition 3.1. *Let ξ be a non-Killing conformal vector field with potential ρ and skew-symmetric tensor ϕ on a complete simply connected complex space form $M(c)$ with $\dim M > 2$. If $\operatorname{div}(\phi) = J\nabla u$ and $\operatorname{div}(J\phi J) = J\nabla v$ for some smooth functions $u, v \in C^\infty(M)$, then $M(c)$ is isometric to the Euclidean complex space $(\mathbb{C}^n, J, \langle , \rangle)$.*

Proof. As $M(c)$ has constant holomorphic curvature c it follows from (2.8) that $\operatorname{Ric}(X, Y) = \frac{c}{2}(n+1)g(X, Y)$ which gives

$$Q(\xi) = \frac{c}{2}(n+1)\xi\ . \tag{3.1}$$

By using that $\operatorname{div}(\phi) = J\nabla u$ and (3.1) in Lemma 2.1 we obtain

$$-(2n-1)\nabla\rho + J\nabla u = Q(\xi) = \frac{c}{2}(n+1)\xi.$$

On taking the divergence in the above equation and using Lemma 2.1, the equation (2.3), we get

$$-(2n-1)\Delta\rho = n(n+1)c\rho. \tag{3.2}$$

On the other hand using that $\operatorname{div}(\phi) = J\nabla u$ and $\operatorname{div}(J\phi J) = J\nabla v$ in Lemma 2.2 we have

$$2(n-1)\nabla\rho = J\nabla(u - v - \operatorname{tr}_g(J\phi)).$$

Consequently, by taking the divergence in the above equation and using Lemma 2.3 we arrive at $2(n-1)\Delta\rho = 0$. Thus as $\dim M > 2$ we obtain $\Delta\rho = 0$, and hence equation (3.2) gives either $\rho = 0$ or $c = 0$. If the potential function $\rho = 0$, by equation (2.1) it would mean that ξ is Killing, which is a contradiction as ξ is a non-Killing vector field. Hence $c = 0$ and this proves that $M(c)$ is isometric to $(\mathbb{C}^n, J, \langle , \rangle)$. $\square$

Now the proof of Theorem 1.1 follows easily from Proposition 3.1. In fact, as already observed in (2.9) a special conformal vector field ξ satisfies $\operatorname{div}(\phi) = -J\nabla f$ and $\operatorname{div}(J\phi J) = J\nabla f$ and hence from Proposition 3.1 we conclude that $M(c)$ is isometric to $(\mathbb{C}^n, J, \langle , \rangle)$. The converse of Theorem 1.1 is trivial since $\xi = \psi + J\psi$, where ψ is position vector field of $\mathbb{C}^n$, is a non-Killing special conformal vector field on $(\mathbb{C}^n, J, \langle , \rangle)$. This ends the proof of Theorem 1.1.

A direct consequence of Theorem 1.1 is the following

Corollary 3.2. *On a hyperbolic complex space form there does not exist a non-Killing special conformal vector field.*

Now we will prove Theorem 1.2. Suppose that ξ is a nontrivial (non-parallel) harmonic special conformal vector field on a $2n$-dimensional Kaehler manifold (M, J, g). Then the skew-symmetric tensor of ξ satisfies $\phi = fJ$ for some smooth function $f \in C^\infty(M)$. In that case equation (2.3) allow us to write

$$\nabla_X \xi = \rho X + fJX, \quad X \in \mathfrak{X}(M). \tag{3.3}$$

Using the above equation we easily get

$$\nabla_X \nabla_Y \xi - \nabla_{\nabla_X Y} \xi = X(\rho)Y + X(f)JY, \quad X, Y \in \mathfrak{X}(M).$$

Also, using the definition of the Laplacian $\Delta\xi$ in the above equation we get

$$\Delta\xi = \nabla\rho + J\nabla f.$$

Since ξ is harmonic, the above equation together with Lemma 2.4 (as $\dim M > 2$) gives $\nabla\rho = \nabla f = 0$, that is, the functions ρ and f are constants. Thus there exist constants c_1, c_2 such that the equation (3.3) acquires the form

$$\nabla_X \xi = c_1 X + c_2 JX, \quad X \in \mathfrak{X}(M). \tag{3.4}$$

Define a smooth function $h \in C^\infty(M)$ by

$$h = \frac{1}{2} g(\xi, \xi).$$

It is easy to see that the gradient of the function h and its Hessian operator are respectively given by

$$\nabla h = c_1\xi - c_2 J\xi, \quad \text{and} \quad \mathcal{H}_h(X) = c_1 \nabla_X \xi - c_2 J \nabla_X \xi, \quad X \in \mathfrak{X}(M). \tag{3.5}$$

Therefore by using (3.4) we get

$$\mathcal{H}_h(X) = (c_1^2 + c_2^2)X = cX, \quad X \in \mathfrak{X}(M). \tag{3.6}$$

Suppose the constant $c = 0$, then by equation (3.4) we shall have $\nabla_X \xi = 0$ for all $X \in \mathfrak{X}(M)$ that will mean that ξ is a parallel vector field which is a contradiction, as ξ is a nontrivial harmonic vector field. Now suppose that the function h is constant, then from the first equation in (3.5) we get $0 = \nabla h = c_1\xi - c_2 J\xi$, and as both c_1 and c_2 cannot be simultaneously zero, we get $\xi = 0$, which again is a contradiction as ξ is a nontrivial harmonic vector field. Thus the equation (3.6) gives that the non-constant function h on the complete and simply connected Kaehler manifold (M, J, g) satisfies

$$\mathcal{H}_h(X, Y) = cg(X, Y), \quad X, Y \in \mathfrak{X}(M)$$

for a nonzero constant c, which is Tashiro's equation (cf. [14]), which confirms that (M, J, g) is isometric to the Euclidean space $\mathbb{C}^n$.

The converse of Theorem 1.2 is trivial, since the vector field decomposes as $\xi = \psi + J\psi$, where ψ is the position vector field of $\mathbb{C}^n$, is a nontrivial harmonic special conformal vector field on $\mathbb{C}^n$. $\square$

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