Subgradient estimates for a nonlinear subparabolic equation on pseudo-Hermitian manifold

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Abstract. Let (M, J, θ) be a closed pseudo-Hermintian $(2n+1)$-manifold. In this paper, we derive the subgradient estimate for positive solutions to a nonlinear subparabolic equation $\frac{\partial u}{\partial t} = \triangle_b u + au \log u + bu$ on $M \times [0, \infty)$, where a, b are two real constants.

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1 Introduction

In the seminal paper of [4], P. Li and S.-T. Yau established the parabolic Li-Yau gradient estimate and Harnack inequality for the positive solution of the heat equation

$$\frac{\partial u}{\partial t}(x,t) = \triangle u(x,t)$$

in a complete Riemannian l-manifold with nonnegative Ricci curvature. Here $\triangle$ is the Laplace-Beltrami operator. Along this line with method of Li-Yau gradient estimate, it is the very first paper of H.-D. Can and S.-T. Yau [3] to consider the heat equation

$$\frac{\partial u}{\partial t}(x,t) = Lu(x,t) \tag{1.1}$$

in a closed l-manifold with a positive measure and a subelliptic operator with respect to the sum squares of vector fields

$$L = \sum_{i=1}^{h} X_i^2 - Y, \quad Y = \sum_{i=1}^{h} c_i X_i,$$

where $X_1, \cdots, X_h$ are smooth vector fields which satisfy Hörmander's condition: the vector fields together with their commutators up to finite order span the tangent space at every point of M. Suppose that $[X_i, [X_j, X_k]]$ can be expressed as linear

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combinations of $X_1, \cdots, X_h$ and their brackets $[X_1, X_2], \cdots, [X_{l-1}, x_h]$. they showed that for the positive solution $u(x,t)$ of (1.1) on $M \times [0, \infty)$, there exist constants C', C'', C''' and $\frac{1}{2} < \lambda < \frac{2}{3}$, such that for any $\delta > 1$, $f = \log u$ satisfies the following gradient estimate

$$\sum_i |X_i f| - \delta f_t + \sum_\alpha (1 + |Y_\alpha f|^2)\lambda - \delta Y f \leq \frac{C'}{t} + C'' + C''' t^{\frac{\lambda}{\lambda-1}},$$

with $\{Y_\alpha\} = \{[X_i, X_j]\}$. In [1], S.-C. Chang, T.-J. Kuo and S.-H. Lai established the CR Cao-Yau type Harmack estimate

$$4\frac{\partial(\log u)}{\partial t} - |\nabla_b \log u|^2 - \frac{1}{3}t[(\log u)_0]^2 + \frac{16}{t} \geq 0,$$

for the positive solution u of the CR heat equation $\frac{\partial u}{\partial t} = \triangle_b u$ in a closed pseudo-Hermitian 3-manifold (M, J, θ) with nonnegative Tanaka-webster curvature and vanishing torsion. Here $\triangle_b$ is the time-independent sub-Laplacian, ∇_b is the subgradient and $\varphi_0 = T\varphi$ for a smooth function φ.

Recently, S.-C. Chang, T.-J. Kuo and S.-H. Lai established the CR Cao-Yau type Harmack estimate

$$\left(1 + \frac{3}{n}\right)\frac{\partial(\log u)}{\partial t} - |\nabla_b \log u|^2 - \frac{n}{3}t[(\log u)_0]^2 + \frac{\frac{9}{n} + 6 + n}{t} \geq 0$$

for the positive solution u of the CR heat equation

$$\frac{\partial u}{\partial t} = \triangle_b u$$

in a closed pseudo-Hermitian $(2n+1)$-manifold (M, J, θ) with $2Ric - (n-2)Tor \geq 0$ and $[\triangle_b, T] = 0$.

On the other hand, for Riemannian case, there are many papers (such as [8, 7] and references therein) to investigate the following nonlinear parabolic equation

$$\frac{\partial u}{\partial t} = \triangle u + au \log u + bu \tag{1.2}$$

on $M \times [0, \infty)$, where (M, g) is a Riemannian manifold, a, b are two real constants. They obtained the gradient estimate for the positive solution of the equation (1.2).

In this paper, we consider the following nonlinear subparabolic equation

$$\frac{\partial u}{\partial t} = \triangle_b u + au \log u + bu \tag{1.3}$$

in a closed pseudohermintian $(2n+1)$-manifold (M, J, θ). We obtain the following results:

Theorem 1.1. *(cf. Theorem 3.1) Let (M, J, θ) be a closed pseudo-Hermitian $(2n+1)$ manifold. Suppose that*

$$2Ric(X, X) - (n-2)Tor(X, X) \geq 0,$$

for all $X \in T_{1,0} \oplus T_{0,1}$*. If* u *is the positive solution of*

$$\frac{\partial u}{\partial t} = \triangle_b u + au \log u. \tag{1.4}$$

with $[\triangle_b, T] = 0$ *on* $M \times [0, \infty)$*, let* $f(x,t) = \log u(x,t)$*. Then we have*

$$|\nabla_b f|^2 - \left(1 + \frac{3}{n}\right)(f_t - af) + \frac{n}{3} t f_0^2 < \frac{1}{t}\left(1 - \frac{a}{2}t\right)\left(\frac{9}{n} + 6 + n\right),$$

where $a \leq 0$ *is a constant.*

Remark 1.1. By replacing u by $e^{\frac{b}{a}} u$, equation (1.3) reduces to equation (1.4).

2 Preliminaries

We first introduced some basic materials in a pseudo-Hermitian $(2n+1)$-manifold (see [5], [6] for more details). Let (M, ξ) be a $(2n+1)$-dimensional, orientable, contact manifold with contract structure ξ. A CR structure compatible with ξ is an endomorphism $J : \xi \to \xi$ such that $J^2 = -1$. We also assume that J satisfies the following integrability condition: If X and Y are in ξ, then so are $[JX, Y] + [X, JY]$ and $J([JX, Y] + [X, JY]) = [JX, Y] - [X, Y]$.

Let $\{T, Z_\alpha, Z_{\bar\alpha}\}$ be a frame of $TM \otimes C$, where Z_α is any local frame of $T_{1,0}$, $Z_{\bar\alpha} = \bar{Z_\alpha} \in T_{0,1}$ and T is the characteristic vector field. Then $\{\theta, \theta^\alpha, \theta^{\bar\alpha}\}$, which is the coframe dual to $\{T, Z_\alpha, Z_{\bar\alpha}\}$, satisfies $d\theta = ih_{\alpha\bar\beta}\theta^\alpha \wedge \theta^{\bar\beta}$ for some positive definite hermitian matrix of function $(h_{\alpha\bar\beta})$. Actually we can always choose Z_α such that $h_{\alpha\bar\beta} = \delta_{\alpha\beta}$; hence, through this note, we assume $h_{\alpha\bar\beta} = \delta_{\alpha\beta}$.

The Levi form $\langle , \rangle_{L_\theta}$ is the Hermitian form on $T_{1,0}$ defined by $\langle Z, W\rangle_{L_\theta} = -i\langle d\theta, Z \wedge \bar W\rangle$. We can extend $\langle , \rangle_{L_\theta}$ to $T_{0,1}$ by defining $\langle \bar Z, \bar W\rangle_{L_\theta} = \langle Z, \bar W\rangle_{L_\theta}$, for all $Z, W\ inT_{0,1}$. The Levi form induces naturally a Hermitian form on the dual bundle of $T_{1,0}$, denoted by $\langle , \rangle_{L^*_\theta}$, and hence on all the induced tensor bundles. Integrating the Hermitian form over M with respect to the volume form $d\mu = \theta \wedge (d\theta)^n$, we get an inner product on the space of sections of each tensor bundle. We denote the inner product by the notation $\langle , \rangle$.

The pseudo-Hermitian connection of (J, θ) is the connection ∇ on $TM \otimes C$ (and extended to tensors) given in terms of a local frame $Z_\alpha \in T_{1,0}$ by

$$\nabla Z_\alpha = \theta_\alpha^\beta \otimes Z_\beta, \quad \nabla Z_{\bar\alpha} = \theta_{\bar\alpha}^{\bar\beta} \otimes Z_{\bar\beta}, \quad \nabla T = 0,$$

where θ_α^β are the 1-forms uniquely determined by the following equations:

$$\begin{aligned} d\theta^\beta &= \theta^\alpha \wedge \theta_\alpha^\beta + \theta \wedge \tau^\beta \\ 0 &= \tau_\alpha \wedge \theta^\alpha, \\ 0 &= \theta_\alpha^\beta + \theta_{\bar\beta}^{\bar\alpha}. \end{aligned}$$

We can write (by Cartan lemma) $\tau_\alpha = A_{\alpha\gamma}\theta^\gamma$ with $A_{\alpha\gamma} = A_{\gamma\alpha}$. The curvature of Tanaka-Webster connection, expressed in terms of the coframe $\{\theta = \theta^0, \theta^\alpha, \theta^{\bar\alpha}\}$, is

$$\Pi_\beta^\alpha = \overline{\Pi_{\bar\beta}^{\bar\alpha}} = d\omega_\beta^\alpha - \omega_\beta^\gamma \wedge \omega_\gamma^\alpha$$

$$\Pi_0^\alpha = \Pi_\alpha^0 = \Pi_0^{\bar\beta} = \Pi_{\bar\beta}^0 = \Pi_0^0 = 0.$$

Webster showed that Π_β^α can be written

$$\Pi_\beta^\alpha = R_{\beta\rho\bar{\sigma}}^\alpha \theta^\rho \wedge \theta^{\bar{\sigma}} + W_{\beta\rho}^\alpha \theta^\rho \wedge \theta - W_{\beta\bar{\rho}}^\alpha \theta^{\bar{\rho}} \wedge \theta + i\theta_\beta \wedge \tau^\alpha - i\tau_\beta \wedge \theta^\alpha,$$

where the coefficients satisfy

$$R_{\beta\bar{\alpha}\rho\bar{\sigma}} = \overline{R_{\alpha\bar{\beta}\sigma\bar{\rho}}} = R_{\bar{\alpha}\beta\bar{\sigma}\rho} = R_{\rho\bar{\alpha}\beta\bar{\sigma}}, \quad W_{\beta\bar{\alpha}\gamma} = W_{\gamma\bar{\alpha}\beta}.$$

Here $R_{\delta\alpha\bar{\beta}}^\gamma$ is the pseudo-Hermitian curvature tensor, $R_{\alpha\bar{\beta}} = R_{\gamma\alpha\bar{\beta}}^\gamma$ is the pseudo-Hermitian Ricci curvature tensor and $A_{\alpha\beta}$ is the torsion tensor. We define Ric and Tor by

$$Ric(X,Y) = R_{\alpha\bar{\beta}} X^\alpha Y^{\bar{\beta}}, \quad Tor(X,Y) = i(A_{\bar{\alpha}\bar{\beta}} X^{\bar{\alpha}} Y^{\bar{\beta}} - A_{\alpha\beta} X^\alpha Y^\beta),$$

for $X = X^\alpha Z_\alpha$, $Y = Y^\beta Z_\beta$ on $T_{1,0}$.

We will denote components of covariant derivatives with indices preceded by comma; thus write $A_{\alpha\beta,\gamma}$. The indices $\{0, \alpha, \bar{\alpha}\}$ indices derivatives with respect to $\{T, Z_\alpha, Z_{\bar{\alpha}}\}$. For derivatives of a scalar function, we will often omit the comma, for instance, $u_\alpha = Z_\alpha u$, $u_{\alpha\bar{\beta}} = Z_{\bar{\beta}} Z_\alpha u - \omega_\alpha^\gamma(Z_{\bar{\beta}}) Z_\gamma u$. For a real function u, the subgradient ∇_b is defined by $\nabla_b u \in \xi$ and $\langle Z, \nabla_b u\rangle = du(Z)$, for all vector fields Z tangent to contact plane. Locally $\nabla_b u = \sum_\alpha u_{\bar{\alpha}} Z_\alpha + \sum_\alpha u_\alpha Z_{\bar{\alpha}}$. We can use the connection to define the subhessian as the complex linear map $(\nabla^H)^2 u : T_{1,0} \oplus T_{0,1} \to T_{1,0} \oplus T_{0,1}$ by

$$(\nabla^H)^2 u(Z) = \nabla_Z \nabla_b u.$$

In particular, $|\nabla_b u|^2 = 2u_\alpha u_{\bar{\alpha}}$, $|\nabla_b^2 u|^2 = 2(u_{\alpha\beta} u_{\bar{\alpha}\bar{\beta}} + u_{\alpha\bar{\beta}} u_{\bar{\alpha}\beta})$ and $\triangle_b u = \sum_\alpha (u_{\alpha\bar{\alpha}} + u_{\bar{\alpha}\alpha})$.

We need the following Lemmas.

Lemma 2.1. *[2] For a smooth real-valued function u and any $\nu > 0$, we have*

$$\triangle_b |\nabla_b u|^2 \geq \frac{1}{n}(\triangle_b u)^2 + n u_0^2 + 2\langle \nabla_b u, \nabla_b \triangle_b u\rangle$$
$$+2(2Ric - (n-2)Tor)((\nabla_b u)_C, (\nabla_b u)_C) - 2v|\nabla_b u_0|^2 - \frac{2}{\nu}|\nabla_b u|^2. \tag{2.1}$$

where $(\nabla_b u)_C = u_{\overline{\alpha}} Z_\alpha$ is the corresponding complex $(1,0)$-vector of $\nabla_b u$.

Lemma 2.2. *Let (M, J, θ) be a pseudo-Hermitian $(2n+1)$-manifold with $[\triangle_b, T] = 0$. If $u(x,t)$ is the positive solution of $\frac{\partial u}{\partial t} = \triangle_b u + au\log u$, then $f = \log u$ satisfies*

$$\triangle_b f_0 - f_{ot} = -af_0 - 2\langle \nabla_b f_0, \nabla_b f\rangle. \tag{2.2}$$

Proof. Since u is the solution of (2.2), we have

$$(\triangle_b - \frac{\partial}{\partial t})f = -af - |\nabla_b f|^2. \tag{2.3}$$

From $[\triangle_b, T] = 0$ and (2.3), we have

$$\triangle_b f_0 - f_{0t} = (\triangle_b f)_0 - f_{t0} = [\triangle_b f - f_t]_0 = -af_0 - 2\langle \nabla_b f_0, \nabla_b f\rangle.$$

□

3 Subgradient estimates for a nonlinear subparabolic equation

In this section, we obtain the following results:

Theorem 3.1. *Let (M, J, θ) be a closed pseudo-Hermitian $(2n+1)$ manifold. Suppose that*

$$2Ric(X, X) - (n-2)Tor(X, X) \geq 0, \tag{3.1}$$

for all $X \in T_{1,0} \oplus T_{0,1}$. If u is the positive solution of

$$\frac{\partial u}{\partial t} = \triangle_b u + au \log u. \tag{3.2}$$

with $[\triangle_b, T] = 0$ on $M \times [0, \infty)$, let $f(x,t) = \log u(x,t)$. Then we have

$$|\nabla_b f|^2 - (1 + \frac{3}{n})(f_t - af) + \frac{n}{3} t f_0^2 < \frac{1}{t}[1 - \frac{a}{2}t][\frac{9}{n} + 6 + n],$$

where $a \leq 0$ is a constant.

Proof. Since u is the positive solution of (3.2), we have

$$(\triangle_b - \frac{\partial}{\partial t}) f(x,t) = -af(x,t) - |\nabla_b f(x,t)|^2 \tag{3.3}$$

Now we define a real-valued function $F(x, t, \alpha, \beta) : M \times [0, T^*] \times R^* \times R^+ \to R$ by

$$F(x, t, \alpha, \beta) = t(|\nabla_b f|^2 + \alpha(f_t - af) + \beta t f_0^2). \tag{3.4}$$

First we differentiate F with respect to the t-variable.

$$F_t = \frac{F}{t} + t[2\langle \nabla_b f, \nabla_b f_t \rangle + \alpha(f_{tt} - af_t) + \beta f_0^2 + 2\beta t f_0 f_{0t}] \tag{3.5}$$

From equation (3.3), we have

$$f_{tt} - af_t = 2\langle \nabla_b f, \nabla_b f_t \rangle + \triangle_b f_t \tag{3.6}$$

From (3.5) and (3.6), we have

$$F_t = \frac{F}{t} + t[2(1+\alpha)\langle \nabla_b f, \nabla_b f_t \rangle + \alpha \triangle_b f_t + \beta f_0^2 + 2\beta t f_0 f_{0t}] \tag{3.7}$$

From Lemma 2.1 and the assumption (3.1), we have

$$\begin{aligned}
&\triangle_b F = t[\triangle_b |\nabla_b f| + \alpha(\triangle_b f_t - a\triangle_b f) + \beta t \triangle_b f_0^2] \\
&\geq t[n f_0^2 + \frac{1}{n}(\triangle_b f)^2 + 2\langle \nabla_b f, \nabla_b \triangle_b f \rangle - \frac{2}{\nu}|\nabla_b f|^2 - 2\nu |\nabla_b f_0|^2 \\
&+\alpha(\triangle_b f_t - a\triangle_b f) + 2\beta t f_0 \triangle_b f_0 + 2\beta t |\nabla_b f_0|^2].
\end{aligned}$$

Taking $\nu = \beta t$, we have

$$\begin{aligned} \triangle_b F &\geq t[nf_0^2 + \frac{1}{n}(\triangle_b f)^2 + 2\langle \nabla_b f, \nabla_b \triangle_b f\rangle - \frac{2}{\beta t}|\nabla_b f|^2 \\ &+ \alpha(\triangle_b f_t - a\triangle_b f) + 2\beta t f_0 \triangle_b f_0]. \end{aligned} \tag{3.8}$$

From (3.7) and (3.8), we have

$$\begin{aligned} &(\triangle_b - \frac{\partial}{\partial t})F \geq -\frac{F}{t} + t[(n-\beta)f_0^2 + \frac{1}{n}(\triangle_b f)^2 + 2\langle \nabla_b f, \nabla_b \triangle_b f\rangle - \frac{2}{\beta t}|\nabla_b f|^2 \\ &-\alpha a \triangle_b f + 2\beta t f_0(\triangle_b f_0 - f_{0t}) - 2(1+\alpha)\langle \nabla_b f, \nabla_b f_t\rangle]. \end{aligned} \tag{3.9}$$

From the lemma 2.2 and the definition of F, we have

$$\begin{aligned} &2\langle \nabla_b f, \nabla_b \triangle_b f\rangle + 2\beta t f_0(\triangle_b f_0 - f_{0t}) - 2(1+\alpha)\langle \nabla_b f, \nabla_b f_t\rangle - \alpha a \triangle_b f \\ &= 2\langle \nabla_b f, \nabla_b [f_t - af - |\nabla_b f|^2]\rangle + 2\beta t f_0(-af - |\nabla_b f|^2)_0 \\ &-2(1+\alpha)\langle \nabla_b f, \nabla_b f_t\rangle - \alpha a \triangle_b f \\ &= -2\alpha\langle \nabla_b f, \nabla_b f_t\rangle - 2a|\nabla_b f|^2 - 2\langle \nabla_b f, \nabla_b |\nabla_b f|^2\rangle - 2a\beta t f_0^2 \\ &-4\beta t\langle \nabla_b f, \nabla_b f_0\rangle - \alpha a \triangle_b f \\ &= -2\alpha\langle \nabla_b f, \nabla_b(\frac{1}{\alpha t}F - \frac{1}{\alpha}|\nabla_b f|^2 - \frac{\beta t}{\alpha}f_0^2 + af)\rangle - 2a|\nabla_b f|^2 \\ &-2\langle \nabla_b f, \nabla_b |\nabla_b f|^2\rangle - 2a\beta t f_0^2 - 4\beta t\langle \nabla_b f, \nabla_b f_0\rangle - \alpha a \triangle_b f \\ &= -\frac{2}{t}\langle \nabla_b f, \nabla_b F\rangle - 2(\alpha+1)a|\nabla_b f|^2 - 2a\beta t f_0^2 - \alpha a \triangle_b f \end{aligned} \tag{3.10}$$

From (3.9) and (3.10), we have

$$\begin{aligned} &(\triangle_b - \frac{\partial}{\partial t})F \geq -\frac{F}{t} - 2\langle \nabla_b f, \nabla_b F\rangle + t[(n-\beta-2a\beta t)f_0^2 \\ &+\frac{1}{n}(\triangle_b f)^2 - \frac{2}{\beta t}|\nabla_b f|^2 - \alpha a \triangle_b f - 2(\alpha+1)a|\nabla_b f|^2] \\ &= -\frac{F}{t} - 2\langle \nabla_b f, \nabla_b F\rangle + t[(n-\beta-2a\beta t)f_0^2 + \frac{1}{n}(\triangle_b f)^2 \\ &-\frac{2}{\beta t}|\nabla_b f|^2 - \alpha a[\frac{F}{\alpha t} - (1+\frac{1}{\alpha})|\nabla_b f|^2 - \frac{\beta t}{\alpha}f_0^2] - 2(\alpha+1)a|\nabla_b f|^2] \\ &= -\frac{F}{t} - 2\langle \nabla_b f, \nabla_b F\rangle - aF + t[(n-\beta-2a\beta t)f_0^2 + \frac{1}{n}(\triangle_b f)^2 \\ &-\frac{2}{\beta t}|\nabla_b f|^2 - \alpha a[\frac{F}{\alpha t} - (1+\frac{1}{\alpha})|\nabla_b f|^2 - \frac{\beta t}{\alpha}f_0^2] - 2(\alpha+1)a|\nabla_b f|^2] \\ &= -\frac{F}{t} - 2\langle \nabla_b f, \nabla_b F\rangle - aF + t[(n-\beta-a\beta t)f_0^2 + \frac{1}{n}(\triangle_b f)^2 \\ &-(\frac{2}{\beta t} + a(1+\alpha))|\nabla_b f|^2] \end{aligned} \tag{3.11}$$

From the definition of F, we have

$$\begin{aligned} (\triangle_b f)^2 &= \left(\frac{F}{\alpha t} - \frac{1+\alpha}{\alpha}|\nabla_b f|^2 - \frac{\beta t}{\alpha}f_0^2\right)^2 \\ &\geq \frac{1}{\alpha^2 t^2}F^2 - \frac{2(1+\alpha)}{\alpha^2 t}F|\nabla_b f|^2 - \frac{2\beta}{\alpha^2}F f_0^2 \end{aligned} \tag{3.12}$$

From (3.11) and (3.12), we have

$$(\triangle_b - \frac{\partial}{\partial t})F \geq \left(\frac{F}{n\alpha^2 t} - \frac{1}{t} - a\right)F - 2\langle \nabla_b F, \nabla_b f\rangle$$
$$+t[(n-\beta-a\beta t - \frac{2\beta}{n\alpha^2}F)f_0^2 + \left(-\frac{2(1+\alpha)}{n\alpha^2 t}F - \frac{2}{\beta t} - a(1+\alpha)\right)|\nabla_b f|^2]. \tag{3.13}$$

For each fixed $T' < \infty$ and each $t \in [0, T']$, let $(p(t), s(t)) \in M \times [0, t]$ be the maximal point of F on $M \times [0, t]$, that is,

$$F(p(t), s(t), \alpha, \beta) = \max_{(x,\mu)\in M\times[0,t]} F(x, \mu, \alpha, \beta).$$

Then we have

$$\nabla_b F(p(t), s(t), \alpha, \beta) = 0, \tag{3.14}$$

$$\triangle_b F(p(t), s(t), \alpha, \beta) \leq 0, \tag{3.15}$$

and

$$\frac{\partial}{\partial t}F(p(t), s(t), \alpha, \beta) \geq 0 \tag{3.16}$$

From (3.13), (3.14), (3.15) and (3.16), we have at $(p(t), s(t))$,

$$0 \geq [\frac{F}{n\alpha^2 s(t)} - \frac{1}{s(t)} - a]F + s(t)[(n-\beta-a\beta s(t) - \frac{2\beta}{n\alpha^2}F)f_0^2$$
$$+(-\frac{2(1+\alpha)}{n\alpha^2 s(t)}F - \frac{2}{\beta s(t)} - a(1+\alpha))|\nabla_b f|^2]. \tag{3.17}$$

Next, we claim that for each fixed $T' < \infty$,

$$F(p(T'), s(T'), -1-\frac{3}{n}, \beta) < (\frac{n}{3\beta} - \frac{a}{2}s(T'))[\frac{9}{n} + 6 + n],$$

where $\alpha = -(1+\frac{3}{n})$ and $0 < \beta < \frac{n}{3}$. Here $(p(T'), s(T')) \in M \times [0, T']$ is the maximal point of F on $M \times [0, T']$.

We prove by contradiction. Suppose not, that is

$$F(p(T'), s(T'), -1-\frac{3}{n}, \beta) - (\frac{n}{3\beta} - \frac{a}{2}s(T'))[\frac{9}{n} + 6 + n] \geq 0.$$

Since $F(p(t), s(t), -1-\frac{3}{n}, \beta) - (\frac{n}{3\beta} - \frac{a}{2}s(t))[\frac{9}{n}+6+n]$ is continuous in the variable t when α, β are fixed and $F(p(0), s(0), -1-\frac{3}{n}, \beta) - (\frac{n}{3\beta} - \frac{a}{2}s(0))[\frac{9}{n}+6+n] = -\frac{n}{3\beta}[\frac{9}{n}+6+n] < 0$, by Intermdiate-value theorem there exists a $t_0 \in (0, T']$ such that

$$F(p(t_0), s(t_0), -1-\frac{3}{n}, \beta) - (\frac{n}{3\beta} - \frac{a}{2}s(t_0))(\frac{9}{n} + 6 + n) = 0,$$

Then we have

$$-\frac{2(1+\alpha)}{n\alpha^2 s(t_0)}F-\frac{2}{\beta s(t_0)}-a(1+\alpha)=\frac{3}{n}(-\frac{s(t_0)}{s(t_0)}+1)a=0, \tag{3.18}$$

$$n-\beta-a\beta s(t_0)-\frac{2\beta}{n\alpha^2}F=\frac{n}{3}-\beta>0, \tag{3.19}$$

and

$$\frac{F}{n\alpha^2 s(t_0)}-\frac{1}{s(t_0)}-a\geq\frac{1}{s(t_0)}[1-\frac{3}{n}\beta]\frac{n}{3\beta}>0. \tag{3.20}$$

From (3.17), (3.18), (3.19) and (3.20), we have

$$\begin{aligned}0&\geq[\frac{F}{n\alpha^2 s(t_0)}-\frac{1}{s(t_0)}-a]F+s(t_0)[(n-\beta-a\beta s(t_0)-\frac{2\beta}{n\alpha^2}F)f_0^2\\&+(-\frac{2(1+\alpha)}{n\alpha^2 s(t_0)}F-\frac{2}{\beta s(t_0)}-a(1+\alpha))|\nabla_b f|^2]>0.\end{aligned}$$

This gives a contradiction. Hence we have

$$F(p(T'),s(T'),-1-\frac{3}{n},\beta)<(\frac{n}{3\beta}-\frac{a}{2}s(T'))[\frac{9}{n}+6+n]\leq(\frac{n}{3\beta}-\frac{a}{2}T')[\frac{9}{n}+6+n]$$

This implies that

$$\max_{(x,t)\in M\times[0,T']}t(|\nabla_b f|^2+\alpha(f_t-af)+\beta t f_0^2)<(\frac{n}{3\beta}-\frac{a}{2}T')[\frac{9}{n}+6+n]$$

When we fix on the set $M\times\{T'\}$, we have

$$T'(|\nabla_b f|^2+\alpha(f_t-af)+\beta T' f_0^2)<(\frac{n}{3\beta}-\frac{a}{2}T')[\frac{9}{n}+6+n]$$

Since T' is arbitrary, we obtain

$$|\nabla_b f|^2+\alpha(f_t-af)+\beta t f_0^2)<(\frac{n}{3\beta t}-\frac{a}{2})[\frac{9}{n}+6+n]$$

Finally let $\beta\to\frac{n}{3}$, then we have

$$|\nabla_b f|^2-(1+\frac{3}{n})(f_t-af)+\frac{n}{3}tf_0^2<\frac{1}{t}[1-\frac{a}{2}t][\frac{9}{n}+6+n].$$

This completes the proof. □

When $a=0$, we have the following results:

Corollary 3.2. *Let* (M,J,θ) *be a closed pseudo-Hermitian* $(2n+1)$ *manifold. Suppose that*

$$2Ric(X,X)-(n-2)Tor(X,X)\geq 0,$$

for all $X \in T_{1,0} \oplus T_{0,1}$. If u is the positive solution of

$$\frac{\partial u}{\partial t} = \triangle_b u$$

with $[\triangle_b, T] = 0$ on $M \times [0, \infty)$, let $f(x,t) = \log u(x,t)$. Then we have

$$|\nabla_b f|^2 - \left(1 + \frac{3}{n}\right) f_t + \frac{n}{3} t f_0^2 < \frac{1}{t}\left(\frac{9}{n} + 6 + n\right).$$

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