Conformal isoparametric spacelike hypersurfaces in conformal space $\mathbb{Q}_1^{n+1}$

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Abstract. In this paper, we study the conformal geometry of conformal isoparametric spacelike hypersurfaces in conformal space $\mathbb{Q}_1^{n+1}$. We obtain the classification of the conformal isoparametric spacelike hypersurfaces in $\mathbb{Q}_1^{n+1}$ with three distinct conformal principal curvatures, one of which is simple, and the classification of the conformal isoparametric spacelike hypersurfaces in $\mathbb{Q}_1^6$.

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1 Introduction

Let $\langle,\rangle_s$ be the Lorentzian inner product with s negative index of the $(n+s)$-dimensional Euclidean space $\mathbb{R}^{n+s}$; we denote

$$\langle X, Y\rangle_s = \sum_{i=1}^{n} x_i y_i - \sum_{i=n+1}^{n+s} x_i y_i, \quad \forall X = (x_i), Y = (y_i) \in \mathbb{R}^{n+s}.$$

Let $\mathbb{R}P^{n+2}$ be the $(n+2)$-dimensional real projective space. The quadric surface $\mathbb{Q}_1^{n+1} = \{[\xi] \in \mathbb{R}P^{n+2} | \langle \xi, \xi\rangle_2 = 0\}$ is called *conformal space.* We denote *the Lorentzian space forms* (the Lorentzian space, the de Sitter sphere and the anti-de Sitter sphere), respectively, as follows:

$$\begin{aligned} \mathbb{R}_1^{n+1} &= (\mathbb{R}^{n+1}, \langle,\rangle_1), \\ \mathbb{S}_1^{n+1} &= \{u \in \mathbb{R}^{n+2} | \langle u, u\rangle_1 = 1\}, \\ \mathbb{H}_1^{n+1} &= \{u \in \mathbb{R}^{n+2} | \langle u, u\rangle_2 = -1\}. \end{aligned}$$

We denote as well $\pi = \{[x] \in \mathbb{Q}_1^{n+1} | x_1 = x_{n+3}\}$, $\pi_+ = \{[x] \in \mathbb{Q}_1^{n+1} | x_{n+3} = 0\}$ and $\pi_- = \{[x] \in \mathbb{Q}_1^{n+1} | x_1 = 0\}$. We shall further consider the conformal diffeomorphisms

$$\begin{aligned} &\sigma_0 : \mathbb{R}_1^n \to \mathbb{Q}_1^{n+1} \backslash \pi, && u \mapsto [(\tfrac{\langle u,u\rangle - 1}{2}, u, \tfrac{\langle u,u\rangle + 1}{2})]; \\ &\sigma_1 : \mathbb{S}_1^{n+1} \to \mathbb{Q}_1^{n+1} \backslash \pi_+, && u \mapsto [(u, 1)]; \\ &\sigma_{-1} : \mathbb{H}_1^{n+1} \to \mathbb{Q}_1^{n+1} \backslash \pi_-, && u \mapsto [(1, u)]. \end{aligned}$$

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From [11], we may regard $\mathbb{Q}_1^{n+1}$ as the common compactified space of $\mathbb{R}_1^{n+1}$, $\mathbb{S}_1^{n+1}$ and $\mathbb{H}_1^{n+1}$, while $\mathbb{R}_1^{n+1}$, $\mathbb{S}_1^{n+1}$ and $\mathbb{H}_1^{n+1}$ are regarded as the subsets of $\mathbb{Q}_1^{n+1}$.

Let $x : M \to \mathbb{Q}_1^{n+1}$ be an n-dimensional immersed conformal regular spacelike hypersurface in the conformal space $\mathbb{Q}_1^{n+1}$. From [9], we know that the conformal metric of the immersion x can be defined by

$$g = \frac{n}{n-1}\left(\sum_{i,j} h_{ij}^2 - nH^2\right)\langle dx, dx\rangle := e^{2\tau}\langle dx, dx\rangle,$$

which is a conformal invariant. Let

$$\Phi = \sum_{i=1}^{n} e^{\tau} C_i\theta_i,\ \mathbf{A} = \sum_{i,j=1}^{n} e^{2\tau} A_{ij}\theta_i \otimes \theta_j,\ \mathbf{B} = \sum_{i,j=1}^{n} e^{2\tau} B_{ij}\theta_i \otimes \theta_j, \mathbf{D} = \mathbf{A} + \lambda\mathbf{B},$$

where λ is a constant. We call Φ, $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{D}$ *the conformal form*, *the conformal Blaschke tensor*, *the conformal second fundamental form* and *the conformal para-Blaschke tensor* of the immersion x, respectively. It is known that Φ, $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{D}$ are conformal invariants.

The conformal geometry of regular hypersurfaces in the conformal space is determined by the conformal metric. The negative index of the conformal space $\mathbb{Q}_1^{n+1}$ is 1. If the negative index is degenerate, then we obtain the Möbius geometry in the unit sphere, which has been studied by many authors (see [1]-[5],[7]-[15]). An eigenvalue of the conformal second fundamental form $\mathbf{B}$, the conformal Blaschke tensor $\mathbf{A}$ and the conformal para-Blaschke tensor $\mathbf{D}$, are respectively called *conformal principal curvature*, *Blaschke eigenvalue* or *para-Blaschke eigenvalue* of the immersion x. A regular spacelike hypersurface $x : M \to \mathbb{Q}_1^{n+1}$ is called *conformal isoparametric spacelike hypersurface*, if $\Phi \equiv 0$ and the conformal principal curvatures of the immersion x are constant.

C.X. Nie et al. studied the conformal geometry of conformal isoparametric spacelike hypersurfaces in the conformal space $\mathbb{Q}_1^{n+1}$ and obtained the following (see [11]):

Theorem 1.1. *If $x : M \to \mathbb{Q}_1^{n+1}$ is a conformal isoparametric spacelike hypersurface with two distinct principal curvatures, then x is conformally equivalent to an open part of the following standard embeddings:*

(i) the Riemannian product $\mathbb{S}^m(c) \times \mathbb{H}^{n-m}(\sqrt{c^2 - r^2})$ in $\mathbb{S}_1^{n+1}(r)$, $c > r$; or

(ii) the Riemannian product $\mathbb{R}^m \times \mathbb{H}^{n-m}(r)$ in $\mathbb{R}_1^{n+1}$; or

(iii) the Riemannian product $\mathbb{H}^m(c) \times \mathbb{H}^{n-m}(\sqrt{r^2 - c^2})$ in $\mathbb{H}_1^{n+1}(r)$, $0 < c < r$; where $r^2 = \frac{n-1}{m(n-m)}$.

Recently, the first author and Su [14] obtained the classification of conformal isoparametric spacelike hypersurfaces in $\mathbb{Q}_1^4$ and $\mathbb{Q}_1^5$. In this paper, we continue to study the topic of conformal isoparametric spacelike hypersurfaces in $\mathbb{Q}_1^{n+1}$. We obtain the classification of the conformal isoparametric spacelike hypersurfaces in $\mathbb{Q}_1^{n+1}$ with three distinct conformal principal curvatures, one of which is simple, and the classification of the conformal isoparametric spacelike hypersurfaces in $\mathbb{Q}_1^6$.

Theorem 1.2. *Let $x : M \to \mathbb{Q}_1^{n+1}$ $(n \geq 3)$ be a conformal isoparametric spacelike hypersurface in $\mathbb{Q}_1^{n+1}$ with three distinct conformal principal curvatures, one of which* *is simple. Then x is conformally equivalent to an open part of the spacelike hypersurface $WP(p,q,c)$ given by Example 3.1, where p,q,c are some constants, $p \geq 1, q \geq 1$, $p+q<n$ and $qc^4+pd^4=(qc^2+pd^2)^2$, $d=\sqrt{c^2-1}$.*

Theorem 1.3. *Let $x : M \to \mathbb{Q}_1^6$ be a conformal isoparametric spacelike hypersurface in $\mathbb{Q}_1^6$. Then*

a) x is conformally equivalent to an open part of the standard embeddings:

(i) the Riemannian product $\mathbb{S}^m(c)\times\mathbb{H}^{5-m}(\sqrt{c^2-r^2})$ in $\mathbb{S}_1^6(r)$, $c>r$, $m=1,2,3,4$, or

(ii) the Riemannian product $\mathbb{R}^m\times\mathbb{H}^{5-m}(r)$ in $\mathbb{R}_1^6$, $m=1,2,3,4$, or

(iii) the Riemannian product $\mathbb{H}^m(c)\times\mathbb{H}^{5-m}(\sqrt{r^2-c^2})$ in $\mathbb{H}_1^6(r)$, $0<c<r$, $m=1,2,3,4$, where $r^2=\frac{4}{m(5-m)}$; or

(iv) the spacelike hypersurface $WP(p,q,c)$ given by Example 3.1, where p,q,c are some constants, $p\geq 1, q\geq 1$, $p+q<5$ and $qc^4+pd^4=(qc^2+pd^2)^2$, $d=\sqrt{c^2-1}$;

b) x is locally a Riemannian product $M_1^m\times M_2^{5-m}$, $m=3,4$, where M_2^{5-m} is a constant curvature Riemannian manifold.

2 Fundamental formulas on conformal geometry

We firstly review the fundamental formulas on conformal geometry of spacelike hypersurfaces in $\mathbb{Q}_1^{n+1}$, and use the following range of indices throughout this paper: $1\leq i,j,k,l,m\leq n$ (for more details, see [11] or [14]).

Let $x : M \to \mathbb{Q}_1^{n+1}$ be an n-dimensional conformal regular spacelike hypersurface with $\Phi\equiv 0$ in $\mathbb{Q}_1^{n+1}$. From the structure equations on M (see [11]), we have

$$\omega_{ij}+\omega_{ji}=0,\quad d\omega_i=\sum_j \omega_{ij}\wedge\omega_j, \tag{2.1}$$

$$e^{2\tau}C_i=H\tau_i-H_i-\sum_j h_{ij}\tau_j,\quad e^{\tau}B_{ij}=h_{ij}-HI_{ij}, \tag{2.2}$$

$$e^{2\tau}A_{ij}=\tau_i\tau_j-\tau_{i,j}-Hh_{ij}-\frac{1}{2}\left(\sum_k \tau^k\tau_k-H^2-\epsilon\right)I_{ij}, \tag{2.3}$$

$$d\omega_{ij}=\sum_k \omega_{ik}\wedge\omega_{kj}-\frac{1}{2}\sum_{k,l}R_{ijkl}\omega_k\wedge\omega_l,\quad R_{ijkl}=-R_{jikl}, \tag{2.4}$$

$$\sum_i B_{ii}=0,\quad \sum_{i,j}B_{ij}^2=\frac{n-1}{n},\quad \mathrm{tr}\mathbf{A}=\frac{1}{2n}(n^2\kappa-1), \tag{2.5}$$

$$A_{ij,k}-A_{ik,j}=B_{ij}C_k-B_{ik}C_j,\quad B_{ij,k}-B_{ik,j}=\delta_{ij}C_k-\delta_{ik}C_j, \tag{2.6}$$

$$C_{i,j}-C_{j,i}=\sum_k (B_{ik}A_{kj}-B_{kj}A_{ki}), \tag{2.7}$$

$$R_{ijkl}=-(B_{ik}B_{jl}-B_{il}B_{jk})+\delta_{ik}A_{jl}+\delta_{jl}A_{ik}-\delta_{il}A_{jk}-\delta_{jk}A_{il}, \tag{2.8}$$

where R_{ijkl} denotes the curvature tensor with respect to the conformal metric g on M. Since the conformal form $\Phi \equiv 0$, we have for all indices i, j, k

$$A_{ij,k} = A_{ik,j}, \quad B_{ij,k} = B_{ik,j}, \quad \sum_k B_{ik}A_{kj} = \sum_k B_{kj}A_{ki}. \tag{2.9}$$

The conformal $(0, 2)$ para-Blaschke tensor is denoted by $\mathbf{D} = \sum_{i,j} D_{ij}\omega_i \otimes \omega_j$,

$$D_{ij} = L_{ij} + \lambda B_{ij}, \quad 1 \le i, j \le n, \tag{2.10}$$

where λ is a constant. From (2.9) and (2.10), we have for all indices i, j, k that $D_{ij,k} = D_{ik,j}$.

3 Some results and examples

From Nie and Wu [10], Shu and Su [14], Nomizu [13], Li and Xie [6], we have the following:

Theorem 3.1. *(see [10]). If $x : M \to \mathbb{Q}_1^{n+1}$ is a conformal regular spacelike hypersurface in $\mathbb{Q}_1^{n+1}$ with parallel conformal second fundamental form, then x is conformally equivalent to an open part of these standard embeddings:*

(i) the Riemannian product $\mathbb{S}^m(a) \times \mathbb{H}^{n-m}(\sqrt{a^2 - r^2})$ in $\mathbb{S}_1^{n+1}(r)$, $a > r$; or

(ii) the Riemannian product $\mathbb{R}^m \times \mathbb{H}^{n-m}(r)$ in $\mathbb{R}_1^{n+1}$; or

(iii) the Riemannian product $\mathbb{H}^m(a) \times \mathbb{H}^{n-m}(\sqrt{r^2 - a^2})$ in $\mathbb{H}_1^{n+1}(r)$, $0 < a < r$, where $r^2 = \frac{n-1}{m(n-m)}$; or

(iv) the spacelike hypersurface $x = \sigma_0 \circ u : \mathbb{S}^p(c) \times \mathbb{R}^+ \times \mathbb{R}^{n-p-q-1} \times \mathbb{H}^q(d) \to \mathbb{Q}_1^{n+1}$ with $d = \sqrt{c^2 - 1}$, $p \ge 1, q \ge 1$, $p + q < n$, where

$$u : \mathbb{S}^p(c) \times \mathbb{R}^+ \times \mathbb{R}^{n-p-q-1} \times \mathbb{H}^q(d) \to \mathbb{R}_1^{n+2} \subset \mathbb{R}_1^{n+1}, \quad u(u', t, u'', u''') = (tu', u'', tu'''),$$

for all $u' \in \mathbb{S}^p(c)$, $t \in \mathbb{R}^+$, $u'' \in \mathbb{R}^{n-p-q-1}$, $u''' \in \mathbb{H}^q(d)$.

Proposition 3.2. *(see [14]) Let $x : M \to \mathbb{Q}_1^{n+1}$ be an n-dimensional conformal isoparametric spacelike hypersurface in $\mathbb{Q}_1^{n+1}$ with constant normalized conformal scalar curvature κ and $\kappa \ne 1$. Then x is an n-dimensional Euclidean isoparametric spacelike hypersurface.*

Proposition 3.3. *(see [13], [6]). Let x be a Euclidean isoparametric spacelike hypersurface in Lorentzian space form. Then x can have at most two distinct Euclidean principal curvatures.*

Example 3.1. (see [10]). For any natural number p, q, $p + q < n$ and real number $c \in (1, +\infty)$ and $d = \sqrt{c^2 - 1}$, consider the immersed hypersurface $u : \mathbb{S}^p(c) \times \mathbb{R}^+ \times \mathbb{R}^{n-p-q-1} \times \mathbb{H}^q(d) \to \mathbb{R}_1^{n+2} \subset \mathbb{R}_1^{n+1}$: $u(u', t, u'', u''') = (tu', u'', tu''')$, $u' \in \mathbb{S}^p(c)$, $t \in \mathbb{R}^+$, $u'' \in \mathbb{R}^{n-p-q-1}$, $u''' \in \mathbb{H}^q(d)$, then $x = \sigma_0 \circ u : \mathbb{S}^p(c) \times \mathbb{R}^+ \times \mathbb{R}^{n-p-q-1} \times \mathbb{H}^q(d) \to \mathbb{Q}_1^{n+1}$ is a conformal regular spacelike hypersurface in $\mathbb{Q}_1^{n+1}$, which is denoted by $WP(p, q, c) = x(\mathbb{S}^p(c) \times \mathbb{R}^+ \times \mathbb{R}^{n-p-q-1} \times \mathbb{H}^q(d))$. From [10], by a direct calculation, we know that $WP(p, q, c)$ has three distinct constant conformal principal curvatures and the conformal second fundamental form is parallel. We may also calculate that $WP(p,q,c)$ is of parallel conformal Blaschke tensor. Thus, the conformal Blaschke eigenvalues are constants, from (2.5), we know that the normalized conformal scalar curvature κ is constant. If $\kappa \neq 1$, from Proposition 3.2 and (2.2), we see that $WP(p,q,c)$ is of three distinct constant Euclidean principal curvatures, this contradicts Proposition 3.3. If $\kappa = 1$, we know that the normalized Euclidean scalar curvature $R = \kappa = 1$. From Gaussian equation $n(n-1)(R-1) = \sum_{i,j} h_{ij}^2 - n^2H^2$, we see that $\sum_{i,j} h_{ij}^2 = n^2H^2$, this is equivalent to $qc^4 + pd^4 = (qc^2 + pd^2)^2$ (see Example 2.1 of [10]).

Example 3.2. (see [14]). Spacelike hypersurface $x : \mathbb{S}^m(c) \times \mathbb{H}^{n-m}(\sqrt{c^2 - r^2}) \to \mathbb{S}_1^{n+1}(r)$, $r < c$. Let $x = (x_1, x_2) \in \mathbb{S}^m(c) \times \mathbb{H}^{n-m}\left(\sqrt{c^2 - r^2}\right) \subset \mathbb{R}_1^{m+1} \times \mathbb{R}_1^{n-m+1}$, $\langle x_1, x_1\rangle = c^2$, $\langle x_2, x_2\rangle = -\left(c^2 - r^2\right)$. By a direct calculation, we see that x has two distinct principal curvatures $\frac{d}{r}$ and $\frac{1}{rd}$ with multiplicities m and $n-m$ and the conformal second fundamental form of x is parallel, where $d = \frac{\sqrt{c^2 - r^2}}{c}$.

Example 3.3. (see [14]). Spacelike hypersurface $x : \mathbb{R}^m \times \mathbb{H}^{n-m}(r) \to \mathbb{R}_1^{n+1}$. Let $x = (x_1, x_2)$, $x_1 \in \mathbb{R}^m$, $x_2 \in \mathbb{H}^{n-m}(r) \subset \mathbb{R}_1^{n-m+1}$, $\langle x_2, x_2\rangle = -r^2$. By a direct calculation, we see that x has two distinct principal curvatures 0 and $-\frac{1}{r}$ with multiplicities m and $n-m$ and the conformal second fundamental form of x is parallel.

Example 3.4. (see [14]). Spacelike hypersurface $x : \mathbb{H}^m(c) \times \mathbb{H}^{n-m}(\sqrt{r^2 - c^2}) \to \mathbb{H}_1^{n+1}(r)$, $0 < c < r$. Let $x = (x_1, x_2) \in \mathbb{H}^m(c) \times \mathbb{H}^{n-m}\left(\sqrt{r^2 - c^2}\right) \subset \mathbb{R}_1^{m+1} \times \mathbb{R}_1^{n-m+1}$, $\langle x_1, x_1\rangle = -c^2$, $\langle x_2, x_2\rangle = -\left(r^2 - c^2\right)$. By a direct calculation, we see that x has two distinct principal curvatures $\frac{d}{r}$ and $-\frac{1}{rd}$ with multiplicities m and $n-m$ and the conformal second fundamental form of x is parallel, where $d = \frac{\sqrt{r^2 - c^2}}{c}$.

4 Proof of theorem 1.2

Throughout this section, we shall make the following convention on the ranges of indices: $1 \leq a, b \leq m_1$, $m_1 + 1 \leq p, q \leq m_1 + m_2$, $m_1 + m_2 + 1 \leq \alpha, \beta \leq m_1 + m_2 + m_3 = n$, $1 \leq i, j, k \leq n$. Let A, B and D denote the $n \times n$-symmetric matrices (A_{ij}), (B_{ij}) and (D_{ij}), respectively. From (2.9) and (2.10), we know that $BA = AB$, $DA = AD$ and $BD = DB$. Thus, we may always choose a local orthonormal basis $\{E_1, E_2, \ldots, E_n\}$ such that

$$A_{ij} = A_i\delta_{ij}, \quad B_{ij} = B_i\delta_{ij}, \quad D_{ij} = D_i\delta_{ij}, \tag{4.1}$$

where A_i, B_i and D_i are the conformal Blaschke eigenvalues, the conformal principal curvatures and the conformal para-Blaschke eigenvalues of the immersion x.

Proof of Theorem 1.2. If the conformal second fundamental form of x is parallel, since x has three distinct conformal principal curvatures, from Theorem 3.1, Example 3.1–Example 3.4, we know that x is conformally equivalent to an open part of the spacelike hypersurface $WP(p,q,c)$ for some constants p, q, c given by Example 3.1.

If the conformal second fundamental form of x is not parallel, denote by B_1, B_2 and B_3 the three distinct constant conformal principal curvatures of x with multiplicities

m_1, m_2 and m_3, from the definition of the covariant derivative of B_{ij} (see (2.7) of [14]), we have

$$\sum_k B_{ij,k}\omega_k = (B_i - B_j)\omega_{ij}, \quad B_{ij,k} = \Gamma^j_{ik}(B_i - B_j), \tag{4.2}$$

where Γ^j_{ik} is the Levi-Civita connection for the conformal metric g given by $\omega_{ij} = \sum_k \Gamma^j_{ik}\omega_k$, $\Gamma^j_{ik} = -\Gamma^i_{jk}$. By (4.2), it follows that for any $a, b, p, q, \alpha, \beta, k$

$$B_{ab,k} = B_{pq,k} = B_{\alpha\beta,k} = 0. \tag{4.3}$$

Since the conformal second fundamental form is not parallel, we see that the only possible non-zero elements in $\{B_{ij,k}\}$ are of the form $\{B_{ap,\alpha}\}$. Since $n \geq 3$, without loss of generality, we may assume that $m_3 = 1$, $m_1 \geq 1$ and $m_2 \geq 1$.

From (2.4) and (2.1), the curvature tensor of x may be given by (see [8])

$$R_{ijkl} = E_l(\Gamma^j_{ik}) - E_k(\Gamma^j_{il}) + \sum_m(\Gamma^j_{im}\Gamma^m_{lk} - \Gamma^j_{im}\Gamma^m_{kl} + \Gamma^m_{ik}\Gamma^j_{ml} - \Gamma^m_{il}\Gamma^j_{mk}). \tag{4.4}$$

Thus, from (4.2) and (4.3), we have

$$\Gamma^p_{ab} = \Gamma^\alpha_{ab} = 0, \quad \Gamma^a_{pq} = \Gamma^\alpha_{pq} = 0, \quad \Gamma^a_{\alpha\beta} = \Gamma^p_{\alpha\beta} = 0, \tag{4.5}$$

$$\Gamma^p_{a\alpha} = \frac{B_{ap,\alpha}}{B_1 - B_2}, \quad \Gamma^a_{\alpha p} = \frac{B_{\alpha a,p}}{B_3 - B_1}, \quad \Gamma^\alpha_{pa} = \frac{B_{p\alpha,a}}{B_2 - B_3}. \tag{4.6}$$

From (4.5) and (4.6), we have

$$\Gamma^a_{nn} = \Gamma^p_{nn} = 0, \quad \Gamma^n_{aa} = \Gamma^n_{pp} = 0, \tag{4.7}$$

$$\Gamma^p_{an} = \frac{B_{ap,n}}{B_1 - B_2}, \quad \Gamma^p_{nb} = \frac{B_{bp,n}}{B_3 - B_2}, \quad \Gamma^n_{bq} = \frac{B_{bq,n}}{B_1 - B_3}, \quad \Gamma^n_{qb} = \frac{B_{bq,n}}{B_2 - B_3}. \tag{4.8}$$

Thus, from (4.4), we have

$$R_{apbq} = \Gamma^p_{an}\Gamma^n_{qb} - \Gamma^p_{an}\Gamma^n_{bq} - \Gamma^n_{aq}\Gamma^p_{nb} = \frac{B_{ap,n}B_{bq,n} + B_{aq,n}B_{bp,n}}{(B_1 - B_3)(B_2 - B_3)}. \tag{4.9}$$

On the other hand, from (2.8), we have

$$R_{apbq} = (-B_aB_p + A_a + A_p)\delta_{ab}\delta_{pq}. \tag{4.10}$$

It follows from (4.9) and (4.10) that

$$\frac{B_{ap,n}B_{bq,n} + B_{aq,n}B_{bp,n}}{(B_1 - B_3)(B_2 - B_3)} = (-B_aB_p + A_a + A_p)\delta_{ab}\delta_{pq}, \tag{4.11}$$

$$\frac{2B_{ap,n}B_{aq,n}}{(B_1 - B_3)(B_2 - B_3)} = (-B_1B_2 + A_a + A_p)\delta_{pq}, \quad \text{if} \quad a = b, \tag{4.12}$$

$$\frac{2B_{ap,n}B_{bp,n}}{(B_1 - B_3)(B_2 - B_3)} = (-B_1B_2 + A_a + A_p)\delta_{ab}, \quad \text{if} \quad p = q, \tag{4.13}$$

$$\frac{2B_{1p,n}B_{1q,n}}{(B_1 - B_3)(B_2 - B_3)} = (-B_1B_2 + A_1 + A_p)\delta_{pq}, \quad \text{if} \quad m_1 = 1. \tag{4.14}$$

Since the conformal second fundamental form is not parallel, we may prove that there exists exactly one p, such that $B_{1p,n} \neq 0$. In fact, if there exist at least two p_1, p_2, $(p_1 \neq p_2)$ such that $B_{1p_1,n} \neq 0$, $B_{1p_2,n} \neq 0$, from (4.14), we have $B_{1p_1,n}B_{1p_2,n} = 0$, this follows that $B_{1p_1,n} = 0$, or $B_{1p_2,n} = 0$, a contradiction. Thus, we know that there exists exactly one p, such that $B_{1p,n} \neq 0$.

If $m_2 = 1$, it follows that $\frac{2B_{am_1+1,n}B_{bm_1+1,n}}{(B_1-B_3)(B_2-B_3)} = (-B_1B_2 + A_a + A_{m_1+1})\delta_{ab}$. The same reason implies that there exists exactly one a, such that $B_{am_1+1,n} \neq 0$.

If $m_1 \geq 2$ and $m_2 \geq 2$, we may prove that there exists exactly one a and exactly one p such that $B_{ap,n} \neq 0$. In fact, if there exist at least two a_1, a_2, $(a_1 \neq a_2)$ such that $B_{a_1p,n} \neq 0$, $B_{a_2p,n} \neq 0$, from (4.13), we see that $B_{a_1p,n}B_{a_2p,n} = 0$, this follows that $B_{a_1p,n} = 0$, or $B_{a_2p,n} = 0$, a contradiction. Thus, we know that there exists exactly one a, such that $B_{ap,n} \neq 0$. By the same reason, we may prove that there exists exactly one p, such that $B_{ap,n} \neq 0$.

Combining the above three cases, we see that if $m_1 \geq 1$ and $m_2 \geq 1$, there exists exactly one a and exactly one p, say a_1 and p_1, such that

$$B_{a_1p_1,n} \neq 0, \quad B_{ap,n} = 0, \quad \text{for } a \neq a_1, \forall p, \quad \text{or for } \forall a, p \neq p_1. \tag{4.15}$$

By (4.10), (4.12) and (4.15), we get

$$R_{a_1p_1a_1p_1} = -B_1B_2 + A_{a_1} + A_{p_1} = \frac{2B^2_{a_1p_1,n}}{(B_1 - B_3)(B_2 - B_3)}, \tag{4.16}$$

$$R_{apap} = -B_1B_2 + A_a + A_p = 0, \quad a \neq a_1, \quad p \neq p_1, \tag{4.17}$$

$$R_{ap_1ap_1} = -B_1B_2 + A_a + A_{p_1} = 0, \quad a \neq a_1, \tag{4.18}$$

$$R_{a_1pa_1p} = -B_1B_2 + A_{a_1} + A_p = 0, \quad p \neq p_1. \tag{4.19}$$

From (4.2), (4.3), (4.4), (2.8), (4.10) and for the reason above, we get

$$R_{a_1na_1n} = -B_1B_2 + A_{a_1} + A_n = \frac{2B^2_{a_1p_1,n}}{(B_1 - B_2)(B_3 - B_2)}, \tag{4.20}$$

$$R_{anan} = -B_1B_2 + A_a + A_n = 0, \quad a \neq a_1, \tag{4.21}$$

$$R_{p_1np_1n} = -B_1B_2 + A_{p_1} + A_n = \frac{2B^2_{a_1p_1,n}}{(B_2 - B_1)(B_3 - B_1)}, \tag{4.22}$$

$$R_{pnpn} = -B_1B_2 + A_p + A_n = 0, \quad p \neq p_1. \tag{4.23}$$

Thus, from (4.16)–(4.23), we see that the normalized conformal scalar curvature $\kappa = \frac{1}{n(n-1)}\sum_{i\neq j} R_{ijij} = 0 \neq 1$. Since (2.2) implies that the matrix (B_{ij}) and (h_{ij}) are commutative, we can choose a local orthonormal basis such that $B_{ij} = B_i\delta_{ij}$ and $h_{ij} = \lambda_i\delta_{ij}$, where λ_i are the Euclidean principal curvatures of x. From (2.2) and Proposition 3.2, we know that x is an n-dimensional Euclidean isoparametric spacelike hypersurface with three distinct Euclidean principal curvatures, this contradicts Proposition 3.3. Thus, the case that the conformal second fundamental form of x is not parallel does not occur. This completes the proof of Theorem 1.2. □

5 Proof of theorem 1.3

Proposition 5.1. *(see [12]). Two regular spacelike hypersurface* $x : M \to \mathbb{Q}^{n+1}_1$ *and* $\tilde{x} : \tilde{M} \to \mathbb{Q}^{n+1}_1$ *in* $\mathbb{Q}^{n+1}_1 (n \geq 3)$ *are conformally equivalent if and only if there* *exists a diffeomorphism* $f : M \to \tilde{M}$ *which preserves the conformal metric* g *and the conformal second fundamental form* $\mathbf{B}$.

From [12], we also know the definition that a spacelike hypersurface with vanishing conformal form is called *a conformal para-isotropic spacelike hypersurface* if there is a function μ such that $\mathbf{A} + \lambda\mathbf{B} + \mu g \equiv 0$. We have the following:

Proposition 5.2. *(see [12]). A conformal para-isotropic spacelike hypersurface in* $\mathbb{Q}_1^{n+1}$ *is conformally equivalent to one of the spacelike hypersurfaces with constant mean curvature and constant scalar curvature in Lorentzian space form.*

Proof of Theorem 1.3. From (2.5), we see that the number γ of distinct conformal principal curvatures can only take the values $\gamma = 2, 3, 4, 5$.

(1) If $\gamma = 2$, from Theorem 1.1, we know that Theorem 1.3 is true.

(2) If $\gamma = 3$, we see that at least one of the conformal principal curvatures is simple. From Theorem 1.2, we know that Theorem 1.3 is true.

(3) If $\gamma = 4$, from Theorem 3.1, Example 3.1–Example 3.4, we know that the conformal second fundamental form of x is not parallel. Let B_1, B_2, B_3, B_4 B_5 be the constant conformal principal curvatures of x. Without loss of generality, we may assume that $B_1 \neq B_2 \neq B_3 \neq B_4 = B_5$. From (4.2), we have

$$B_{ii,k} = 0, \quad B_{45,k} = 0, \quad \text{for all } \ i, k, \quad \omega_{ij} = \sum_k \frac{B_{ij,k}}{B_i - B_j}\omega_k, \quad \text{for } \ B_i \neq B_j. \tag{5.1}$$

By the similar method in [4], we have the following Lemmas (see Lemma 3.1 and Lemma 3.2 in [4]):

Lemma 5.3. *Under the assumptions above, we have*

$$\frac{B_{12,4}B_{12,5}}{(B_1 - B_2)(B_4 - B_2)} = \frac{B_{13,4}B_{13,5}}{(B_1 - B_3)(B_3 - B_4)}, \tag{5.2}$$

$$\frac{B_{12,4}B_{12,5}}{(B_2 - B_1)(B_4 - B_1)} = \frac{B_{23,4}B_{23,5}}{(B_2 - B_3)(B_3 - B_4)}. \tag{5.3}$$

Lemma 5.4. *Let* i, j, k *be the three distinct elements of* $\{1, 2, 3\}$ *with arbitrarily given order. Then*

$$R_{ijij} = \frac{2B_{12,3}^2}{(B_k - B_i)(B_k - B_j)} + \frac{2(B_{ij,4}^2 + B_{ij,5}^2)}{(B_4 - B_i)(B_4 - B_j)}, \tag{5.4}$$

$$R_{i4i4} = \frac{2B_{ij,4}^2}{(B_j - B_i)(B_j - B_4)} + \frac{2B_{ik,4}^2}{(B_k - B_i)(B_k - B_4)}, \tag{5.5}$$

$$R_{i5i5} = \frac{2B_{ij,5}^2}{(B_j - B_i)(B_j - B_5)} + \frac{2B_{ik,5}^2}{(B_k - B_i)(B_k - B_5)}. \tag{5.6}$$

Lemma 5.5. *Under the assumptions above, we have*

(*i*) *for any distinct* $i, j \in \{1, 2, 3\}$ *and any distinct* $\alpha, \beta \in \{4, 5\}$, *if* $B_{12,3}B_{ij,\alpha} \neq 0$, *then* $B_{12,\beta} = B_{13,\beta} = B_{23,\beta} = 0$*;*

(*ii*) $B_{12,4}B_{12,5} = B_{13,4}B_{13,5} = B_{23,4}B_{23,5} = 0$.

Proof. (*i*) Without loss of generality, we may only prove that for any distinct $i, j \in \{1,2,3\}$, if $B_{12,3}B_{ij,4} \neq 0$, then $B_{12,5} = B_{13,5} = B_{23,5} = 0$. In fact, if $B_{12,5} \neq 0$, from the definition of the covariant derivative of L_{ij} and B_{ij} (see (2.6) and (2.7) of [14]), we have

$$A_{ij,k} = E_k(A_i)\delta_{ij} + \Gamma^j_{ik}(A_i - A_j), \quad B_{ij,k} = E_k(B_i)\delta_{ij} + \Gamma^j_{ik}(B_i - B_j). \tag{5.7}$$

Thus, from (5.7), we see that for any distinct $i, j \in \{1,2,3\}$, $\frac{A_{12,3}}{B_{12,3}} = \frac{A_1-A_2}{B_1-B_2} = \frac{A_1-A_3}{B_1-B_3} = \frac{A_2-A_3}{B_2-B_3}$, $\frac{A_{ij,4}}{B_{ij,4}} = \frac{A_i-A_j}{B_i-B_j} = \frac{A_i-A_4}{B_i-B_4} = \frac{A_j-A_4}{B_j-B_4}$, $\frac{A_{12,5}}{B_{12,5}} = \frac{A_1-A_2}{B_1-B_2} = \frac{A_1-A_5}{B_1-B_5} = \frac{A_2-A_5}{B_2-B_5}$. If $i = 2, j = 3$, we see that there is a function λ such that

$$\frac{A_1 - A_2}{B_1 - B_2} = \frac{A_1 - A_3}{B_1 - B_3} = \frac{A_2 - A_4}{B_2 - B_4} = \frac{A_1 - A_5}{B_1 - B_5} = -\lambda. \tag{5.8}$$

Thus, from (5.8), we also see that there is another function μ such that

$$A_1 + \lambda B_1 = A_2 + \lambda B_2 = A_3 + \lambda B_3 = A_4 + \lambda B_4 = A_5 + \lambda B_5 = -\mu. \tag{5.9}$$

Thus, we see that x is a conformal para-isotropic spacelike hypersurface, and from [12], we know that λ and μ are constant. From Proposition 5.2, we know that x is conformally equivalent to one of the spacelike hypersurfaces with constant mean curvature and constant scalar curvature in Lorentzian space form, which, from [12], is also a conformal para-isotropic spacelike hypersurface denoted by $\tilde{x}$. From Proposition 5.1, we know that $\tilde{x}$ also has four distinct constant conformal principal curvatures $\tilde{B}_i (i = 1,2,3,4)$. Since $\tilde{x}$ has constant mean curvature and constant scalar curvature, from Gaussian equation and $e^{\tilde{\tau}}\tilde{B}_i = \tilde{\lambda}_i - \tilde{H}$, we see that $\tilde{x}$ is a Euclidean isoparametric spacelike hypersurface with four distinct Euclidean principal curvatures $\tilde{\lambda}_i (i = 1,2,3,4)$ in Lorentzian space form, this contradicts Proposition 3.3. Thus, we must have $B_{12,5} = 0$. By the similar reason, we may prove that $B_{13,5} = 0$ and $B_{23,5} = 0$.

(*ii*) Suppose that $B_{12,4}B_{12,5} \neq 0$, by Lemma 5.3, we have $B_{13,4}B_{13,5} \neq 0$ and $B_{23,4}B_{23,5} \neq 0$. By the similar method in the proof of (i), we shall conclude. □

Now, we return to consider the case $\gamma = 4$, since the conformal second fundamental form is not parallel, from (5.1), we should notice that the possible nonzero elements of $B_{ij,k}, 1 \le i,j,k \le 5$, may be $\{B_{12,3},\ B_{12,4},\ B_{12,5},\ B_{13,4},\ B_{13,5},\ B_{23,4},\ B_{23,5}\}$.

We may consider two cases: $B_{12,3} = 0$ and $B_{12,3} \neq 0$.

Case (*i*). If $B_{12,3} = 0$, since $\mathbf{B}$ is not parallel, we know that there is at least one nonzero element in $\{B_{12,4}, B_{12,5}, B_{13,4}, B_{13,5}, B_{23,4}, B_{23,5}\}$, without loss of generality, we may assume that $B_{12,4} \neq 0$. By Lemma 5.5, we have $B_{12,5} = 0$ and there are at most two nonzero elements in $\{B_{13,4}, B_{13,5}, B_{23,4}, B_{23,5}\}$.

Subcase (*i*). If $B_{13,4} = B_{13,5} = B_{23,4} = B_{23,5} = 0$, since $B_{12,4} \neq 0$, $B_{12,3} = 0$ and $B_{12,5} = 0$, from (5.4), (5.6) and (2.8), we have

$$A_2 + A_5 - B_2B_5 = 0, \quad A_3 + A_5 - B_3B_5 = 0, \tag{5.10}$$

$$A_2 + A_4 - B_2B_4 = \frac{2B_{12,4}^2}{(B_1 - B_2)(B_1 - B_4)}, \quad A_3 + A_4 - B_3B_4 = 0. \tag{5.11}$$

From (5.10) and (5.11), we have $A_2 - A_3 - (B_2 - B_3)B_4 = \frac{2B_{12,4}^2}{(B_1-B_2)(B_1-B_4)}$, $A_2 - A_3 - (B_2 - B_3)B_5 = 0$. Since $B_4 = B_5$, we see that $\frac{2B_{12,4}^2}{(B_1-B_2)(B_1-B_4)} = 0$, that is $B_{12,4} = 0$, a contradiction. Thus, subcase (i) does not occur.

Subcase (ii). If exactly one of $\{B_{13,4}, B_{13,5}, B_{23,4}, B_{23,5}\}$ is nonzero, the symmetry of indices 1 and 2 implies that we need only to consider two cases: $B_{23,4} \neq 0$ with $B_{13,4} = B_{13,5} = B_{23,5} = 0$, or $B_{23,5} \neq 0$ with $B_{13,4} = B_{13,5} = B_{23,4} = 0$.

If $B_{23,4} \neq 0$ with $B_{13,4} = B_{13,5} = B_{23,5} = 0$, since $B_{12,4} \neq 0$, $B_{12,3} = B_{12,5} = 0$, from Lemma 5.4, we have

$$R_{1212} = \frac{2B_{12,4}^2}{(B_4 - B_1)(B_4 - B_2)}, \quad R_{1414} = \frac{2B_{12,4}^2}{(B_2 - B_1)(B_2 - B_4)}, \tag{5.12}$$

$$R_{1313} = 0, \quad R_{1515} = 0, \quad R_{2323} = \frac{2B_{23,4}^2}{(B_4 - B_2)(B_4 - B_3)}, \tag{5.13}$$

$$R_{2424} = \frac{2B_{12,4}^2}{(B_1 - B_2)(B_1 - B_4)} + \frac{2B_{23,4}^2}{(B_3 - B_2)(B_3 - B_4)}, \quad R_{2525} = 0, \tag{5.14}$$

$$R_{3434} = \frac{2B_{23,4}^2}{(B_2 - B_3)(B_2 - B_4)}, \quad R_{3535} = 0. \tag{5.15}$$

From (5.1), we have $\omega_{15} = \omega_{25} = \omega_{35} = 0$. From (2.8), we know that if three of $\{i, j, k, l\}$ are either the same or distinct, then

$$R_{ijkl} = 0. \tag{5.16}$$

By (2.4), (5.16), $\omega_{ij} = \sum\limits_k \Gamma_{ik}^j \omega_k$, $\omega_{15} = \omega_{25} = \omega_{35} = 0$ and $R_{1515} = R_{2525} = R_{3535} = 0$, we obtain $0 = d\omega_{15} - \sum_k \omega_{1k} \wedge \omega_{k5} = -\omega_{14} \wedge \omega_{45} = -\Gamma_{24}^1 \omega_2 \wedge \omega_{45}$, $0 = -\omega_{24} \wedge \omega_{45} = -(\Gamma_{14}^2 \omega_1 + \Gamma_{34}^2 \omega_3) \wedge \omega_{45}$, $0 = -\omega_{34} \wedge \omega_{45} = -\Gamma_{24}^3 \omega_2 \wedge \omega_{45}$, this follows that $\omega_{45} = 0$. Combining $\omega_{15} = \omega_{25} = \omega_{35} = 0$, we obtain $R_{4545} = 0$. From (5.12)–(5.15) and $R_{4545} = 0$, we have $\kappa = \frac{1}{20} \sum_{i \neq j} R_{ijij} = \frac{1}{20}\{R_{1212} + R_{1313} + R_{1414} + R_{1515} + R_{2323} + R_{2424} + R_{2525} + R_{3434} + R_{3535} + R_{4545}\} = 0 \neq 1$. From (2.2) and Proposition 3.2, we know that x is a Euclidean isoparametric spacelike hypersurface with four distinct Euclidean principal curvatures, this contradicts Proposition 3.3.

If $B_{23,5} \neq 0$ with $B_{13,4} = B_{13,5} = B_{23,4} = 0$, since $B_{12,4} \neq 0$, $B_{12,3} = B_{12,5} = 0$, from Lemma 5.4, we have $R_{3434} = 0$. By (5.1), we have $\omega_{13} = \omega_{15} = \omega_{34} = 0$. Thus, from (2.4), $\omega_{ij} = \sum\limits_k \Gamma_{ik}^j \omega_k$, (5.16) and $R_{3434} = 0$, we obtain $0 = -\omega_{32} \wedge \omega_{24} - \omega_{35} \wedge \omega_{54} = -\Gamma_{35}^2 \Gamma_{21}^4 \omega_5 \wedge \omega_1 - \Gamma_{32}^5 \omega_2 \wedge \omega_{54}$, this implies that $\Gamma_{35}^2 \Gamma_{21}^4 = \frac{B_{32,5} B_{24,1}}{(B_3-B_2)(B_2-B_4)} = 0$, a contradiction. Thus, subcase (ii) does not occur.

Subcase (iii). If exactly two of $\{B_{13,4}, B_{13,5}, B_{23,4}, B_{23,5}\}$ are nonzero, the symmetry of indices 1 and 2 and (ii) of Lemma 5.5 imply that we need only to consider three cases: $B_{23,4} \neq 0$, $B_{13,4} \neq 0$ with $B_{13,5} = B_{23,5} = 0$, or $B_{23,4} \neq 0$, $B_{13,5} \neq 0$ with $B_{13,4} = B_{23,5} = 0$, or $B_{23,5} \neq 0$, $B_{13,5} \neq 0$ with $B_{13,4} = B_{23,4} = 0$.

If $B_{23,4} \neq 0$, $B_{13,4} \neq 0$ with $B_{13,5} = B_{23,5} = 0$, since $B_{12,4} \neq 0$, $B_{12,3} = B_{12,5} = 0$, from (5.1), we see that $\omega_{12} = \frac{B_{12,4}}{B_1-B_2}\omega_4$, $\omega_{13} = \frac{B_{13,4}}{B_1-B_3}\omega_4$, $\omega_{14} = \frac{B_{12,4}}{B_1-B_4}\omega_2 + \frac{B_{13,4}}{B_1-B_4}\omega_3$, $\omega_{15} = 0$, $\omega_{23} = \frac{B_{23,4}}{B_2-B_3}\omega_4$, $\omega_{24} = \frac{B_{12,4}}{B_2-B_4}\omega_1 + \frac{B_{23,4}}{B_2-B_4}\omega_3$, $\omega_{25} = 0$, $\omega_{34} = \frac{B_{13,4}}{B_3-B_4}\omega_1 +$

$\frac{B_{23,4}}{B_3-B_4}\omega_2$, $\omega_{35}=0$. From (2.4) and (2.1), we have

$$
\begin{aligned}
-R_{2323}\omega_2\wedge\omega_3 = & -\frac{1}{2}\sum_{k,l}R_{23kl}\omega_k\wedge\omega_l = d\omega_{23}-\sum_k\omega_{2k}\wedge\omega_{k3}\\
= & \frac{B_{23,4}}{B_2-B_3}d\omega_4+\frac{dB_{23,4}}{B_2-B_3}\wedge\omega_4-\omega_{21}\wedge\omega_{13}-\omega_{24}\wedge\omega_{43}\\
= & \frac{B_{23,4}}{B_2-B_3}\omega_1\wedge\Big(\frac{B_{12,4}}{B_1-B_4}\omega_2+\frac{B_{13,4}}{B_1-B_4}\omega_3\Big)\\
& +\frac{B_{23,4}}{B_2-B_3}\omega_2\wedge\Big(\frac{B_{12,4}}{B_2-B_4}\omega_1+\frac{B_{23,4}}{B_2-B_4}\omega_3\Big)\\
& +\frac{B_{23,4}}{B_2-B_3}\omega_3\wedge\Big(\frac{B_{13,4}}{B_3-B_4}\omega_1+\frac{B_{23,4}}{B_3-B_4}\omega_2\Big)\\
& +\frac{B_{23,4}}{B_2-B_3}\omega_5\wedge\omega_{54}+\frac{dB_{23,4}}{B_2-B_3}\wedge\omega_4\\
& +\Big\{\frac{B_{12,4}}{B_2-B_4}\omega_1+\frac{B_{23,4}}{B_2-B_4}\omega_3\Big\}\wedge\Big\{\frac{B_{13,4}}{B_3-B_4}\omega_1+\frac{B_{23,4}}{B_3-B_4}\omega_2\Big\}.
\end{aligned}
\tag{5.17}
$$

Comparing the coefficients of $\omega_1\wedge\omega_2$ and $\omega_1\wedge\omega_3$ on both sides of the above equation, we obtain $\frac{1}{(B_2-B_3)(B_1-B_4)}-\frac{1}{(B_2-B_3)(B_2-B_4)}+\frac{1}{(B_2-B_4)(B_3-B_4)}=0$, $\frac{1}{(B_2-B_3)(B_1-B_4)}-\frac{1}{(B_2-B_3)(B_3-B_4)}-\frac{1}{(B_2-B_4)(B_3-B_4)}=0$, this implies $\frac{3}{(B_2-B_4)(B_3-B_4)}=0$, a contradiction.

If $B_{23,4}\neq 0$, $B_{13,5}\neq 0$ with $B_{13,4}=B_{23,5}=0$, since $B_{12,4}\neq 0$, $B_{12,3}=B_{12,5}=0$, from (5.7) and for the reason in the proof of Lemma 5.5, we see that (5.9) holds for λ and μ, that is, x is a conformal para-isotropic spacelike hypersurface. By reasoning as in the proof of Lemma 5.5 again, we have a contradiction.

If $B_{23,5}\neq 0$, $B_{13,5}\neq 0$ with $B_{13,4}=B_{23,4}=0$, since $B_{12,4}\neq 0$, $B_{12,3}=B_{12,5}=0$, by reasoning as in the proof of Lemma 5.5, we see that x is a conformal para-isotropic spacelike hypersurface and we also have a contradiction. Thus, subcase (iii) does not occur.

To sum up, we know that case (i) does not occur.

Case (*ii*). If $B_{12,3}\neq 0$, by Lemma 5.3 and Lemma 5.5, we see that there are at most three nonzero elements in $\{B_{12,4},B_{12,5},B_{13,4},B_{13,5},B_{23,4},B_{23,5}\}$.

Subcase (*i*). If $B_{12,4}=B_{12,5}=B_{13,4}=B_{13,5}=B_{23,4}=B_{23,5}=0$, since $B_{12,3}\neq 0$, from Lemma 5.4 and (2.8), we have

$$A_1+A_2-B_1B_2=\frac{2B_{12,3}^2}{(B_3-B_1)(B_3-B_2)},\tag{5.18}$$

$$A_1+A_3-B_1B_3=\frac{2B_{12,3}^2}{(B_2-B_1)(B_2-B_3)},\tag{5.19}$$

$$A_2+A_3-B_2B_3=\frac{2B_{12,3}^2}{(B_1-B_2)(B_1-B_3)},\tag{5.20}$$

$$A_1+A_4-B_1B_4=0,\quad A_2+A_4-B_2B_4=0,\quad A_3+A_4-B_3B_4=0,\tag{5.21}$$

$$A_1+A_5-B_1B_5=0,\quad A_2+A_5-B_2B_5=0,\quad A_3+A_5-B_3B_5=0.\tag{5.22}$$

Since $B_4 = B_5$, from (5.21) and (5.22), we get $A_4 = A_5$. From (5.18)–(5.21) we obtain

$$(5.23)\qquad (B_1 - B_2)(B_3 - B_4) = \frac{2(B_1 + B_2 - 2B_3)B_{12,3}^2}{(B_2 - B_1)(B_2 - B_3)(B_1 - B_3)},$$

$$(5.24)\qquad 2A_1 - B_1B_2 - B_1B_3 + B_2B_3 = \frac{4B_{12,3}^2}{(B_3 - B_1)(B_1 - B_2)}.$$

From (5.23), we see that $B_{12,3}$ is constant. Thus, from (5.24), (5.18)–(5.22), we know that A_1, A_2, A_3, A_4, A_5 are constants. By (5.21), we see that $A_1 - B_4B_1 = A_2 - B_4B_2 = A_3 - B_4B_3 = -A_4$. On the other hand, we have $A_4 - B_4B_4 = A_5 - B_4B_5 = \textit{constant} =: \nu$. We may prove that $\nu \neq -A_4$. In fact, if $\nu = -A_4$, denote by $\mathbf{D} = \mathbf{A} + (-B_4)\mathbf{B}$ the conformal para-Blaschke tensor of the immersion x, we see that x is a conformal para-isotropic spacelike hypersurface. By reasoning as in the proof of Lemma 5.5, we have a contradiction. Thus, we know that x must be a conformal spacelike hypersurface with two distinct constant conformal para-Blaschke eigenvalues. Let ζ and η be the two distinct constant conformal para-Blaschke eigenvalues of x with multiplicities m and $5 - m$ respectively. From the definition of the covariant derivative of D_{ij}, we have $\sum_k D_{ij,k}\omega_k = (D_i - D_j)\omega_{ij}$. Thus $D_{ij,k} = 0$ for $1 \leq i, j \leq m$, or $m + 1 \leq i, j \leq 5$. From the symmetry of $D_{ij,k}$, we see that $D_{ij,k} = 0$ for all i, j, k, that is, the conformal para-Blaschke tensor of x is parallel. Thus, we have $\omega_{ij} = 0$, for $1 \leq i \leq m$, $m + 1 \leq j \leq 5$. Hence, we know that the distributions of the eigenspaces with respect to ζ and η are integrable. Since the number of distinct conformal para-Blaschke eigenvalues of x is two, we see that x is locally a Riemannian product $M_1^m \times M_2^{5-m}$, where M_1^m and M_2^{5-m} are the Riemannian integrable manifold corresponding to ζ and η respectively. Since $\omega_{ij} = 0$, for $1 \leq i \leq m$, $m+1 \leq j \leq 5$, we have $R_{ijij} = 0$ for $1 \leq i \leq m$, $m+1 \leq j \leq 5$. Thus, from (2.8) and $\mathbf{D} = \mathbf{A} + (-B_4)\mathbf{B}$, we have $(B_i - B_4)(B_j - B_4) - (\zeta + \eta) - B_4^2 = 0$ for $1 \leq i \leq m$, $m + 1 \leq j \leq 5$.

If $m = 1$, we have $(B_1 - B_4)(B_{j_1} - B_{j_2}) = 0$ for $2 \leq j_1$, $j_2 \leq 5$, $j_1 \neq j_2$. Since $B_1 \neq B_4$, we obtain that $B_2 = B_3 = B_4 = B_5$, a contradiction.

If $m = 2$, we have $(B_{i_1} - B_{i_2})(B_j - B_4) = 0$ for $1 \leq i_1, i_2 \leq 2$, $i_1 \neq i_2$, $3 \leq j \leq 5$. Since $B_1 \neq B_2$, we obtain that $B_3 = B_4 = B_5$, a contradiction. Thus, we must have $m \geq 3$. From (2.8), we may easily obtain that $R_{ijkl} = (2\eta + B_4^2)(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk})$ for $m + 1 \leq i, j, k, l \leq 5$, that is, M_2^{5-m} is a constant curvature Riemannian manifold.

Subcase (ii). If exactly one of $\{B_{12,4}, B_{12,5}, B_{13,4}, B_{13,5}, B_{23,4}, B_{23,5}\}$ is nonzero, without loss of generality, we may assume that $B_{12,4} \neq 0$. Since $B_{12,3} \neq 0$, $B_{12,5} = B_{13,4} = B_{13,5} = B_{23,4} = B_{23,5} = 0$, from (5.5), (5.6) and (2.8), we have $A_2 + A_4 - B_2B_4 = \frac{2B_{12,4}^2}{(B_1-B_2)(B_1-B_4)}$, $A_2+A_5-B_2B_5 = 0$, $A_3+A_4-B_3B_4 = 0$, $A_3+A_5-B_3B_5 = 0$, this implies that $\frac{2B_{12,4}^2}{(B_1-B_2)(B_1-B_4)} = 0$ and $B_{12,4} = 0$, a contradiction. Thus, subcase (ii) does not occur.

Subcase (iii). If exactly two of $\{B_{12,4}, B_{12,5}, B_{13,4}, B_{13,5}, B_{23,4}, B_{23,5}\}$ are nonzero, without loss of generality, we may assume that $B_{12,4} \neq 0$. By Lemma 5.5, we have $B_{12,5} = B_{13,5} = B_{23,5} = 0$. Thus exactly one of $\{B_{13,4}, B_{23,4}\}$ is nonzero, without loss of generality, we may assume that $B_{13,4} \neq 0$, $B_{23,4} = 0$. From (5.1), (2.4), (2.1) and for the reason above, we see that

$$
\begin{aligned}
(5.25)\qquad -R_{2323}\omega_2\wedge\omega_3 &= -\frac{1}{2}\sum_{k,l}R_{23kl}\omega_k\wedge\omega_l = d\omega_{23}-\sum_k\omega_{2k}\wedge\omega_{k3}\\
&=\frac{dB_{12,3}}{B_2-B_3}\wedge\omega_1+\frac{B_{12,3}}{B_2-B_3}\Big(\frac{B_{12,3}}{B_1-B_2}\omega_3+\frac{B_{12,4}}{B_1-B_2}\omega_4\Big)\wedge\omega_2\\
&+\frac{B_{12,3}}{B_2-B_3}\Big(\frac{B_{12,3}}{B_1-B_3}\omega_2+\frac{B_{13,4}}{B_1-B_3}\omega_4\Big)\wedge\omega_3\\
&+\frac{B_{12,3}}{B_2-B_3}\Big(\frac{B_{12,4}}{B_1-B_4}\omega_2+\frac{B_{13,4}}{B_1-B_4}\omega_3\Big)\wedge\omega_4\\
&+\Big(\frac{B_{12,3}}{B_1-B_2}\omega_3+\frac{B_{12,4}}{B_1-B_2}\omega_4\Big)\wedge\Big(\frac{B_{12,3}}{B_1-B_3}\omega_2+\frac{B_{13,4}}{B_1-B_3}\omega_4\Big).
\end{aligned}
$$

Comparing the coefficients of $\omega_2\wedge\omega_4$ and $\omega_3\wedge\omega_4$ on both sides of the above equation, we obtain $\frac{1}{(B_2-B_3)(B_1-B_2)}-\frac{1}{(B_2-B_3)(B_1-B_4)}+\frac{1}{(B_1-B_2)(B_1-B_3)}=0$, $\frac{1}{(B_2-B_3)(B_1-B_3)}-\frac{1}{(B_2-B_3)(B_1-B_4)}-\frac{1}{(B_1-B_2)(B_1-B_3)}=0$, this implies $\frac{3}{(B_1-B_2)(B_1-B_3)}=0$, a contradiction. Thus, subcase (iii) does not occur.

Subcase (iv). If exactly three of $\{B_{12,4},\ B_{12,5},\ B_{13,4},\ B_{13,5},\ B_{23,4},\ B_{23,5}\}$ are nonzero, we may consider the following cases:

If all of $B_{12,5}, B_{13,5}, B_{23,5}$ are zero, then it must have $B_{12,4}\neq 0$, $B_{13,4}\neq 0$, $B_{23,4}\neq 0$. From (5.1) , Lemma 5.4 and (2.8), we have $\omega_{15}=\omega_{25}=\omega_{35}=0$ and $R_{1515}=0$, $R_{2525}=0$, $R_{3535}=0$. Therefore, from (2.4) and (4.3), we obtain $0=d\omega_{15}-\sum_k\omega_{1k}\wedge\omega_{k5}=-\omega_{14}\wedge\omega_{45}=-(\Gamma^1_{24}\omega_2+\Gamma^1_{34}\omega_3)\wedge\omega_{45}$, $0=-\omega_{24}\wedge\omega_{45}=-(\Gamma^2_{14}\omega_1+\Gamma^2_{34}\omega_3)\wedge\omega_{45}$, $0=-\omega_{34}\wedge\omega_{45}=-(\Gamma^3_{14}\omega_1+\Gamma^3_{24}\omega_2)\wedge\omega_{45}$, this follows that $\omega_{45}=0$. Combining $\omega_{15}=\omega_{25}=\omega_{35}=0$, we obtain $R_{4545}=0$. From (5.4), (5.5) and $R_{i5i5}=0, i=1,2,3,4$, we get $\kappa=\frac{1}{20}\sum_{i\neq j}R_{ijij}=0\neq 1$. From (2.2) and Proposition 3.2, we know that x is a Euclidean isoparametric spacelike hypersurface with four distinct Euclidean principal curvatures, this contradicts Proposition 3.3.

If two of $\{B_{12,5},\ B_{13,5},\ B_{23,5}\}$ are zero, without loss of generality, we may assume that $B_{12,5}=B_{13,5}=0$ and $B_{23,5}\neq 0$. From (ii) of Lemma 5.5, we must have $B_{23,4}=0$. Thus, it must follow that $B_{12,4}\neq 0$, $B_{13,4}\neq 0$. From (5.7) and for the reason in the proof of Lemma 5.5, we see that x is a conformal para-isotropic spacelike hypersurface and we have a contradiction.

If one of $\{B_{12,5},\ B_{13,5},\ B_{23,5}\}$ is zero, without loss of generality, we may assume that $B_{12,5}=0$, $B_{13,5}\neq 0$ and $B_{23,5}\neq 0$. From (ii) of Lemma 5.5, we must have $B_{13,4}=B_{23,4}=0$. Thus, it must follow that $B_{12,4}\neq 0$. By reasoning as in the proof of Lemma 5.5, we see that x is a conformal para-isotropic spacelike hypersurface and we have a contradiction.

To sum up, we know that case (ii) does not occur.

(4) If $\gamma=5$, from Theorem 3.1, Example 3.1–Example 3.4, we know that $\mathbf{B}$ is not parallel. Without loss of generality, we may assume that $B_{12,3}\neq 0$. Since $B_1\neq B_2\neq B_3\neq B_4\neq B_5$, from (4.2), we have $B_{ii,k}=0$ for all i,k.

By a similar method as in the proof of [4], we have the following (see Lemma 4.1 in [4]):

Lemma 5.6. *Let B_1, B_2, B_3, B_4, B_5 be the constant conformal principal curvatures of $x: M\to\mathbb{Q}^6_1$ with $B_1\neq B_2\neq B_3\neq B_4\neq B_5$ and i,j,k,l,s be the five distinct* *elements of* $\{1,2,3,4,5\}$ *with arbitrarily given order. Then*

$$R_{ijij} = \frac{2B_{ij,k}^2}{(B_k - B_i)(B_k - B_j)} + \frac{2B_{ij,l}^2}{(B_l - B_i)(B_l - B_j)} + \frac{2B_{ij,s}^2}{(B_s - B_i)(B_s - B_j)}. \tag{5.26}$$

Now, we return to consider the case $\gamma = 5$, since $\mathbf{B}$ is not parallel, we may consider the following two cases:

Case (i). If $B_{12,3} \neq 0$ and all of $\{B_{12,4},\ B_{12,5},\ B_{13,4},\ B_{13,5},\ B_{23,4},\ B_{23,5}\}$ are zero, in this case, we may prove that at most one of $\{B_{14,5}, B_{24,5}, B_{34,5}\}$ is zero. In fact, without loss of generality, if $B_{14,5} = B_{24,5} = 0$, by Lemma 5.6, we obtain $\kappa = \frac{1}{20}\sum_{i\neq j} R_{ijij} = 0 \neq 1$. From (2.2) and Proposition 3.2, we know that x is a Euclidean isoparametric spacelike hypersurface with five distinct Euclidean principal curvatures, this contradicts Proposition 3.3.

We now assume that $B_{24,5} \neq 0, B_{34,5} \neq 0$. Since $B_{12,3} \neq 0$, by the similar method as in the proof of Lemma 5.5, we see that there exist λ and ν such that $A_2 + \lambda B_2 = A_3 + \lambda B_3 = A_4 + \lambda B_4 = A_5 + \lambda B_5$, $A_1 + \nu B_1 = A_2 + \nu B_2 = A_3 + \nu B_3$, this implies that $\lambda = \nu$ and

$$A_1 + \lambda B_1 = A_2 + \lambda B_2 = A_3 + \lambda B_3 = A_4 + \lambda B_4 = A_5 + \lambda B_5. \tag{5.27}$$

From (5.27), we see that x is a conformal para-isotropic spacelike hypersurface. By reasoning as in the proof of Lemma 5.5, we have a contradiction.

Case (ii). If $B_{12,3} \neq 0$ and at least one of $\{B_{12,4},\ B_{12,5},\ B_{13,4},\ B_{13,5},\ B_{23,4},\ B_{23,5}\}$ is nonzero, without loss of generality, we may assume that $B_{12,4} \neq 0$. We consider the following two subcases:

Subcase (i). If all of $\{B_{12,5},\ B_{13,5},\ B_{23,5},\ B_{14,5},\ B_{24,5},\ B_{34,5}\}$ are zero, since $B_{12,3} \neq 0$ and $B_{12,4} \neq 0$, by Lemma 5.6, we obtain $\kappa = \frac{1}{20}\sum_{i\neq j} R_{ijij} = 0 \neq 1$. From (2.2) and Proposition 3.2, we know that x is a Euclidean isoparametric spacelike hypersurface with five distinct Euclidean principal curvatures, this contradicts Proposition 3.3.

Subcase (ii). If at least one of $\{B_{12,5},\ B_{13,5},\ B_{23,5},\ B_{14,5},\ B_{24,5},\ B_{34,5}\}$ is nonzero, without loss of generality, we may assume that $B_{12,5} \neq 0$. Since $B_{12,3} \neq 0$ and $B_{12,4} \neq 0$, by the similar method as in the proof of Lemma 5.5, we see that x is a conformal para-isotropic spacelike hypersurface and we have a contradiction. This completes the proof of Theorem 1.3. □

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