Theorems on conformal mappings of complete Riemannian manifolds and their applications

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Abstract. We prove several Liouville-type non-existence theorems for conformal mappings of complete Riemannian manifolds. As well, we provide applications of these results to General Relativity and to the theory of conharmonic transformations.

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1 Subharmonic and superharmonic functions

Let (M, g) be an n-dimensional ($n \geq 2$) Riemannian manifold. We recall that $f \in C^2M$ is *subharmonic* (resp. *superharmonic* or harmonic) if $\Delta f \geq 0$ (resp. $\Delta f \leq 0$ or $\Delta f = 0$) for the Laplace-Beltrami operator $\Delta f = div\,(grad\, f)$. In particular, if (M, g) is compact, then every harmonic (subharmonic or superharmonic) functions is constant by Hopf's theorem [1].

We prove the following Lemma on superharmonic functions, which consists of two statements that are analogous to two Yau propositions on subharmonic functions (see [2]). Yau has stated in [2, p. 660] that on a complete Riemannian manifold (M, g), each subharmonic function $u \in C^2M$, whose gradient has integrable norm on (M, g), must be harmonic. Secondly, he has shown in [7, p. 663] that on a complete Riemannian manifold, each non-negative subharmonic function $u \in C^2M$ such that $\int_M u^p dVol_g < \infty$ for some $1 < p < \infty$, must be constant. In particular, if the volume of (M, g) is infinite, then $u = 0$.

Lemma 1.1. *If (M, g) is a connected complete Riemannian manifold (without boundary), then any superharmonic function $\varphi \in C^2M$ with $\|grad\,\varphi\| \in L^1(M, g)$ is harmonic and each non-positive superharmonic function $\varphi \in C^2M$ such that $\varphi \in L^p(M, g)$ for some $1 < p < \infty$ must be constant. In particular, if the volume of (M, g) is infinite, then $\varphi = 0$.*

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Proof. On the one hand, if we assume that $u = -\varphi$ for any superharmonic function $\varphi \in C^2M$ then the conditions $\Delta\varphi \leq 0$ and $\|grad\ \varphi\| \in L^1(M, g)$, which must be satisfy for the super-harmonic function φ can be written in the form $\Delta u \geq 0$ and $\|grad\ u\| \in L^1(M, g)$. In this case, using the Yau statement for subharmonic functions we conclude that $\Delta u = 0$ and hence $\varphi = -u$ is a harmonic function. On the other hand, the function $u = -\varphi$ for any superharmonic function $\varphi \in C^2M$ which satisfies the conditions $\varphi \leq 0$, $\Delta\varphi \leq 0$ and $\int_M |\varphi|^p \, dVol_g < \infty$ for some $1 < p < \infty$ must be satisfied the following conditions $u \geq 0$, $\Delta u \geq 0$ and $\int_M u^p dVol_g < \infty$ for some $1 < p < \infty$. Therefore, u is a constant function and hence $\varphi = -u$ is a constant function too. It is obvious that if the volume of (M, g) is infinite, then $\varphi = 0$. □

2 Conformal diffeomorphisms of complete Riemannian manifolds

Let (M, g) and $\left(\bar{M}, \bar{g}\right)$ be pseudo-Riemannian or Riemannian manifolds such that dim $M =$ dim $\bar{M} = n$ for any $n \geq 3$. Then a diffeomorphism $f : (M, g) \to \left(\bar{M}, \bar{g}\right)$ is called *conformal* if it preserves angles between any pair curves. In this case, $\bar{g} = e^{2\sigma} g$ for some scalar function σ (see [2, p. 663]). If the function σ is a constant then f is a *homothetic mapping*. In particular, if $\sigma = 0$, f is an *isometric mapping*.

If $\sigma \in C^2M$ then for each pair of corresponding points $x \in M$ and $\overline{x} = f(x) \in \bar{M}$ we have the equation (see [3, p. 90])

$$e^{2\sigma} \bar{s} = s - 2(n-1)\Delta\sigma - (n-1)(n-2)\|grad\ \sigma\|^2, \tag{2.1}$$

where s and $\bar{s}$ denote the scalar curvatures of (M, g) and $\left(\bar{M}, \bar{g}\right)$, respectively. In the case when (M, g) and $\left(\bar{M}, \bar{g}\right)$ are Riemannian manifolds we can formulate the following Liouville-type non-existence theorem.

Theorem 2.1. *Let (M, g) be an n-dimensional $(n \geq 3)$ complete Riemannian manifold and $f : (M, g) \to \left(\bar{M}, \bar{g}\right)$ be a conformal diffeomorphism onto another Riemannian manifold $\left(\bar{M}, \bar{g}\right)$ such that $\bar{g} = e^{2\sigma} g$ and $\bar{s} \geq e^{-2\sigma} s$ for some function $\sigma \in C^2M$ and the scalar curvatures s and $\bar{s}$ of (M, g) and $\left(\bar{M}, \bar{g}\right)$, respectively. Then the following propositions are true.*

1. *If $\|grad\ \sigma\| \in L^1(M, g)$, then f is a homothetic mapping.*
2. *If σ is non-positive function and $\sigma \in L^p(M, g)$ for some $1 < p < \infty$ then f is a homothetic mapping. In particular, if the volume of (M, g) is infinite, then f is an isometric mapping.*

Proof. If $f : (M, g) \to \left(\bar{M}, \bar{g}\right)$ is a conformal diffeomorphism a connected complete Riemannian manifold (M, g) onto another Riemannian manifold $\left(\bar{M}, \bar{g}\right)$ such that $\bar{g} = e^{2\sigma} g$ for some function $\sigma \in C^2M$, then from (2.1) we obtain

$$2(n-1)\Delta\sigma = s - e^{2\sigma}\bar{s} - (n-1)(n-2)\|grad\ \sigma\|^2. \tag{2.2}$$

Let $s \leq e^{2\sigma}\bar{s}$ then (2) shows $\Delta\sigma \leq 0$. It means that σ is a superharmonic function. By the condition of our theorem, the gradient of σ has integrable norm on (M, g) and we obtain from (2.2) that $\Delta\sigma = 0$ (see our Lemma). In this case, σ is a harmonic function. Since $n \geq 3$, we see from (2.2) that σ is constant. In the other hand, if σ is a non-positive function such that $s \leq e^{2\sigma}\bar{s}$ and $\sigma \in L^P(M,g)$ for some $1 < p < \infty$ then using the Lemma we can conclude that σ is a constant function. It is obvious that if the volume of (M,g) is infinite, then $\sigma = 0$ (see our Lemma). The proof of the theorem is complete. □

In particular, if we assume that $s \geq 0$ and $\bar{s} \leq 0$ in the condition of our theorem, then the inequality $s \geq \lambda^2\bar{s}$ must be satisfied. Then, as a result the proofs of the theorem, we can conclude that $s = \bar{s} = 0$. Therefore we have

Corollary 2.2. *Let (M,g) be an n-dimensional $(n \geq 0)$ complete Riemannian manifold and $f : (M,g) \to (\bar{M},\bar{g})$ be a conformal diffeomorphism onto another Riemannian manifold $(\bar{M},\bar{g})$ such that $\bar{g} = e^{2\sigma}g$ for some function $\sigma \in C^2M$, $s \geq 0$ and $\bar{s} \leq 0$ for the scalar curvatures s and $\bar{s}$ of (M,g) and $(\bar{M},\bar{g})$, respectively. If the one of the following conditions holds:*

1. *$\|grad\,\sigma\| \in L^1(M,g)$,*
2. *$\sigma \in L^p(M,g)$ for some $1 < p < \infty$ and $\sigma \leq 0$,*

then f is a homothetic mapping and $s = \bar{s} = 0$. If in the second case the volume of (M,g) is infinite, then f is an isometric mapping.

Let $\sigma = \log\lambda$ for some positive scalar function $\lambda \in C^2M$ then

$$\Delta\sigma = \lambda^{-1}\Delta\lambda - \lambda^{-2}\|grad\,\lambda\|^2, \qquad \|grad\,\sigma\|^2 = \lambda^{-2}\|grad\,\lambda\|^2.$$

In this case, (2.2) can be rewritten in the following equivalent form

$$2(n-1)\lambda\Delta\lambda = \lambda^2\left(s - \lambda^2\bar{s}\right) - (n-1)(n-4)\|grad\,\lambda\|^2. \tag{2.3}$$

If $s \geq \lambda^2\bar{s}$ for $n \leq 4$ then from (2.3) we obtain that $\lambda\Delta\lambda \geq 0$. On the other hand, Yau has proved in [2, p. 664] that if a smooth function $\lambda \in C^2M$ on a complete Riemannian manifold (M,g) such that $\lambda\Delta\lambda \geq 0$, then either $\int_M |\lambda|^p dV_g = \infty$ for all $p \neq 1$ or $\lambda = constant$. Therefore, in the case when (M,g) and $(\bar{M},\bar{g})$ are Riemannian manifolds we have

Theorem 2.3. *Let (M,g) be an n-dimensional $(n = 3,4)$ complete Riemannian manifold and $f : (M,g) \to (\bar{M},\bar{g})$ be a conformal diffeomorphism onto another Riemannian manifold $(\bar{M},\bar{g})$ such that $\bar{g} = \lambda^2 g$ and $s \geq \lambda^2\bar{s}$ for some positive function $\lambda \in C^2M$ and for the scalar curvatures s and $\bar{s}$ of (M,g) and $(\bar{M},\bar{g})$, respectively. If $\lambda \in L^p(M,g)$ for some $p \neq 1$, then f is a homothetic mapping.*

In particular, if we assume that $s \geq 0$ and $\bar{s} \leq 0$ in the condition of Theorem 2.3, then one can verify that in this case f is a homothetic mapping and $s = \bar{s} = 0$. Therefore, we have

Corollary 2.4. *Let (M,g) be an n-dimensional $(n = 3,4)$ complete Riemannian manifold and $f : (M,g) \to (\bar{M},\bar{g})$ be a conformal diffeomorphism onto another Riemannian manifold $(\bar{M},\bar{g})$ such that $\bar{g} = \lambda^2 g$ for some positive function $\lambda \in C^2M$ and $\lambda \in L^p(M,g)$ for some $p \neq 1$. If $s \geq 0$ and $\bar{s} \leq 0$ for the scalar curvatures s and $\bar{s}$ of (M,g) and $(\bar{M},\bar{g})$, respectively, then f is a homothetic mapping and $s = \bar{s} = 0$.*

If we assume that $\lambda = u^{\frac{2}{n-2}}$, then (2.3) immediately gives

$$\frac{4(n-1)}{n-2}\Delta u = s\, u - \bar{s}\, u^{\frac{n+2}{n-2}}. \tag{2.4}$$

In the case of the Riemannian manifolds (M, g) and $(\bar{M}, \bar{g})$, the equation (2.4) is the classical *Yamabe equation* (see [5, p. 39]). The equation (2.4) can be written in the form

$$\frac{4\,(n-1)}{n-2}\Delta u = u\left(s - \lambda^2 \bar{s}\right). \tag{2.5}$$

Then for $s \geq \lambda^2 \bar{s}$, from (2.4) we obtain that $\Delta u \geq 0$. On the other hand, Yau has shown in [2, p. 663] that if u is a non-negative subharmonic function defined on a complete Riemannian manifold (M, g), then $\int_M u^p dV_g = \infty$ for all $p > 1$, unless u =constant. Therefore, in the case when (M, g) and $(\bar{M}, \bar{g})$ are Riemannian manifolds, we have the following Liouville-type non-existence theorem.

Theorem 2.5. *Let (M, g) be a n-dimensional $(n \geq 3)$ complete Riemannian manifold and $f : (M, g) \to \left(\bar{M}, \bar{g}\right)$ be a conformal diffeomorphism onto another Riemannian manifold $\left(\bar{M}, \bar{g}\right)$ such that $\bar{g} = \lambda^2 g$ and $\lambda^{(n-2)/2} \in L^p(M, g)$ for some positive function $\lambda \in C^2 M$ and for some $p \neq 1$. If $s \geq \lambda^2 \bar{s}$ for the scalar curvatures s and $\bar{s}$ of (M, g) and $\left(\bar{M}, \bar{g}\right)$, respectively, then f is a homothetic mapping.*

In particular, if we assume that $s \geq 0$ and $\bar{s} \leq 0$ in the condition of Theorem 2.5, then we can prove that f is a homothetic mapping and $s = \bar{s} = 0$. Therefore we have

Corollary 2.6. *Let (M, g) be a n-dimensional $(n \geq 3)$ complete Riemannian manifold and $f : (M, g) \to \left(\bar{M}, \bar{g}\right)$ be a conformal diffeomorphism onto another Riemannian manifold $\left(\bar{M}, \bar{g}\right)$ such that $\bar{g} = \lambda^2 g$ and $\lambda^{(n-2)/2} \in L^p(M, g)$ for some positive function $\lambda \in C^2 M$ for some $p \neq 1$. If $s \geq 0$ and $\bar{s} \leq 0$ for the scalar curvatures s and $\bar{s}$ of (M, g) and $\left(\bar{M}, \bar{g}\right)$, respectively, then f is a homothetic mapping and $s = \bar{s} = 0$.*

3 An application to the theory of conharmonic transformations

A mapping $f : (M, g) \to (M, \bar{g})$ is called *conharmonic transformation* (Ishi, [4]) if it is a conformal transformation, i.e., $\bar{g} = e^{2\sigma} g$ for some scalar function $\sigma \in C^2 M$ satisfying the equation

$$\Delta \sigma = -\frac{n-2}{2}\left\| grad\ \sigma \right\|^2 \tag{3.1}$$

for any $n \geq 3$. The conharmonic transformations introduced by Ishi are a subgroup of the group of conformal transformations which preserve the harmonicity of certain class of smooth functions (see [5]). From (3.1) we conclude that σ is a superharmonic function. Then the following Corollary is obvious from Theorem 2.1.

Corollary 3.1. *Let $f : (M, g) \to (M, \bar{g})$ be a conharmonic transformation of an n-dimensional $(n \geq 3)$ complete Riemannian manifold (M, g), i.e. $\bar{g} = e^{2\sigma} g$ for* *some function $\sigma \in C^2M$ which satisfies the equation* (3.1). *If σ has a gradient with integrable norm on (M, g), then the function σ is constant and f is a homothetic transformation.*

Let $\sigma = \log \lambda$ for some positive scalar function $\lambda \in C^2M$ then (3.1) can be rewritten in the following equivalent form

$$2\lambda\Delta\lambda = (n-4)\,\|grad\ \lambda\|^2\,. \tag{3.2}$$

In this case, we can formulate a proposition that is an analogue of Theorem 2.5.

Corollary 3.2. *Let $f : (M, g) \to (M, \bar{g})$ be a conharmonic transformation of an n-dimensional $(n \geq 4)$ complete Riemannian manifold (M, g), i.e. $\bar{g} = \lambda^2 g$ for some positive function $\lambda \in C^2M$ which satisfies the equation* (3.2). *If $\lambda \in L^p(M, g)$ for some $p \neq 1$, then f is a homothetic mapping.*

In particular, for $n = 4$ from (3.2) we obtain that $\Delta\lambda = 0$. Then λ is a positive harmonic function on a complete Riemannian manifold (M, g). We can easily state the following

Theorem 3.3. *Let $f : (M, g) \to (M, \bar{g})$ be a conharmonic transformation of a n-dimensional Riemannian manifold (M, g) such that $\bar{g} = \lambda^2 g$, then for the case $n = 4$ the function λ is harmonic.*

Remark 3.1. Corollaries 3.1 and 3.2 generalize Proposition 4.7 from [6] on conharmonic transformations of compact manifolds.

4 An application to General Relativity

In this paragraph we give an application of our results to General Relativity using the classical Bochner technique for Lorentzian geometry (see, for example, [7]). Let (M, g) be a compact space-time, i.e. a four-dimensional compact Lorentzian manifold (M, g). For $n = 4$, the equation (2.3) can be rewritten in the form

$$6\Delta\lambda = \lambda\left(s - \lambda^2\bar{s}\right). \tag{4.1}$$

In this case, using Green's divergence theorem from (4.1), we obtain the integral formula

$$\int_M \lambda\left(s - \lambda^2\bar{s}\right) dV_g = 0. \tag{4.2}$$

It's obvious that the conditions $s > \lambda^2\bar{s}$, or $s < \lambda^2\bar{s}$ contrast with (4.1). Therefore, we can formulate the following non-existence theorem.

Theorem 4.1. *Let (M, g) be a compact space-time. There does not exist any conformal transformation $f : (M, g) \to (M, \bar{g})$ such that $\bar{g} = \lambda^2 g$ and $s > \lambda^2\bar{s}$ (or $s < \lambda^2\bar{s}$) for some positive function $\lambda \in C^2M$ and the scalar curvatures s and $\bar{s}$ of (M, g) and $(M, \bar{g})$, respectively.*

Moreover, we have the following

Corollary 4.2. *Let (M, g) be a compact space-time. There does not exist any conformal transformation $f : (M, g) \to (M, \bar{g})$ such that $\bar{g} = \lambda^2 g$, $s > 0$ and $\bar{s} < 0$ (or $s > 0$ and $\bar{s} < 0$) for some positive function $\lambda \in C^2 M$ and the scalar curvatures s and $\bar{s}$ of (M, g) and $(M, \bar{g})$, respectively.*

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