Orlicz mixed affine surface areas

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Abstract. In the paper, our main aim is to generalize the classical $L_p$-Blaschke addition and $L_p$-affine surface areas to the Orlicz space. Under the framework of Orlicz-Brunn-Minkowski theory, we find a geometric operation call it Orlicz $L_\psi$-Blaschke addition. A new affine geometric quantity with respect to the operation is introduced and call it Orlicz $L_\psi$-mixed affine surface area. The fundamental notions and conclusions of the $L_p$-Blaschke addition and $L_p$-affine surface areas, and Minkowski’s and Brunn-Minkowski’s inequalities for the $L_p$-affine surface areas are extended to an Orlicz setting. The new related concepts and inequalities of $L_{pq}$-mixed affine surface areas are also derived.

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Key words: convex body; $L_p$-Blaschke addition; $L_p$-affine surface areas; Orlicz $L_\psi$-Blaschke addition; Orlicz $L_\psi$-mixed affine surface areas; Orlicz Brunn-Minkowski theory.

1 Introduction

One of the most important operations in geometry is vector addition. As an operation between sets $K$ and $L$, defined by

$$K + L = \{x + y : x \in K, y \in L\},$$

it is usually called Minkowski addition and combine volume play an important role in the Brunn-Minkowski theory. During the last few decades, the theory has been extended to $L_p$-Brunn-Minkowski theory. The first, a set called as $L_p$ addition, introduced by Firey in [3] or [4]. Denoted by $+_p$, for $1 \leq p \leq \infty$, defined by

$$h(K +_p L, x)^p = h(K, x)^p + h(L, x)^p,$$  \hspace{1cm} (1.1)

for all $x \in \mathbb{R}^n$ and compact convex sets $K$ and $L$ in $\mathbb{R}^n$ containing the origin. When $p = \infty$, (1.1) is interpreted as $h(K +_\infty L, x) = \max\{h(K, x), h(L, x)\}$, as is customary. Here the functions are the support functions. If $K$ is a nonempty closed (not necessarily bounded) convex set in $\mathbb{R}^n$, then

$$h(K, x) = \max\{x \cdot y : y \in K\},$$
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for \( x \in \mathbb{R}^n \), defines the support function \( h(K, x) \) of \( K \). A nonempty closed convex set is uniquely determined by its support function. \( L_p \) addition and inequalities are the fundamental and core contents in the \( L_p \) Brunn-Minkowski theory. For some important results and more information from this theory, we refer to [8], [9], [10], [11], [17], [18], [20], [21], [24], [25], [26], [27], [28], [31], [32], [34], [36], [38], [39], [43] and the references therein. In recent years, a new extension of \( L_p \)-Brunn-Minkowski theory is to Orlicz-Brunn-Minkowski theory, initiated by Lutwak et al [29] and [30]. Gardner, Hug and Weil [6] constructed a general framework for the Orlicz-Brunn-Minkowski theory, not only introduced the Orlicz addition but also made clear for the first time the relation to Orlicz spaces and norms. The Orlicz addition of convex bodies was also introduced and extend the \( L_p \)-Brunn-Minkowski inequality to the Orlicz-Brunn-Minkowski inequality (see [40]). The Orlicz centroid inequality for convex bodies was introduced and extend the \( L_p \)-Brunn-Minkowski inequality (see [40]). The Orlicz centroid inequality for convex bodies was introduced in [53]. The Orlicz-Brunn-Minkowski theory and its dual theory have attracted people’s attention. The other articles recent advance these theories can be found in literatures [7], [13], [14], [15], [16], [19], [33], [37], [41], [42], [44], [45], [46], [47], [48], [49], [50], [51] and [52].

A body in \( \mathbb{R}^n \) is a compact set equal to the closure of its interior. A set \( K \) is called a convex body if it is compact and convex subset with non-empty interiors. Let \( K_n \) denote the class of convex bodies in \( \mathbb{R}^n \). Let \( K_n^o \) denote the class of convex bodies containing the origin in their interiors in \( \mathbb{R}^n \). A convex body \( K \in K_n \) was said to have a positive continuous curvature function \( f_p(K, \cdot) : S^{n-1} \to [0, \infty) \), if \( S_p(K, \cdot) \), is absolutely continuous with respect to spherical Lebesgue measure, \( S \), and (see e.g. [22])

\[
\frac{dS_p(K, \cdot)}{dS} = f_p(K, \cdot),
\]

almost everywhere with respect to \( S \), and where \( S_p(K, \cdot) \) denotes the positive Borel measure on \( S^{n-1} \) (see [22]). The subset of \( K_n \) consisting of convex bodies which have a positive continuous curvature function will be denoted by \( F_n \). The subset of \( K_n^o \) consisting of convex bodies which have a positive continuous curvature function will be denoted by \( F_n^o \). The class of the origin-symmetric convex bodies with positive and continuous curvature function in \( \mathbb{R}^n \) will be denoted by \( F_n^s \). Lutwak [22] introduced the \( L_p \)-affine surface areas: For \( p \geq 1 \), the \( L_p \)-affine surface area of \( K \in F_n^o \), denoted by \( \Omega_p(K) \), defined by

\[
\Omega_p(K) = \int_{S^{n-1}} f_p(K, u)^{\frac{1}{p}} dS(u).
\]

Moreover, the mixed affine surface areas of convex bodies was introduced in [23]. The classical \( L_p \)-Blaschke addition of convex bodies \( K, L \in F_n^o \), denoted by \( K \oplus_p L \), defined by (see [21])

\[
dS_p(K \oplus_p L, \cdot) = dS_p(K, \cdot) + dS_p(L, \cdot).
\]

In the paper, our main aim is to generalize the \( L_p \)-affine surface area \( \Omega_p(K) \) and the \( L_p \)-Blaschke addition \( \oplus_p \) to the Orlicz space. Under the framework of Orlicz-Brunn-Minkowski theory, we first introduce Orlicz \( L_\psi \)-Blaschke addition \( \oplus_\psi \). On this basis, we introduce a new affine geometric quantity call it Orlicz \( L_\psi \)-mixed affine surface area. The fundamental notions and conclusions of the \( L_p \)-Blaschke addition and \( L_p \)-affine surface areas, and Minkowski’s and Brunn-Minkowski’s inequalities for
the $L_p$-affine surface areas are extended to an Orlicz setting. The related concepts and inequalities of $L_{pq}$-mixed affine surface areas of convex bodies are also derived. The new Orlicz $L_p$-Minkowski and Orlicz $L_p$-Brunn-Minkowski inequalities in special case yield the well-known $L_p$-Minkowski and $L_p$-Brunn-Minkowski inequalities, and yield new $L_{pq}$-Minkowski and $L_{pq}$-Brunn-Minkowski inequalities.

In Section 3, we introduce a notion of Orlicz $L_\varphi$-Blaschke addition of convex bodies $K, L \in \mathcal{F}_s^n$, denoted by $K \overset{\varphi}{+} L$, defined by

$$
\psi \left( \frac{f_p(K, u)}{f_p(K \overset{\varphi}{+} L, u)} \cdot \frac{f_p(L, u)}{f_p(K \overset{\varphi}{+} L, u)} \right) = 1, \quad (1.5)
$$

for $u \in S^{n-1}$ and $p \geq 1$, if $f_p(K, u) + f_p(L, u) > 0$, and by $f_p(K \overset{\varphi}{+} L, u) = 0$, if $f_p(K, u) = f_p(L, u) = 0$. Here $\psi \in \Phi_2$, and $\Phi_2$ denotes the set of convex functions $\varphi : [0, \infty)^2 \to [0, \infty)$ that are increasing in each variable and satisfy $\varphi(0, 0) = 0$ and $\varphi(1, 0) = \varphi(0, 1) = 1$. When $\psi(x_1, x_2) = x_1^q + x_2^q$ and $q = 1$, the Orlicz $L_\varphi$-Blaschke addition (1.5) becomes to

$$
f_p(K \overset{\varphi}{+} L, \cdot) = f_p(K, \cdot) + f_p(L, \cdot).
$$

This is just the $L_p$-Blaschke addition $\overset{\varphi}{+}$ defined in (1.4). The particular instance of interest corresponds to using (1.5) with $\psi(x_1, x_2) = \psi_1(x_1) + \varepsilon \psi_2(x_2)$ for $\varepsilon > 0$ and some $\psi_1, \psi_2 \in \Phi$, where $\Phi$ is the set of convex functions $\psi_1, \psi_2 : [0, \infty) \to (0, \infty)$ that are increasing and satisfy $\psi_1(0) = \psi_2(0) = 0$ and $\psi_1(1) = \psi_2(1) = 1$.

Comply with the basic spirit ofAleksandrov [1], Fenchel and Jessen [2] introduction of mixed quermassintegrals, and introduction of Lutwak’s [22] $p$-affine surface areas, we are based on the study of the first order Orlicz variational of the $L_p$-affine surface areas. In Section 4, we prove that the first order Orlicz variation of the $L_p$-affine surface areas can be expressed as: For $K, L \in \mathcal{F}_s^n$, $\psi_1, \psi_2 \in \Phi$, $p \geq 1$ and $\varepsilon > 0$,

$$
\frac{d}{d\varepsilon} \bigg|_{\varepsilon=0^+} \Omega_p(K \overset{\varphi}{+}\varepsilon \cdot L) = \frac{n}{n+p} \cdot \frac{1}{(\psi_1)'(1)} \cdot \Omega_{\psi_2,p}(K,L), \quad (1.6)
$$

where $(\psi_1)'(1)$ denotes the value of left derivative of convex function $\psi_1$ at point 1. In this first order variational equation (1.6), we find a new geometric quantity. Based on this, we extract the required geometric quantity, denoted by $\Omega_{\psi_2,p}(K,L)$ and call it Orlicz $L_{\psi_2}$-mixed affine surface area of $K, L \in \mathcal{F}_s^n$, defined by for $p \geq 1$

$$
\Omega_{\psi_2,p}(K,L) = \frac{n+p}{n} \cdot (\psi_1)'(1) \cdot \frac{d}{d\varepsilon} \bigg|_{\varepsilon=0^+} \Omega_p(K \overset{\varphi}{+} \psi \cdot L), \quad (1.7)
$$

where $\psi_1, \psi_2 \in \Phi$. We also prove the new affine geometric quantity has an integral representation.

$$
\Omega_{\psi,p}(K,L) = \int_{S^{n-1}} \psi \left( \frac{f_p(L, u)}{f_p(K, u)} \cdot f_p(K, u) \right) dS(u), \quad (1.8)
$$

where $\psi \in \Phi$ and $p \geq 1$. In Section 5, we establish an Orlicz $L_\psi$-Minkowski inequality for the Orlicz $L_\psi$-mixed affine surface areas: If $K, L \in \mathcal{F}_s^n$, $p \geq 1$ and $\psi \in \Phi$, then

$$
\Omega_{\psi,p}(K,L) \geq \Omega_p(K) \cdot \psi \left( \left( \frac{\Omega_p(L)}{\Omega_p(K)} \right)^{\frac{n+p}{n}} \right), \quad (1.9)
$$
If $\psi$ is strictly convex, equality holds if and only if $K$ and $L$ are homothetic.

When $\psi(t) = t^{\frac{n+p}{n}}$, $i \geq n + p$, $i \in \mathbb{R}$ and $p \geq 1$, (1.9) becomes the following $L_p$-Minkowski inequality for $p$-mixed affine surface area, which was established in [36]. If $K, L \in F^n_p$, $p \geq 1$, $i \in \mathbb{R}$ and $i \geq n + p$, then
\[
\Omega_{p,i}(K, L) \geq \Omega_p(K)^{\frac{n-p}{n}} \cdot \Omega_p(L)^{\frac{n}{n-p}},
\]
with equality if and only if $K$ and $L$ are homothetic, and where
\[
\Omega_{p,i}(K, L) = \int_{S^{n-1}} f_p(K, u)^{\frac{n-p}{n}} f_p(L, u)^{\frac{n}{n-p}} dS(u).
\]

In Section 6, we establish an Orlicz $L_p$-Brunn-Minkowski inequality for the Orlicz $L_p$-Blaschke addition and the $L_p$ affine surface areas. If $K, L \in F^n_p$, $p \geq 1$ and $\psi \in \Psi_2$, then
\[
1 \geq \psi \left( \left( \frac{\Omega_p(K)}{\Omega_p(K+qL)} \right)^{\frac{n+p}{n}} , \left( \frac{\Omega_p(L)}{\Omega_p(K+qL)} \right)^{\frac{n+p}{n}} \right).
\]
If $\psi$ is strictly convex, equality holds if and only if $K$ and $L$ are homothetic.

When $\psi(s, t) = s^q + t^q$ and $q = 1$, (1.11) becomes the following $L_p$-Brunn-Minkowski inequality for $p$-mixed affine surface area, which was established in [36]. If $K, L \in F^n_p$ and $p \geq 1$, then
\[
\Omega_p(K^p L) \geq \Omega_p(K)^{\frac{n+p}{n}} \Omega_p(L)^{\frac{n+p}{n}},
\]
with equality if and only if $K$ and $L$ are homothetic.

2 Preliminaries

The setting for this paper is $n$-dimensional Euclidean space $\mathbb{R}^n$. The support function of convex body $K$ is homogeneous of degree 1, that is (see e.g. [35]),
\[
h(K, ru) = rh(K, u),
\]
for all $u \in S^{n-1}$ and $r > 0$. If $K \in K^n$ and $A \in GL(n)$, then for all $x \in \mathbb{R}^n$ (see e.g. [5], p.17)
\[
h(AK, x) = h(K, A^t x).
\]
Let $\delta$ denote the Hausdorff metric, as follows, if $K, L \in K^n$, then
\[
\delta(K, L) = \|h(K, u) - h(L, u)\|_\infty.
\]
For $K_i \in F^n_p$, $i = 1, \ldots, m$, define the real numbers $R_{K_i}$ and $r_{K_i}$ by
\[
R_{K_i} = \max_{u \in S^{n-1}} f_p(K_i, u), \quad r_{K_i} = \min_{u \in S^{n-1}} f_p(K_i, u).
\]
Obviously, $0 < r_{K_i} < R_{K_i}$, for all $K_i \in F^n_p$, and writing $R = \max\{R_{K_i}\}$ and $r = \min\{r_{K_i}\}$, where $i = 1, \ldots, m$. 
2.1 \( L_p \)-curvature function

A convex body \( K \in \mathcal{K}^n \) was said to have first order positive continuous curvature function \( f(K, \cdot) : S^{n-1} \to [0, \infty) \), if \( S(K, \cdot) \), is absolutely continuous with respect to spherical Lebesgue measure, \( S \), and
\[
\frac{dS(K, \cdot)}{dS} = f(K, \cdot),
\]
almost everywhere with respect to \( S \). A convex body \( K \in \mathcal{K}^n \) was said to have a positive continuous curvature function \( f_p(K, \cdot) : S^{n-1} \to [0, \infty) \), if \( S_p(K, \cdot) \), is absolutely continuous with respect to spherical Lebesgue measure, \( S \), and
\[
\frac{dS_p(K, \cdot)}{dS} = f_p(K, \cdot),
\]
almost everywhere with respect to \( S \). It is easily seen that a body in \( \mathcal{K}^n \) has a positive continuous curvature function if and only if the body belongs to \( \mathcal{F}^n_o \). Obviously, for \( K \in \mathcal{F}^n_o \) and \( p \geq 1 \)
\[
f_p(K, \cdot) = h(K, \cdot)^{1-p} f(K, \cdot).
\]
Suppose \( K \in \mathcal{F}^n_o \). If \( p \geq 1 \) and \( A \in SL(n) \), then
\[
f_p(AK, u) = f_p(K, Au), \tag{2.1}
\]
for \( u \in S^{n-1} \) (see [22]).

2.2 \( L_p \)-mixed affine surface areas

If \( K_1, \ldots, K_n \in \mathcal{F}^n_o \), the \( L_p \)-mixed affine surface area of \( K_1, \ldots, K_n \), denoted by \( \Omega_p(K_1, \ldots, K_n) \), defined by Lutwak (see [22])
\[
\Omega_p(K_1, \ldots, K_n) = \int_{S^{n-1}} (f_p(K_1, u) \cdots f_p(K_n, u)) \frac{n-1}{p} dS(u). \tag{2.2}
\]
If \( K_1 = \cdots = K_n = K \), then the \( L_p \)-mixed affine surface area \( \Omega(K_1, \ldots, K_n) \) is written as \( \Omega_p(K) \). Obviously, for \( K \in \mathcal{F}^n_o \) and \( p \geq 1 \),
\[
\Omega_p(K) = \int_{S^{n-1}} f_p(K, u) \frac{n-1}{p} dS(u). \tag{2.3}
\]
This is just the \( L_p \)-affine surface area \( \Omega_p(K) \) stated in the introduction.

2.3 \( L_{pq} \)-mixed affine surface areas

When \( \psi(x_1, x_2) = x_1^q + x_2^q \) and \( q \geq 1 \), the Orlicz \( L_\psi \)-Blaschke addition \( +_\psi \) becomes a new addition in \( L_p \)-space, denoted by \( +_{pq} \), and call as \( L_{pq} \)-Blaschke addition of convex bodies \( K, L \in \mathcal{F}^n_o \)
\[
f_p(K +_{pq} L, u)^q = f_p(K, u)^q + f_p(L, u)^q, \tag{2.4}
\]
for \( u \in S^{n-1} \) and \( p \geq 1 \). Obviously, when \( q = 1 \), \( L_{pq} \)-Blaschke addition becomes \( L_p \)-Blaschke addition. The following result follows immediately form (2.4) with \( p, q \geq 1 \).
\[
\frac{q(n+p)}{n} \lim_{\varepsilon \to 0^+} \frac{\Omega_p(K +_{pq_\varepsilon} L) - \Omega_p(L)}{\varepsilon} = \int_{S^{n-1}} f_p(K, u) \frac{p^n}{n^{n-1}} \frac{q-1}{p} f_p(L, u)^q dS(u).
\]
Definition 2.1 Let $K, L \in \mathcal{F}_s^n$ and $p, q \geq 1$, $L_{pq}$-mixed affine surface area of $K$ and $L$, denoted by $\Omega_{p,q}(K,L)$, defined by

$$\Omega_{p,q}(K,L) = \int_{S^{n-1}} f_p(K,u) \frac{u^q}{u^n} \frac{u^q}{u^n} dS(u). \quad (2.5)$$

Obviously, when $K = L$, the $L_{pq}$-mixed affine surface area $\Omega_{p,q}(K,K)$ becomes the $L_p$ affine surface area $\Omega_p(K)$. This integral representation (2.5), together with Hölder inequality, immediately gives:

Proposition 2.2 ($L_{pq}$-Minkowski inequality) If $K, L \in \mathcal{F}_s^n$ and $p, q \geq 1$, then

$$\Omega_{p,q}(K,L) \frac{u^q}{u^n} \frac{u^q}{u^n} \geq \Omega_p(K) \frac{u^q}{u^n} \frac{u^q}{u^n} \Omega_p(L)^q, \quad (2.6)$$

with equality if and only if $K$ and $L$ are homothetic.

Proposition 2.3 ($L_{pq}$-Brunn-Minkowski inequality) If $K, L \in \mathcal{F}_s^n$ and $p, q \geq 1$, then

$$\Omega_p(K\overset{t}{+}p,L) \frac{u^q}{u^n} \frac{u^q}{u^n} \geq \Omega_p(K) \frac{u^q}{u^n} \frac{u^q}{u^n} \Omega_p(L)^q,$$

(2.7)

with equality if and only if $K$ and $L$ are homothetic.

Proof From (2.4) and (2.5), it is easily seen that the $L_{pq}$-mixed affine surface area is linear with respect to the $L_{pq}$-Blaschke addition, and together with inequality (2.6) show that for $p, q \geq 1$

$$\Omega_{p,q}(Q, K\overset{t}{+}pL) = \Omega_{p,q}(Q, K) + \Omega_{p,q}(Q, L) \geq \Omega_p(Q) \frac{u^q}{u^n} \frac{u^q}{u^n} \Omega_p(K)^q + \Omega_p(L)^q,$$

with equality if and only if $K$ and $L$ are homothetic.

Take $K\overset{t}{+}pL$ for $Q$, recall that $\Omega_{p,q}(Q, Q) = \Omega_p(Q)$, inequality (2.7) follows easy. This completes the proof. \square

3 Orlicz $L_\psi$-Blaschke addition

Throughout the paper, the standard orthonormal basis for $\mathbb{R}^n$ will be $\{e_1, \ldots, e_n\}$. Let $\Phi_n, n \in \mathbb{N}$, denote the set of convex function $\psi : [0, \infty)^n \to (0, \infty)$ that are increasing in each variable and satisfy $\psi(0) = 0$ and $\psi(e_j) = 1$, $j = 1, \ldots, n$. When $n = 1$, we shall write $\Phi$ instead of $\Phi_1$. The left derivative and right derivative of a real-valued function $f$ are denoted by $(f)_l$ and $(f)_r$, respectively. We first define the Orlicz $L_\psi$-Blaschke addition.

Definition 3.1 Let $m \geq 2, \psi \in \Phi_m, K_j \in \mathcal{F}_s^n$ and $j = 1, \ldots, m$, Orlicz $L_\psi$-Blaschke addition of $K_1, \ldots, K_m$, denoted by $\overset{\psi}{+}(K_1, \ldots, K_m)$, defined by

$$f_p(\overset{\psi}{+}(K_1, \ldots, K_m), u) = \inf \left\{ \lambda > 0 : \psi \left( \frac{f_p(K_1, u)}{\lambda}, \ldots, \frac{f_p(K_m, u)}{\lambda} \right) \leq 1 \right\}, \quad (3.1)$$

for $u \in S^{n-1}$.

What’s worth mentioning here is $\overset{\psi}{+}(K_1, \ldots, K_m) \in \mathcal{F}_s^n$. The proof of this argument can be found in Lemma 3.7.

Equivalently, the Orlicz $L_\psi$-Blaschke addition $\overset{\psi}{+}(K_1, \ldots, K_m)$ can be defined implicitly by

$$\psi \left( \frac{f_p(K_1, u)}{f_p(\overset{\psi}{+}(K_1, \ldots, K_m), u)}, \ldots, \frac{f_p(K_m, u)}{f_p(\overset{\psi}{+}(K_1, \ldots, K_m), u)} \right) = 1, \quad (3.2)$$
if \( f_p(K_1, u) + \cdots + f_p(K_m, u) > 0 \), and by \( f_p(\hat{\psi}(K_1, \ldots, K_m), u) = 0 \), if \( f_p(K_1, u) = \cdots = f_p(K_m, u) = 0 \) for all \( u \in S^{n-1} \). An important special case is obtained when

\[
\psi(x_1, \ldots, x_m) = \sum_{j=1}^{m} \psi_j(x_j),
\]

for some fixed \( \psi_j \in \Phi \) such that \( \psi_1(1) = \cdots = \psi_m(1) = 1 \), and write \( \hat{\psi}(K_1, \ldots, K_m) = K_1^1 \psi \cdots \hat{\psi} K_m \). This means that \( K_1^1 \psi \cdots \hat{\psi} K_m \) is defined either by

\[
f_p(K_1^1 \psi \cdots \hat{\psi} K_m, u) = \inf \left\{ \lambda > 0 : \sum_{j=1}^{m} \psi_j \left( \frac{f_p(K_j, u)}{\lambda} \right) \leq 1 \right\},
\]

(3.3)

for all \( u \in S^{n-1} \), or by the corresponding special case of (3.2).

**Lemma 3.2** The Oorticz \( L_p^\omega \)-Blaschke addition \( \hat{\psi} : (F_\omega^n)^m \to F_\omega^n \) is monotonic and has the identity property.

**Proof** Suppose \( K_j \subset L_j \), \( j = 1, \ldots, m \), where \( K_j, L_j \in F_\omega^n \). If \( f_p(K_1, u) = \cdots = f_p(K_m, u) = 0 \), then \( f_p(\hat{\psi}(K_1, \ldots, K_m), u) = 0 \). \( \hat{\psi}(L_1, K_2, \ldots, L_m, u) \). If \( f_p(K_1, u) + \cdots + f_p(K_m, u) > 0 \), then \( f_p(L_1, u) + \cdots + f_p(K_m, u) > 0 \), by using (3.1), and in view of \( K_1 \subset L_1 \) and the fact that \( \psi \) is increasing in the first variable, we obtain

\[
\psi \left( \frac{f_p(L_1, u)}{f_p(\hat{\psi}(L_1, K_2, \ldots, K_m), u)}, \frac{f_p(K_2, u)}{f_p(\hat{\psi}(K_1, K_2, \ldots, K_m), u)}, \ldots, \frac{f_p(K_m, u)}{f_p(\hat{\psi}(K_1, K_2, \ldots, K_m), u)} \right)
\]

\[
= 1
\]

\[
= \psi \left( \frac{f_p(L_1, u)}{f_p(\hat{\psi}(L_1, K_2, \ldots, K_m), u)}, \frac{f_p(K_2, u)}{f_p(\hat{\psi}(K_1, K_2, \ldots, K_m), u)}, \ldots, \frac{f_p(K_m, u)}{f_p(\hat{\psi}(K_1, K_2, \ldots, K_m), u)} \right)
\]

which again implies that \( f_p(\hat{\psi}(K_1, K_2, \ldots, K_m), u) \leq f_p(\hat{\psi}(L_1, K_2, \ldots, L_m, u)) \).

By repeating this argument for each of the other \( m - 1 \) variables, we have \( f_p(\hat{\psi}(K_1, \ldots, K_m), u) \leq f_p(\hat{\psi}(L_1, \ldots, L_m, u)) \).

The identity property is obvious from (3.2). This completes the proof. \( \square \)

**Lemma 3.3** The Oorticz \( L_p^\omega \)-Blaschke addition \( \hat{\psi} : (F_\omega^n)^m \to F_\omega^n \) is \( \text{SL}(n) \)-covariant.

**Proof** From (2.1), (3.1) and let \( A \in \text{SL}(n) \), we obtain

\[
f_p(\hat{\psi}(AK_1, AK_2, \ldots, AK_m), u)
\]

\[
= \inf \left\{ \lambda > 0 : \psi \left( \frac{f_p(AK_1, u)}{\lambda}, \frac{f_p(AK_2, u)}{\lambda}, \ldots, \frac{f_p(AK_m, u)}{\lambda} \right) \leq 1 \right\}
\]

\[
= \inf \left\{ \lambda > 0 : \psi \left( \frac{f_p(K_1, A^u)}{\lambda}, \frac{f_p(K_2, A^u)}{\lambda}, \ldots, \frac{f_p(K_m, A^u)}{\lambda} \right) \leq 1 \right\}
\]

\[
= f_p(\hat{\psi}(K_1, \ldots, K_m), A^u)
\]

\[
= f_p(\hat{\psi}(K_1, \ldots, K_m), u).
\]
This shows Orlicz $L_\psi$-Blaschke addition $\hat{+}_\psi$ is SL($n$) covariant. This completes the proof.

**Lemma 3.4** If $K_1, \ldots, K_m \in \mathcal{F}_\psi^n$ and $\psi \in \Phi$, then
\[
\psi\left(\frac{f_p(K_1, u)}{t}\right) + \cdots + \psi\left(\frac{f_p(K_m, u)}{t}\right) = 1
\]
if and only if
\[
f_p(\hat{+}_\psi(K_1, \ldots, K_m), u) = t
\]

**Proof** This follows immediately from Definition 3.1.

**Lemma 3.5** If $K_1, \ldots, K_m \in \mathcal{F}_\psi^n$ and $\psi \in \Phi$, then
\[
\frac{r}{\psi^{-1}\left(\frac{1}{m}\right)} \leq f_p(\hat{+}_\psi(K_1, \ldots, K_m), u) \leq \frac{R}{\psi^{-1}\left(\frac{1}{m}\right)}.
\]
for all $u \in S^{n-1}$.

**Proof** Suppose $f_p(\hat{+}_\psi(K_1, \ldots, K_m), u) = t$. From Lemma 3.4 and noting that $\psi$ is increasing on $(0, \infty)$, we have
\[
1 = \psi\left(\frac{f_p(K_1, u)}{t}\right) + \cdots + \psi\left(\frac{f_p(K_m, u)}{t}\right) \\
\leq \psi\left(\frac{R_{K_1}}{t}\right) + \cdots + \psi\left(\frac{R_{K_m}}{t}\right) \leq m\psi\left(\frac{R}{t}\right).
\]
Noting that $\psi^{-1}$ is increasing on $(0, \infty)$, we obtain the upper bound for $f_p(\hat{+}_\psi(K_1, \ldots, K_m), u)$:
\[
t \leq \frac{R}{\psi^{-1}\left(\frac{1}{m}\right)}.
\]
On the other hand, from the Lemma 3.4, together with the convexity and the fact $\psi$ is increasing on $(0, \infty)$, we have
\[
1 = \psi\left(\frac{f_p(K_1, u)}{t}\right) + \cdots + \psi\left(\frac{f_p(K_m, u)}{t}\right) \\
\geq m\psi\left(\frac{f_p(K_1, u) + \cdots + f_p(K_m, u)}{mt}\right) \geq m\psi\left(\frac{r}{t}\right).
\]
Hence, we obtain the lower estimate:
\[
t \geq \frac{r}{\psi^{-1}\left(\frac{1}{m}\right)}.
\]
This completes the proof.

**Lemma 3.6** The Orlicz $L_\psi$-Blaschke addition $\hat{+}_\psi : (\mathcal{F}_\psi^n)^m \to \mathcal{F}_\psi^n$ is continuous.

**Proof** To see this, indeed, let $K_{ij} \in \mathcal{F}_\psi^n, i \in \mathbb{N} \cup \{0\}, j = 1, \ldots, m$, be such that $K_{ij} \to K_{0j}$ as $i \to \infty$. Let
\[
f_p(\hat{+}_\psi(K_{i1}, \ldots, K_{im}), u) = t_i.
\]
Then Lemma 3.5 shows
\[ \frac{r_{ij}}{\psi^{-1}\left(\frac{1}{m}\right)} \leq t_i \leq \frac{R_{ij}}{\psi^{-1}\left(\frac{1}{m}\right)}, \]
where \( r_{ij} = \min\{r_{K_{ij}}\} \) and \( R_{ij} = \max\{R_{K_{ij}}\} \). Since \( K_{ij} \to K_0 \), we have \( R_{K_{ij}} \to R_{K_0} \), and \( r_{K_{ij}} \to r_{K_0} > 0 \), and thus there exist \( a, b \) such that \( 0 < a \leq t_i \leq b < \infty \) for all \( i \). To show that the bounded sequence \( \{t_i\} \) converges to \( f_p(\varphi\psi(K_{01}, \ldots, K_{0m}), u) \), we show that every convergent subsequence of \( \{t_i\} \) converges to \( f_p(\varphi\psi(K_{01}, \ldots, K_{0m}), u) \).

Denote any subsequence of \( f \) we show that every convergent subsequence of \( f \) converges to \( f_p(\varphi\psi(K_{01}, \ldots, K_{0m}), u) \).

Let \( u_i \in S^{n-1} \), for any subsequence \( \{u_i\} \subset S^{n-1} \) such that \( u_i \to u_0 \) as \( i \to \infty \).

Let
\[ f_p(K_{\varphi\psi}L, u_i) = \lambda_i. \]

Then Lemma 3.5 shows
\[ \frac{r}{\psi^{-1}\left(\frac{1}{r}\right)} \leq \lambda_i \leq \frac{R}{\psi^{-1}\left(\frac{1}{r}\right)}, \]
where \( R = \max\{R_K, R_L\} \) and \( r = \min\{r_K, r_L\} \).

Since \( K, L \in F^n \), we have \( 0 < r_K \leq R_K < \infty \) and \( 0 < r_L \leq R_L < \infty \), and thus there exist \( a, b \) such that \( 0 < a \leq \lambda_i \leq b < \infty \) for all \( i \). To show that the bounded sequence \( \{\lambda_i\} \) converges to \( f_p(K_{\varphi\psi}L, u_0) \), we show that every convergent subsequence of \( \{\lambda_i\} \) converges to \( f_p(K_{\varphi\psi}L, u_0) \). Denote any subsequence of \( \{\lambda_i\} \) by \( \{\lambda_i\} \) as well, and suppose that for this subsequence, we have
\[ \lambda_i \to \lambda_0. \]

Obviously \( a \leq \lambda_0 \leq b \). From (3.4) and note that \( \psi_1, \psi_2 \) are continuous functions, so \( \psi_1^{-1} \) is continuous, we obtain
\[ \lambda_i \to \frac{f_p(K, u_0)}{\psi_1^{-1}\left(1 - \varepsilon\psi_2\left(\frac{f_p(L, u_0)}{\lambda_0}\right)\right)} \]
as \( i \to \infty \). Hence
\[ \psi_1\left(\frac{f_p(K, u_0)}{\lambda_0}\right) + \varepsilon\psi_2\left(\frac{f_p(L, u_0)}{\lambda_0}\right) = 1. \]
Therefore
\[ \lambda_0 = f_p(K + \psi \cdot L, u_0). \]

Namely
\[ f_p(K + \psi \cdot L, u_i) \to f_p(K + \psi \cdot L, u_0). \]
as \( i \to \infty. \)

This shows that \( K + \psi \cdot L \in \mathcal{F}_a^\cdot. \)

Next, we define the Orlicz \( L_\psi \)-Blaschke linear combination on the case \( m = 2. \)

**Definition 3.8** Orlicz \( L_\psi \)-Blaschke linear combination \( \psi(K, L, \alpha, \beta) \) for \( K, L \in \mathcal{F}_a^\cdot, \) and \( \alpha, \beta \geq 0 \) (not both zero), defined by
\[
\alpha \cdot \psi_1 \left( \frac{f_p(K, u)}{f_p(K + \psi(K, L, \alpha, \beta), u)} \right) + \beta \cdot \psi_2 \left( \frac{f_p(L, u)}{f_p(K + \psi(K, L, \alpha, \beta), u)} \right) = 1, \tag{3.4}
\]
if \( \alpha f_p(K, u) + \beta f_p(L, u) > 0 \), and by \( f_p(K + \psi(K, L, \alpha, \beta), u) = 0 \), if \( \alpha f_p(K, u) + \beta f_p(L, u) = 0 \), for all \( u \in S^{n-1}. \)

We shall write \( K + \psi \cdot L \) instead of \( K + \psi(K, L, 1, 1) \), and call the Orlicz \( L_\psi \)-Blaschke addition of \( K \) and \( L \).

### 4 Orlicz \( L_\psi \)-mixed affine surface areas

In order to define Orlicz \( L_\psi \)-mixed affine surface area, we need the following lemmas.

**Lemma 4.1** If \( K, L \in \mathcal{F}_a^\cdot, \varepsilon > 0 \) and \( \psi \in \Phi, \) then
\[ K + \psi \cdot L \to K \quad \text{as } \varepsilon \to 0^+. \]

**Proof** From (3.4) and noting that \( \psi_2, \psi_1^{-1} \) and \( f_p \) are continuous functions, we obtain
\[ f_p(K + \psi \cdot L, u) \to f_p(K, u) \]
as \( \varepsilon \to 0. \) Since \( \psi_1^{-1} \) is continuous, \( \psi_2 \) is bounded and in view of \( \psi_1^{-1}(1) = 1, \) we have
\[ \psi_1^{-1} \left( 1 - \varepsilon \psi_2 \left( \frac{f_p(L, u)}{f_p(K + \psi \cdot L, u)} \right) \right) \to 1 \]
as \( \varepsilon \to 0. \) This yields
\[ f_p(K + \psi \cdot L, u) \to f_p(K, u) \]
as \( \varepsilon \to 0^+. \) This completes the proof. \( \square \)

**Lemma 4.2** If \( K, L \in \mathcal{F}_a^\cdot, p \geq 1 \) and \( \psi, \psi_2 \in \Phi, \) then
\[
\frac{d}{d\varepsilon} \bigg|_{\varepsilon=0^+} f_p(K + \psi \cdot L, u \frac{\partial}{\partial \varepsilon}) \frac{n+p}{n} = \frac{n}{n+p} \cdot \frac{1}{(\psi_1^{-1}(1)) \cdot \psi_2 \left( \frac{f_p(L, u)}{f_p(K, u)} \right)} \cdot f_p(K, u) \frac{n+p}{n}. \tag{4.2}
\]
Proof From (3.4), we have
\[
\frac{df_p(K + \phi \varepsilon \cdot L, u)}{d\varepsilon} = \frac{f_p(K, u) \cdot \frac{d\psi_1(y)}{dy}}{\psi_1^{-1} \left( 1 - \varepsilon \psi_2 \left( \frac{f_p(L, u)}{f_p(K + \phi \varepsilon \cdot L, u)} \right) \right)} \times \\
\left[ \psi_2 \left( \frac{f_p(L, u)}{f_p(K + \phi \varepsilon \cdot L, u)} \right) - \varepsilon \cdot \frac{d\psi_2(z)}{dz} \cdot \frac{f_p(L, u)}{f_p(K + \phi \varepsilon \cdot L, u)^2} \cdot \frac{df_p(K + \phi \varepsilon \cdot L, u)}{d\varepsilon} \right],
\]
(4.3)
where
\[y = 1 - \varepsilon \psi_2 \left( \frac{f_p(L, u)}{f_p(K + \phi \varepsilon \cdot L, u)} \right),\]
and noting that \(y \to 1^-\) as \(\varepsilon \to 0^+\), and
\[z = \frac{f_p(L, u)}{f_p(K + \phi \varepsilon \cdot L, u)}.\]

From (4.1), (4.3) and notice that \(\psi_1^{-1}, \psi_2\) are continuous functions and \(\psi_1^{-1}(1) = 1\), we obtain for \(p \geq 1\)
\[
\lim_{\varepsilon \to 0^+} \frac{f_p(K + \phi \varepsilon \cdot L, u)}{\varepsilon} = \frac{n + p}{n + p} f_p(K, u) \cdot \frac{f_p(L, u)}{f_p(K, u)} \cdot \frac{f_p(K, u)}{\psi_2(1)} \cdot \frac{f_p(K, u) \frac{n}{n+1}}{f_p(K, u) \frac{n}{n+1} dS(u)}.
\]
This completes the proof. \(\square\)

Lemma 4.3 If \(K, L \in \mathcal{F}_p^\alpha\), \(p \geq 1\) and \(\psi \in \Phi_2\), then
\[
\frac{n + p}{n} \cdot (\psi_1')'(1) \cdot \frac{d}{d\varepsilon} \Omega_{\psi, \phi \varepsilon \cdot L} = \int_{S_{n-1}} \psi_2 \left( \frac{f_p(L, u)}{f_p(K, u)} \right) \cdot f_p(K, u) \frac{n}{n+1} dS(u).
\]
(4.4)

Proof This follows immediately from Lemma 4.2 and (1.3). \(\square\)

Denoting by \(\Omega_{\psi, \phi \varepsilon \cdot L}\), for any \(\psi \in \Phi\) and \(p \geq 1\), the integral on the right-hand side of (4.4) with \(\psi_2\) replaced by \(\psi\), we see that either side of the equation (4.4) is equal to \(\Omega_{\psi_2, \phi \varepsilon \cdot L}\) and hence this new Orlicz \(L_\psi\)-mixed affine surface area \(\Omega_{\psi, \phi \varepsilon \cdot L}\) has been born.

Definition 4.4 For \(K, L \in \mathcal{F}_p^\alpha\), \(\psi \in \Phi\) and \(p \geq 1\), Orlicz \(L_\psi\)-mixed affine surface area of \(K\) and \(L\), denoted by \(\Omega_{\psi, \phi \varepsilon \cdot L}\), defined by
\[
\Omega_{\psi, \phi \varepsilon \cdot L} := \int_{S_{n-1}} \psi \left( \frac{f_p(L, u)}{f_p(K, u)} \right) \cdot f_p(K, u) \frac{n}{n+1} dS(u).
\]
(4.5)
Obviously, when $K = L$ and $p \geq 1$, the Orlicz $L_q$-mixed affine surface area $\Omega_{\psi,q}(K,L)$ becomes the $L_p$-affine surface area $\Omega_{\psi}(K)$. When $\psi(t) = t^q$ and $q \geq 1$, the Orlicz $L_q$-mixed affine surface area $\Omega_{\psi,q}(K,L)$ becomes the $L_p$-mixed affine surface area $\Omega_{p,q}(K,L)$ stated in the Section 2. When $(t) = t^q$ and $q \geq 1$, the Orlicz $L_q$-mixed affine surface area $\Omega_{\psi,q}(K,L)$ becomes the well-known $i$-th $L_p$-mixed affine surface area $\Omega_{p,i}(K,L)$.

**Lemma 4.5** If $K,L \in F_n$, $p \geq 1$ and $\psi_1, \psi_2 \in \Phi$, then

\[
\Omega_{\psi_1,\psi_2}(K,L) = \frac{n + p}{n} \cdot (\psi_1)'(1) \cdot \frac{d}{d\varepsilon}_{\varepsilon=0^+} \Omega_p(K^\psi \cdot L).
\]  

**Proof** This follows immediately from Lemma 4.3 and (4.5). \hfill \Box

**Lemma 4.6** If $K,L \in F_n$, $\psi \in \Phi$ and any $A \in SL(n)$, then for $\varepsilon > 0$

\[
A(K^\psi \cdot L) = (AK)^\psi \cdot (AL).
\]  

**Proof** For any $A \in SL(n)$, from (2.1) and (3.4), we obtain

\[
f_p((AK^\psi \cdot AL), u) = \inf \left\{ \lambda > 0 : \psi \left( \frac{f_p(AK,u)}{\lambda} \right) + \varepsilon \psi \left( \frac{f_p(AL,u)}{\lambda} \right) \leq 1 \right\}
\]

\[
= \inf \left\{ \lambda > 0 : \psi \left( \frac{f_p(K,A^t u)}{\lambda} \right) + \varepsilon \psi \left( \frac{f_p(L, A^t u)}{\lambda} \right) \leq 1 \right\}
\]

\[
= f_p(K^{(\psi_{\varepsilon} \cdot L}, A^t u)
\]

This completes the proof. \hfill \Box

**Lemma 4.7** If $K,L \in F_n$, $p \geq 1$ and $\psi \in \Phi$, then for $A \in SL(n)$,

\[
\Omega_{\psi}(AK,AL) = \Omega_{\psi,p}(K,L).
\]  

**Proof** From Lemma 4.5 and Lemma 4.6, we have for $A \in SL(n)$,

\[
\Omega_{\psi_1,\psi_2}(AK,AL) = \frac{n + p}{n} \cdot (\psi_1)'(1) \cdot \frac{d}{d\varepsilon}_{\varepsilon=0^+} \Omega_p(K^\psi \cdot AL)
\]

\[
= \frac{n + p}{n} \cdot (\psi_1)'(1) \cdot \frac{d}{d\varepsilon}_{\varepsilon=0^+} \Omega_p(A(K^\psi \cdot L))
\]

\[
= \frac{n + p}{n} \cdot (\psi_1)'(1) \cdot \frac{d}{d\varepsilon}_{\varepsilon=0^+} \Omega_p(K^\psi \cdot L)
\]

\[
= \Omega_{\psi_2,p}(K,L).
\]

This completes the proof. \hfill \Box

5 Orlicz $L_q$-Minkowski inequality

In this section, we need define a Borel measure in $S^{n-1}$, denoted by $\Omega_{n,p}(K,\nu)$, call it $L_p$-curvature measure of convex body $K$. 
**Definition 5.1** Let $K \in \mathcal{F}^n_a$ and $p \geq 1$, the $L_p$-curvature measure, denoted by $\Omega_{n,p}(K,v)$, defined by

$$d\Omega_{n,p}(K,v) = \frac{f_p(K,v)}{\Omega_p(K)} \, dS(v). \quad (5.1)$$

**Lemma 5.2** (Jensen’s inequality) Let $\mu$ be a probability measure on a space $X$ and $g : X \to I \subseteq \mathbb{R}$ is a $\mu$-integrable function, where $I$ is a possibly infinite interval. Suppose the equality holds in (5.3).

If $\phi : I \to \mathbb{R}$ is a convex function, then

$$\int_X \phi(g(x)) \, d\mu(x) \geq \phi \left( \int_X g(x) \, d\mu(x) \right). \quad (5.2)$$

If $\phi$ is strictly convex, equality holds if and only if $g(x)$ is constant for $\mu$-almost all $x \in X$ (see [12, p.165]).

**Lemma 5.3** Let $0 < a \leq \infty$ be an extended real number, and let $I = [0,a]$ be a possibly infinite interval. Suppose that $\psi : I = [0,a] \to [0,\infty)$ is convex and increasing with $\psi(0) = 0$. If $K,L \in \mathcal{F}^n_a$ are such that $L \subseteq \text{int}(aK)$, then for $p \geq 1$,

$$\frac{1}{\Omega_p(K)} \int_{S^{n-1}} \psi \left( \frac{f_p(L,u)}{f_p(K,u)} \right) \frac{\overline{f_p}(K,u)^{\frac{1}{p}}}{\Omega_p(K)} \, dS(u) \geq \psi \left( \left( \frac{\Omega_p(L)}{\Omega_p(K)} \right)^{\frac{a+p}{pa}} \right). \quad (5.3)$$

If $\psi$ is strictly convex, equality holds if and only if $K$ and $L$ are homothetic.

**Proof** Since $L \subseteq \text{int}(aK)$, so we have $f_p(L,u)/f_p(K,u) \in I$ for all $u \in S^{n-1}$. For $K \in \mathcal{F}^n_a$, $p \geq 1$ and any $u \in S^{n-1}$, noting that

$$\int_{S^{n-1}} f_p(K,u) \frac{\overline{f_p}(K,u)^{\frac{1}{p}}}{\Omega_p(K)} \, dS(u) = 1,$$

hence the $L_p$-curvature measure $\Omega_{n,p}(K,u)$ is a probability measure on $S^{n-1}$. Hence, from (5.1) and by using Jensen’s inequality and Hölder’s inequality, and in view of $\psi$ is increasing, we obtain

$$\frac{1}{\Omega_p(K)} \int_{S^{n-1}} \psi \left( \frac{f_p(L,u)}{f_p(K,u)} \right) \frac{\overline{f_p}(K,u)^{\frac{1}{p}}}{\Omega_p(K)} \, dS(u)$$

$$= \int_{S^{n-1}} \psi \left( \frac{f_p(L,u)}{f_p(K,u)} \right) d\Omega_{n,p}(K,u)$$

$$\geq \psi \left( \left( \frac{f_p(L,u)}{f_p(K,u)} \right)^{\frac{1}{pa}} \frac{\overline{f_p}(L,u) \, dS(u)}{\Omega_p(K)} \right)$$

$$\geq \psi \left( \left( \frac{\Omega_p(L)}{\Omega_p(K)} \right)^{\frac{a+p}{pa}} \right) = \psi \left( \left( \frac{\Omega_p(L)}{\Omega_p(K)} \right)^{\frac{a+p}{pa}} \right).$$

Next, we discuss the equal condition of (5.3). Suppose the equality holds in (5.3). When $\psi$ is strictly convex, form the equality condition of Jensen’s inequality, then $f_p(L,u)/f_p(K,u)$ must be a constant, namely: $f_p(L,u)$ and $f_p(K,u)$ are proportional, this yields that $K$ and $L$ must be homothetic. On the other hand, form the equality condition of Hölder’s inequality, it follows that $K$ and $L$ must be homothetic. Combine these, this yields that the equality holds in (5.3) must $K$ and $L$ be homothetic.
Conversely, suppose that $K$ and $L$ are homothetic, i.e. there exist $\lambda > 0$ such that $f_p(L, u) = \lambda f_p(K, u)$ for all $u \in S^{n-1}$. Hence

$$\frac{1}{\Omega_p(K)} \int_{S^{n-1}} \psi \left( f_p(L, u) \right) f_p(K, u)^{n/p - 1} dS(u) = \frac{1}{\Omega_p(K)} \int_{S^{n-1}} \psi \left( \left( \frac{\Omega_p(L)}{\Omega_p(K)} \right)^{\frac{n+p}{n}} f_p(K, u)^{n/p - 1} dS(u) \right. \left. \right) = \psi \left( \left( \frac{\Omega_p(L)}{\Omega_p(K)} \right)^{\frac{n+p}{n}} \right).$$

This implies the equality in (5.3) holds.

**Theorem 5.4** (Orlicz $L_p$-Minkowski inequality) If $K, L \in F^n_s$, $p \geq 1$ and $\psi \in \Phi$, then

$$\Omega_{\psi,p}(K, L) \geq \Omega_p(K) \cdot \psi \left( \left( \frac{\Omega_p(L)}{\Omega_p(K)} \right)^{\frac{n+p}{n}} \right).$$

(5.4)

If $\psi$ is strictly convex, equality holds if and only if $K$ and $L$ are homothetic.

**Proof** This follows immediately from (4.5) and Lemma 5.3 with $a = \infty$. When $\psi(t) = t^q$ and $q \geq 1$, we have the following inequality.

**Corollary 5.5** If $K, L \in F^n_s$ and $p, q \geq 1$, then

$$\Omega_{p,q}(K, L)^{\frac{n}{n+p}} \geq \Omega_p(K)^{\frac{n}{n+p}} q \Omega_p(L)^q;$$

(5.5)

with equality if and only if $K$ and $L$ are homothetic.

This is just $L_{pq}$-Minkowski inequality proved in the Section 2.

**Theorem 5.6** Let $K, L \in M \subset F^n_s$, $p \geq 1$ and $\psi \in \Phi$, and if either

$$\Omega_{\psi,p}(Q, K) = \Omega_{\psi,p}(Q, L), \text{ for all } Q \in M$$

(5.6)

or

$$\frac{\Omega_{\psi,p}(K, Q)}{\Omega_p(K)} = \frac{\Omega_{\psi,p}(L, Q)}{\Omega_p(L)}, \text{ for all } Q \in M,$$

(5.7)

then $K = L$.

**Proof** Suppose (5.6) hold. Taking $K$ for $Q$, then from (4.5) and (5.4), we obtain

$$\Omega_{p}(K) = \Omega_{\psi,p}(K, L) \geq \Omega_p(K) \psi \left( \left( \frac{\Omega_p(L)}{\Omega_p(K)} \right)^{\frac{n+p}{n}} \right)$$

with equality if and only if $K$ and $L$ are homothetic. Hence

$$1 \geq \psi \left( \left( \frac{\Omega_p(L)}{\Omega_p(K)} \right)^{\frac{n+p}{n}} \right),$$

with equality if and only if $K$ and $L$ are homothetic. Since $\psi$ is increasing function on $(0, \infty)$, this follows that

$$\Omega_{p}(K) \geq \Omega_p(L),$$
with equality if and only if $K$ and $L$ are homothetic. On the other hand, if taking $L$ for $Q$, we similar get $\Omega_p(K) \leq \Omega_p(L)$, with equality if and only if $K$ and $L$ are homothetic. Hence $\Omega_p(K) = \Omega_p(L)$, and $K$ and $L$ are homothetic, it follows that $K$ and $L$ must be equal.

Suppose (5.7) hold. Taking $L$ for $Q$, then from (4.5) and (5.4), we obtain

$$
\Omega_p(K, L) \geq \psi\left(\left(\frac{\Omega_p(L)}{\Omega_p(K)}\right)^{\frac{1}{n+p}}\right),
$$

with equality if and only if $K$ and $L$ are homothetic. Hence

$$
1 = \psi\left(\left(\frac{\Omega_p(L)}{\Omega_p(K)}\right)^{\frac{1}{n+p}}\right),
$$

with equality if and only if $K$ and $L$ are homothetic. Since $\psi(t) = t^q$ and $q \geq 1$, Corollary 5.6 becomes the following result.

**Corollary 5.7** Let $K, L \in \mathcal{M} \subset F_n$, and $p, q \geq 1$, and if either

$$
\Omega_p(K, Q) = \Omega_p(L, Q), \text{ for all } Q \in \mathcal{M}
$$

or

$$
\frac{\Omega_p(K, Q)}{\Omega_p(K)} = \frac{\Omega_p(L, Q)}{\Omega_p(L)}, \text{ for all } Q \in \mathcal{M},
$$

then $K = L$.

### 6 Orlicz $L_\psi$-Brunn-Minkowski inequality

**Lemma 6.1** If $K, L \in F_n$, $p \geq 1$, and $\psi_1, \psi_2 \in \Phi$, then for $\varepsilon > 0$

$$
\Omega_p(K + \varepsilon \cdot \psi \cdot L) = \Omega_{\psi_1, p}(K + \varepsilon \cdot \psi \cdot L, K) + \varepsilon \Omega_{\psi_2, p}(K + \varepsilon \cdot \psi \cdot L, L).
$$

**Proof** From (1.3), (3.4) and (4.5), we have for any $Q \in F_n$

$$
\begin{align*}
\Omega_{\psi_1, p}(Q, K) + \varepsilon \Omega_{\psi_2, p}(Q, L) &= \int_{S^{n-1}} \left(\psi_1 \left(\frac{f_p(K, u)}{f_p(Q, u)}\right) + \varepsilon \psi_2 \left(\frac{f_p(L, u)}{f_p(Q, u)}\right)\right) f_p(Q, u)^{\frac{n}{n+p}} dS(u) \\
&= \int_{S^{n-1}} \psi_1 \left(\frac{f_p(K, u)}{f_p(Q, u)}\right) f_p(Q, u)^{\frac{n}{n+p}} dS(u) + \varepsilon \int_{S^{n-1}} \psi_2 \left(\frac{f_p(L, u)}{f_p(Q, u)}\right) f_p(Q, u)^{\frac{n}{n+p}} dS(u) = \Omega_p(Q).
\end{align*}
$$

Putting $Q = K + \varepsilon \cdot \psi \cdot L$ in (6.2), (6.2) changes (6.1).
Theorem 6.2 (Orlicz $L_\psi$-Brunn-Minkowski inequality) If $K, L \in \mathcal{F}_n$, $p \geq 1$ and $\psi \in \Phi_2$, then for $\varepsilon > 0$

$$1 \geq \psi \left( \left( \frac{\Omega_p(K)}{\Omega_p(K + \varepsilon L)} \right)^{\frac{n+p}{n}}, \left( \frac{\Omega_p(L)}{\Omega_p(K + \varepsilon L)} \right)^{\frac{n+p}{n}} \right).$$

(6.3)

If $\psi$ is strictly convex, equality holds if and only if $K$ and $L$ are homothetic.

Proof From (5.4) and Lemma 6.1, we have

$$\Omega_p(K + \varepsilon L) = \Omega_{p,1}(K + \varepsilon L, K) + \varepsilon \Omega_{p,2}(K + \varepsilon L, L)$$

$$\geq \Omega_p(K + \varepsilon L) \left( \psi_1 \left( \left( \frac{\Omega_p(K)}{\Omega_p(K + \varepsilon L)} \right)^{\frac{n+p}{n}} \right) + \varepsilon \psi_2 \left( \left( \frac{\Omega_p(L)}{\Omega_p(K + \varepsilon L)} \right)^{\frac{n+p}{n}} \right) \right)$$

$$= \Omega_p(K + \varepsilon L) \psi \left( \left( \frac{\Omega_p(K)}{\Omega_p(K + \varepsilon L)} \right)^{\frac{n+p}{n}}, \left( \frac{\Omega_p(L)}{\Omega_p(K + \varepsilon L)} \right)^{\frac{n+p}{n}} \right).$$

This is just the inequality (6.3). From the equality condition of (5.4), if follows that if $\psi$ is strictly convex, equality in (6.3) holds if and only if $K$ and $L$ are homothetic.

□

When $\psi(x_1, x_2) = x_1^p + x_2^q$ and $q \geq 1$, we have following result.

Corollary 6.3 If $K, L \in \mathcal{F}_n$ and $p, q \geq 1$, then

$$\Omega_p(K + p q L) \leq \Omega_p(K)^{\frac{q(n+p)}{n}} + \Omega_p(L)^{\frac{q(n+p)}{n}},$$

(6.4)

with equality if and only if $K$ and $L$ are homothetic.

This is just $L_{pq}$-Brunn-Minkowski inequality proved in the Section 2. When $q = 1$, (6.4) becomes the $L_p$-Brunn-Minkowski inequality for $p$-mixed affine surface area stated in the Introduction.

Corollary 6.4 If $K, L \in \mathcal{F}_n$, $p \geq 1$ and $\psi \in \Phi$, then

$$\Omega_{\psi,p}(K, L) \geq \Omega_p(K) \cdot \psi \left( \left( \frac{\Omega_p(L)}{\Omega_p(K)} \right)^{\frac{n+p}{n}} \right).$$

(6.5)

If $\psi$ is strictly convex, equality holds if and only if $K$ and $L$ are homothetic.

Proof Let

$$K_\varepsilon = K + \varepsilon L.$$
From (4.6) and in view of the Orlicz-Brunn-Minkowski inequality (6.3), we obtain

\[
\frac{n}{n+p} \cdot \frac{1}{\psi_1(1)} \cdot \Omega_{\psi_2,p}(K,L) = \frac{d}{de} \bigg|_{e=0^+} \Omega_p(K_e)
\]

\[
= \lim_{\varepsilon \to 0^+} \frac{1 - \frac{\Omega_p(K)}{\Omega_p(K_{\varepsilon})}}{\psi_1(1) - \psi_1\left(\left(\frac{\Omega_p(K)}{\Omega_p(K_{\varepsilon})}\right)^{\frac{n+p}{n}}\right)} \cdot 1 - \psi_1\left(\left(\frac{\Omega_p(K)}{\Omega_p(K_{\varepsilon})}\right)^{\frac{n+p}{n}}\right) \times \Omega_p(K_{\varepsilon})
\]

\[
= \lim_{t \to 1^-} \frac{1-t}{\psi_1(1)} \cdot \lim_{\varepsilon \to 0^+} \frac{1 - \psi_1\left(\left(\frac{\Omega_p(K)}{\Omega_p(K_{\varepsilon})}\right)^{\frac{n+p}{n}}\right)}{\varepsilon} \times \lim_{\varepsilon \to 0^+} \Omega_p(K_{\varepsilon}) \geq \frac{n}{n+p} \cdot \frac{1}{\psi_1(1)} \cdot \lim_{\varepsilon \to 0^+} \psi_2\left(\left(\frac{\Omega_p(L)}{\Omega_p(K_{\varepsilon})}\right)^{\frac{n+p}{n}}\right) \cdot \lim_{\varepsilon \to 0^+} \Omega_p(K_{\varepsilon})
\]

\[
= \frac{n}{n+p} \cdot \frac{1}{\psi_1(1)} \cdot \psi_2\left(\left(\frac{\Omega_p(L)}{\Omega_p(K)}\right)^{\frac{n+p}{n}}\right) \cdot \Omega_p(K).
\]

(6.6)

Obviously, from (6.6), (6.5) yields. If \( \psi \) is strictly convex, from the equality condition of (6.3), it follows that the equality holds in (6.5) if and only if \( K \) and \( L \) are homothetic. This proof is complete. \( \square \)

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