On kernels of second-order elliptic operators defined by Stein-Weiss operators acting on covariant tensors

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Abstract. The article is devoted to the study of the global geometry of symmetric and skew-symmetric higher order tensors on complete Riemannian manifolds using second-order elliptic operators, which are constructed on the basis of Stein-Weiss operators.

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1 Introduction

We consider a real vector bundle $E \to M$ on a differentiable $C^\infty$-manifold $M$ of dimension $n \geq 2$ with a linear connection $\nabla : C^\infty(E) \to C^\infty(T^*M \otimes E)$ and a Lie group $G$, acting in the fibers of the bundles $T^*M \otimes E$ and $E$. Let $\text{Diff}(E, T^*M \otimes E)$ denote a $C^\infty$-module of first order linear differential operators $D : C^\infty E \to C^\infty(T^*M \otimes E)$ on the space $C^\infty(E)$ of smooth sections of $E$.

E. Stein and G. Weiss introduced in [20] the generalized gradient (in short, G-gradient), as the differential operator $D \in \text{Diff}(E, T^*M \otimes E)$, which is the projection of the covariant derivative $\nabla s$ on the pointwise $G$-irreducible subbundle of the bundle $T^*M \otimes E$ for any section $s \in C^\infty(E)$. For example, Maxwell and Dirac equations, are based on these Stein-Weiss gradients (e.g., [20]). Later on, G-gradients were called Stein-Weiss operators (see [6]). We will also use this terminology.

Let $g$ be a Riemannian metric on $M$, then on any real vector bundle $E \to M$ there exists a Riemannian metric, which we also denote by $g$. In this case, any Stein-Weiss differential operator $D$ admits a formal adjoint operator $D^*$ defined using $g$ (see [3, p. 34]). Based on this fact, we are interested in a special class of second order differential operators $D^*D$, from which many geometric statements can be derived. In [6, 17], they studied ellipticity of second order differential operators $D^*D$.

Our starting point is the following statement: If $D$ is a differential operator of order $k$ with injective symbol, then $D^*D$ is elliptic. We also consider an elliptic differential operator $\Delta_E = \bar{\Delta} + t \Re$ (of the Weitzenböck decomposition form) for a suitable constant $t$, see [9], acting on $C^\infty(E)$, where $\bar{\Delta} = \nabla^* \nabla$ is the rough or Bochner Laplacian.
denotes the formal adjoint of \( \nabla \) with respect to \( g \) (e.g., [3, p. 53] and [16, p. vii]), and \( \mathcal{R} \) is a smooth symmetric endomorphism of \( E \) depending linearly in a known way on the curvature \( R^\nabla \) of the connection \( \nabla \) on \( E \). An example of a bundle to which the above reasoning applies is the space of differential \( p \)-forms, where the role of \( \Delta_E \) is played by the Hodge-De Rham Laplacian \( \Delta_H \). A smooth section \( s \in C^\infty (E) \) is called \( \Delta_E \)-harmonic if \( \Delta_E s = 0 \) (see [16, p. 104]). Below, we consider the relationship between the operators \( \Delta_E \) and \( D^* D \) and give examples of such harmonic sections.

The article has the following structure. In Section 2, we review the properties of Stein-Weiss operators \( D \) defined on differential \( p \)-forms \((1 \leq p \leq n-1)\) and corresponding second order elliptical operators \( D^* D \), and also the geometry of tensors lying in kernels of such operators. In Sections 3 and 4, we extend the results of [21, 22, 25] for symmetric \( p \)-tensors \((p \geq 2)\). In Sections 5 and 6, we study the global geometry of traceless symmetric conformal Killing tensors and Codazzi tensors using second-order elliptic operators based on Stein-Weiss operators and the approach of a short article [24], where the question was investigated for tensors of order \( p = 2 \).

2 Stein-Weiss operators on differential forms

Let a linear group \( \text{GL}(n, \mathbb{R}) \) act in the fibers of tensor bundles over \( M \). Let \( C^\infty \Lambda^p M \) denote the space of \( C^\infty \)-sections of the bundle of \( p \)-forms on \( M \) for \( 1 \leq p \leq n-1 \), and \( d : C^\infty \Lambda^p M \to C^\infty \Lambda^{p+1} M \) the exterior derivative operator (see [3, p. 21]). There is a pointwise \( \text{GL}(n, \mathbb{R}) \)-irreducible decomposition \( T^* M \otimes \Lambda^p M = \Lambda^{p+1} M \oplus \ker \Lambda^{p+1} \) for the pointwise algebraic alternation operator \( \Lambda^p : T^* M \otimes \Lambda^p M \to \Lambda^{p+1} M \). As a consequence, we have the following pointwise \( \text{GL}(n, \mathbb{R}) \)-irreducible decomposition:

\[
\nabla \omega = L_1 \omega + L_2 \omega
\]

for any \( \omega \in C^\infty \Lambda^p M \), where \( L_1 = (p+1)^{-1} d \) and \( L_2 = \nabla - (p+1)^{-1} d \) (see [21]). Due to [20, 11], these \( L_1 \) and \( L_2 \) are \( \text{GL}(n, \mathbb{R}) \)-gradients, or, Stein-Weiss operators, defined on \( C^\infty \Lambda^p M \). The kernels \( L_1 \) and \( L_2 \) consist of closed \( p \)-forms and Killing \( p \)-forms, respectively, and the last ones, for (pseudo-)Riemannian manifolds, are called Killing-Yano tensors (see [26, p. 559]). For a Riemannian manifold \((M, g)\), the decomposition (2.1) is pointwise orthogonal, i.e., \( g(L_1 \omega, L_2 \omega) = 0 \) for any \( \omega \in C^\infty \Lambda^p M \).

Note that \( d : C^\infty \Lambda^p M \to C^\infty \Lambda^{p+1} M \) has a formally adjoint operator \( d^* : C^\infty \Lambda^{p+1} M \to C^\infty \Lambda^p M \) with respect to Riemannian metric on \( M \), called codifferential (see [3, c. 54]). Thus, for \( L_2 \) there exists a formally adjoint operator \( L_2^* = p(p+1)^{-1} d^* \).

Using these operators, we build the second order differential operator

\[
L_2^* L_2 = p(p+1)^{-1} (\bar{\Delta} - (p+1)^{-1} d^* d).
\]

The main symbol \( \sigma(L_2^* L_2)(\xi, \omega_x) \) of the operator (2.2) has the form

\[
\sigma(L_2^* L_2)(\xi, \omega_x) = -\frac{p}{p+1} \left( \frac{1}{p+1} \|\xi\|^2 \omega_x + \frac{1}{p+1} \xi \wedge (i \xi \omega_x) \right)
\]

according to the following formulas (see [3, p. 461]):

\[
\sigma(\nabla)(\xi, \omega_x) = \xi \otimes \omega_x, \quad \sigma(\nabla^*)(\xi, \omega_x) = -i \xi \theta_x, \\
\sigma(d)(\xi, \omega_x) = \xi \wedge \omega_x, \quad \sigma(d^*)(\xi, \omega_x) = -i \xi \omega_x
\]
for all \( \xi \in R^x Mu \setminus \{0\} \), \( \omega_\xi \in \Lambda^r(T^*_x M) \) and \( \theta_x \in T^*_x M \otimes \Lambda^r(T^*_x M) \) at each point \( x \in M \). From (2.3) we obtain the following inequality:

\[
-g(\sigma(L^*_2 L_2)(\xi, \omega_x), \omega_x) = \frac{p}{(p+1)^2} \left( \rho g(\xi, \xi) \omega_x + g(\iota_\xi \omega_x, \iota_\xi \omega_x) \right) > 0
\]

for any nonzero \( \xi \) and \( \omega_x \). Thus, (2.2) is an elliptic operator (see [3, p. 462]). On a compact manifold \( M \), the kernel of \( L^*_2 L_2 \) consists of Killing-Yano \( p \)-tensors (see [23]), because of the inequality \( \int_M g(L^*_2 L_2 \omega, \omega) dV_g = \int_M g(L_2 \omega, L_2 \omega) dV_g \geq 0 \), where \( dV_g \) is the volume form of \( g \); moreover, according to [3, p. 464], as a consequence of ellipticity of \( L^*_2 L_2 : C^\infty \Lambda^p M \to C^\infty \Lambda^{p+1} M \) we get the decomposition \( C^\infty \Lambda^p M = \ker L^*_2 \oplus \text{Im} \ L_2 \) with respect to the \( L^2 \)-global scalar product on \( (M, g) \), defined by \( \langle \omega, \omega' \rangle = \frac{1}{p} \int_M g(\omega, \omega') dV_g \), where \( \omega, \omega' \in C^\infty \Lambda^p M \). As the result, we get

**Proposition 2.1.** For any \( \omega \in C^\infty \Lambda^p M \) and its \( SL(n, \mathbb{R}) \)-gradients \( L_1 \omega = (p+1)^{-1} d \omega \) and \( L_2 \omega = \nabla \omega - (p+1)^{-1} d \omega \) on \( \Lambda^p M \) the orthogonal decomposition (2.1) holds. If \( (M, g) \) is compact, then the orthogonal decomposition \( C^\infty \Lambda^{p+1} M = \ker L^*_2 \oplus \text{Im} \ L_2 \) holds. Moreover, \( L^*_2 L_2 \) in (2.2) is a nonnegative definite elliptic operator, whose kernel is a finite-dimensional vector space over \( \mathbb{R} \) consisting of Killing-Yano \( p \)-tensors.

Bourguignon [5] studied first order natural differential operators on the spaces of \( C^\infty \)-sections of bundle of \( \Lambda^p M \) on \( (M, g) \) with the structural group \( O(n, \mathbb{R}) \) and the Levi-Civita connection \( \nabla \) (see the theory in [13]). By definition, if the symbols of these operators are projectors on pointwise \( O(n, \mathbb{R}) \)-irreducible subbundles of \( T^* M \otimes \Lambda^p M \), they are called fundamental. Fundamental differential operators of Bourguignon are Stein-Weiss operators. Bourguignon proved that \( T^* M \otimes \Lambda^p M \) is decomposed into three pointwise \( O(n, \mathbb{R}) \)-irreducible subbundles. Based on this fact, Bourguignon defined fundamental operators \( d \) and \( d^* \) and indicated the existence of a third fundamental operator. He also noted that apart from the case \( p = 1 \), the third fundamental operator does not have a simple geometric interpretation. As a consequence, this allows for each \( \omega \in C^\infty \Lambda^p M \) to obtain an expansion of \( \nabla \omega \in C^\infty (T^* M \otimes \Lambda^p M) \) in the sum of three pointwise \( O(n, \mathbb{R}) \)-irreducible components

\[
\nabla \omega = G_1 \omega + G_2 \omega + G_3 \omega.
\]

Then, all three Stein-Weiss operators were found explicitly in [22]:

\[
(2.5) \quad G_1 = (p + 1)^{-1} d, \quad G_2 = (n - p + 1)^{-1} g \wedge d^*, \quad G_3 = \nabla - G_1 - G_2,
\]

and it was proved in [27] that the kernel of \( G_3 \) consists of conformal Killing \( p \)-forms.

Further, in [23], the operator \( G^*_3 \) formally conjugated to \( G_3 \) on \( (M, g) \) was found, the following second order differential operator was constructed and studied:

\[
G^*_3 G_3 = \frac{p}{p + 1} \left( \Delta - \frac{1}{p + 1} d^* d - \frac{1}{n - p + 1} d d^* \right).
\]

For \( n = 2p \) we get \( G^*_3 G_3 = \frac{p}{p + 1} (\Delta - \frac{1}{p + 1} \Delta H) \) for the Hodge-de Rham Laplacian \( \Delta_H = d^* d + d d^* \) (see [16, p. 260]). The Hodge-de Rham Laplacian \( \Delta_H \) admits the Wittenbäck decomposition (e.g., [3, p. 57]) \( \Delta_H = \Delta + \mathfrak{R} \), where \( \mathfrak{R} \) depends linearly in a known way on the curvature tensor and the Ricci tensor \( \text{Ric} \) of \( \nabla \). Moreover, for \( n = 2p \) we get the equality \( G^*_3 G_3 = (\frac{p}{p + 1})^2 (\Delta - \frac{1}{p} \mathfrak{R}) \), where \( \Delta_L = \Delta - p^{-1} \mathfrak{R} \) is the Lichnerovich Laplacian (see [9]). Thus, the following is valid.
Proposition 2.2. Let for each differential $p$-form $\omega \in C^\infty \Lambda^p \mathcal{M}$ the expansion of its covariant derivative $\nabla \omega \in C^\infty (T^* \mathcal{M} \otimes \Lambda^p \mathcal{M})$ in the sum (2.4) of pointwise $O(n, \mathbb{R})$-irreducible components with Stein-Weiss operators (2.5) hold. Then for $n = 2p$ the operator $p^{-2}(p + 1)G_3^*G_3$ is the Lichnerovich Laplacian.

The Bochner-Weitzenböck formula (e.g., [16, p. 106]), can be rewritten as

$$\frac{1}{2} \Delta \| \omega \|^2 = -g(\Delta_H \omega, \omega) - g(\nabla(\omega), \omega) + \| G_1 \omega \|^2 + \| G_2 \omega \|^2 + \| G_3 \omega \|^2.$$ 

The operator $G_3^*G_3$ is elliptic for $2 \leq p \leq n - 1$ (see [18, 10], where it lacks the normalizing factor $p(p + 1)^{-1}$ calculated in [23]): on a compact $(\mathcal{M}, g)$ the kernel of $G_3^*G_3$ is formed by conformal Killing $p$-forms.

3 The Stein-Weiss operator on symmetric tensors

Let $C^\infty \mathcal{S}^p \mathcal{M}$ be the space of $C^\infty$-sections of the bundle $\mathcal{S}^p \mathcal{M}$ of symmetric $p$-tensors on $\mathcal{M}$. Consider $T_x \mathcal{M}$ at any point $x \in \mathcal{M}$ as an $n$-dimensional vector space $V$ with the structure group $GL(n, \mathbb{R})$. Let $\mathcal{S}^p V$ denote the $p$-th symmetric power of the space $V^*$ dual to $V$. The fiber of $T^* \mathcal{M} \otimes \mathcal{S}^p \mathcal{M}$ is the tensor space $V^* \otimes \mathcal{S}^p V$, which will be regarded as the representation space of $GL(n, \mathbb{R})$. Define an endomorphism $\mathcal{S}^{p+1} : V^* \otimes \mathcal{S}^p V \rightarrow \mathcal{S}^{p+1} V \subset V^* \otimes \mathcal{S}^p V$, called the Young symmetrizer, see [1], by

$$(\mathcal{S}^{p+1}(\phi))_{i_0, i_1, \ldots, i_p} := \phi_{(i_0 i_1 \ldots i_{p-1} i_p)} = \frac{1}{p + 1}(\phi_{i_0 i_1 \ldots i_{p-1} i_p} + \phi_{i_1 i_2 \ldots i_{p-1} i_0} + \ldots + \phi_{i_p i_0 i_1 \ldots i_{p-1}})$$

for components $\phi_{i_0 i_1 \ldots i_{p-1} i_p} = \phi(e_{i_0}, e_{i_1}, \ldots, e_{i_p})$ of any $\phi \in V^* \otimes \mathcal{S}^p V$ in any basis $e_1, \ldots, e_n$ of $V$. The endomorphism $\mathcal{S}^{p+1}$ is $GL(n, \mathbb{R})$-invariant and $\mathcal{S}^{p+1}(\mathcal{S}^{p+1}(\phi)) = \mathcal{S}^{p+1}(\phi)$, i.e., $\mathcal{S}^{p+1}$ is an idempotent in $V^* \otimes \mathcal{S}^p V$. Thus, the $GL(n, \mathbb{R})$-invariant decomposition of $V^* \otimes \mathcal{S}^p V$ into a direct sum $V^* \otimes \mathcal{S}^p V = \text{Im} \mathcal{S}^{p+1} \otimes \ker \mathcal{S}^{p+1}$ of two subspaces $V^* \otimes \mathcal{S}^p V$ holds, where $\text{Im} \mathcal{S}^{p+1} = \mathcal{S}^{p+1} V$, and $\ker \mathcal{S}^{p+1} := \text{Im}(\text{id} - \mathcal{S}^{p+1})$ consists of tensors of the form $\phi - \mathcal{S}^{p+1}(\phi)$.

Lemma 3.1. Let $GL(n, \mathbb{R})$ act on fibers of tensor bundles on $\mathcal{M}$. Then the following pointwise $GL(n, \mathbb{R})$-irreducible decomposition holds:

$$(3.1) \quad T^* \mathcal{M} \otimes \mathcal{S}^p \mathcal{M} = \mathcal{S}^{p+1} \mathcal{M} \oplus \ker \mathcal{S}^{p+1}.$$ 

Proof. The first component of the expansion $\mathcal{S}^{p+1} V$ for $V = T_x \mathcal{M}$ and any point $x \in \mathcal{M}$ is irreducible $GL(n, \mathbb{R})$ - a module. To find $GL(n, \mathbb{R})$-irreducible subspaces in $\mathcal{S}^{p+1} V$, we need a list of all correctly filled $(n, p + 1)$-Young schemes, which in this case contains only one simple scheme $\begin{bmatrix} 1 & 2 & \ldots & p & p + 1 \end{bmatrix}$.

Thus, there are no other $GL(n, \mathbb{R})$-irreducible subspaces in $\mathcal{S}^{p+1} V$ other than $\mathcal{S}^{p+1} V$. To determine what weights with respect to the maximal tori (diagonal matrices) have elements of $\ker \mathcal{S}^{p+1}$, we decompose $V^* \otimes \mathcal{S}^p V$ into weighted spaces, where the weight vectors are tensors of the form

$$\bar{\varphi}_{(i_1, \ldots, i_l)}(k, i_1, \ldots, i_l) = \begin{cases} 1, & \text{if } l = k \text{ and } i_1, \ldots, i_l \text{ is a permutation of } j_1, \ldots, j_l, \\ 0, & \text{otherwise.} \end{cases}$$
The above tensor has weight \( \text{diag}(t_1, \ldots, t_n) \mapsto t_k t_{i_1} \ldots t_{i_p} \). Then the maximum weight with respect to the order of domination \( \lambda \geq \mu \iff \forall m : \sum_{i=1}^{m} \lambda_i \geq \sum_{i=1}^{m} \mu_i \) has tensor \( \phi^{(2,1, \ldots, 1)} \neq 0 \), since \( \phi^{(1,1, \ldots, 1)} = 0 \). The weight of this nonzero vector is \( (p, 1, 0, \ldots, 0) \). It follows that \( \ker S^{p+1} \equiv V((p, 1, 0, \ldots, 0)) \). Since the module \( S^{p+1}V \) is \( \text{GL}(n, \mathbb{R}) \)-irreducible, the decomposition (3.1) is also \( \text{GL}(n, \mathbb{R}) \)-irreducible.

Based on the above, we conclude that there are only two Stein-Weiss differential operators defined on the space of sections \( C^\infty S^p M \) of \( S^p M \). We define the first-order linear differential operator \( \delta^* : C^\infty S^p M \to C^\infty S^{p+1} M \) by means of the equality

\[
\delta^* \varphi = (p + 1) S^{p+1} (\nabla \varphi).
\]

It has the following form in local coordinates \( x^1, \ldots, x^n \):

\[
(\delta^* \varphi)_{k_1 \ldots k_p-1} := \nabla_k \varphi_{i_1 \ldots i_p} + \cdots + \nabla_{i_p} \varphi_{t_1 \ldots t_{p-1} k}
\]

where \( \nabla_k = \nabla_{\partial_k / \partial x^k} \), and \( \varphi \in C^\infty S^p M \). The value on \( \xi \in C^\infty T^*_x M \) of the symbol \( \sigma(\delta^*) \) of the operator \( \delta^* \) is a homomorphism

\[
\sigma(\delta^*)(\xi, x) : \varphi_x \in \mathcal{S}^p(T_x M) \to (p + 1) \xi \otimes \varphi_x \in \mathcal{S}^{p+q}(T_x M),
\]

according to the law of symmetric multiplication \( \varphi_x \otimes \varphi'_x = \mathcal{S}^{p+q}(\varphi_x \otimes \varphi'_x) \) for the pointwise defined symmetric multiplication \( \mathcal{S}^{p+q} : \mathcal{S}^p(T_x M) \otimes \mathcal{S}^q(T_x M) \to \mathcal{S}^{p+q}(T_x M) \) and any tensors \( \varphi \in C^\infty S^p M \) and \( \varphi' \in C^\infty S^q M \). Therefore, \( P_1 = (p + 1)^{-1} \delta^* \) is the first Stein-Weiss operator defined as symmetrization of the covariant derivative.

Consider further an operator of the form \( P_2 = \nabla - (p + 1)^{-1} \delta^* \). The value of its symbol \( \sigma(P_2) \) on any 1-form \( \xi \in C^\infty T^* M \) is the homomorphism

\[
\sigma(P_2)(\xi, x) : \varphi_x \in \mathcal{S}^p(T_x M) \to (\xi \otimes \varphi_x - (p + 1) \xi \otimes \varphi_x) \in \ker \mathcal{S}^{p+1}(T_x M)
\]

defined at any point \( x \in M \). Thus, the second operator will be \( P_2 \).

Since for any \( \varphi \in C^\infty S^p M \) there is a pointwise \( \text{GL}(n, \mathbb{R}) \)-irreducible decomposition

\[
\nabla \varphi = P_1 \varphi + P_2 \varphi,
\]

then due to Stein-Weiss approach in [20], the above \( P_1 \) and \( P_2 \) are Stein-Weiss operators on the space of symmetric \( p \)-tensors, because \( P_1 \varphi \) and \( P_2 \varphi \) are pointwise \( \text{GL}(n, \mathbb{R}) \)-irreducible components of the decomposition of \( \nabla \varphi \). Thus, we get

**Proposition 3.2.** Let \( M \) be a smooth \( n \)-dimensional \( (n \geq 2) \) manifold with a linear connection \( \nabla \) without torsion. Then there are two Stein-Weiss differential operators

\[
P_1 = \frac{1}{p + 1} \delta^* \text{ and } P_2 = \nabla - \frac{1}{p + 1} \delta^*
\]

on the space of sections \( C^\infty S^p M \).

The kernel of \( P_1 \) consists of *symmetric Killing \( p \)-tensors*, that is, tensor fields \( \varphi \in C^\infty S^p M \) such that \( \mathcal{S}^{p+1}(\nabla \varphi) = 0 \). The kernel of \( P_2 \) consists of *Codazzi \( p \)-tensors* \( \varphi \in C^\infty S^p M \), for which \( \nabla \varphi \in C^\infty S^{p+1} M \). According to [3, p. 35], the operator \( \delta^* : C^\infty S^p M \to C^\infty S^{p+1} M \) has the formally adjoint operator \( \delta : C^\infty S^{p+1} M \to C^\infty S^p M \), called *divergence* and defined by the equality \( \delta \varphi = -\text{trace}_g \nabla \varphi \). Here, the trace is given by the formula \( \text{trace}_g \varphi(a_3, \ldots, a_p) = \sum_{i=1}^{n} \varphi(e_i, a_3, \ldots, a_p) \) for any vectors \( a_3, \ldots, a_p \) and orthonormal basis \( \{e_1, \ldots, e_n\} \) of \( T_x M \) at any point \( x \in M \). Therefore, the formally adjoint to \( P_1 \) operator has the form \( P_1^* = (p + 1)^{-1} \delta \). Let us construct a second-order differential operator \( P_1^* P_1 = (p + 1)^{-2} \delta^* \). The operator \( P_1^* P_1 : C^\infty S^p M \to C^\infty S^p M \) is elliptic, since its principal symbol satisfies

\[
-g(\sigma(P_1^* P_1)(\xi, x) \varphi_x, \varphi_x) = g(\xi, \xi) g(\varphi_x, \varphi_x) - (p + 1) g(\xi \otimes \varphi_x, \xi, \varphi_x) \]

\[
= g(\xi, \xi) \cdot g(\varphi_x, \varphi_x) + p \cdot g(i_\xi \varphi_x, i_\xi \varphi_x) > 0
\]
for any $\xi \in T^*_{2}M \setminus \{0\}$ and nonzero $\varphi_\xi$ at any point $x \in M$. Thus, on a compact $(M, g)$ the kernel of $P_1^* P_1$ is a finite-dimensional vector space over $\mathbb{R}$. A local estimate for the dimension of this space was found in [2]:
\[
\dim_{\mathbb{R}} \ker P_1^* P_1 \leq C_p^{n+p} C_p^{n+p-1} - C_{p+1}^{n+p} C_{p-1}^{n+p-1},
\]
where the equality is attained on the Euclidean sphere. Since $\int_M g(P_1^* P_1 \varphi, \varphi) \, dV_g = \int_M g(P_1 \varphi, P_1 \varphi) \, dV_g \geq 0$, the kernel of $P_1^* P_1$ consists of symmetric Killing tensors $\varphi \in C^\infty S^p M$. By [3, p. 464], the following orthogonal decomposition is valid:
\[
(3.4) \quad C^\infty S^{p+1} M = \ker P_1^* \oplus \text{Im} P_1
\]
for the $L^2$-global scalar product on a compact $(M, g)$. Summing up, we formulate

**Proposition 3.3.** For any tensor field $\varphi \in C^\infty S^p M$ there is a pointwise orthogonal decomposition (3.2), where $P_1 = \frac{1}{p+1} \delta^*$ and $P_2 = \nabla - \frac{1}{p+1} \delta^*$. On a compact manifold $(M, g)$, the second-order differential operator $P_1^* P_1 = (p+1)^{-2} \delta^*$ is a nonnegative elliptic operator, whose kernel is a finite-dimensional vector space of symmetric Killing $p$-tensors. Moreover, the orthogonal decomposition (3.4) is valid.

If $(M, g)$ is a compact Riemannian manifold of nonpositive sectional curvature, then $\ker P_1^* P_1$ consists of parallel symmetric $p$-tensors, that is, tensors $\varphi$ obeying the condition $\nabla \varphi = 0$ (see [7]). If, in addition, $M$ is connected and there is a point at which all sectional curvatures are negative, then $\ker P_1^* P_1$ consists of symmetric $p$-tensors of the form $C \cdot g^k$ for some real constant $C$ (see also [7]).

**4 The Stein-Weiss operators on traceless symmetric tensors**

Bourguignon studied first order **natural differential operators** on the spaces of $C^\infty$-sections of the bundle $S^2_0 M$ of symmetric traceless 2-tensors on $(M, g)$, e.g., [6]. The symbols of such operators are projectors onto pointwise $O(n, \mathbb{R})$-irreducible sub-bundles of $T^* M \otimes S^2_0 M$. The following decomposition is valid:
\[
T^* M \otimes S^2_0 M = \text{Pr}_{S^2_0 M}(T^* M \otimes S^2_0 M) \oplus \text{Pr}_{T^* M}(T^* M \otimes S^2_0 M) \oplus \text{Pr}_{\ker S^2 \cap \ker \text{trace}_g}(T^* M \otimes S^2_0 M).
\]
As a consequence, we have the pointwise $O(n, \mathbb{R})$-irreducible decomposition
\[
(4.1) \quad \nabla \varphi = D_1 \varphi + D_2 \varphi + D_3 \varphi
\]
for any traceless symmetric 2-form, or, the field of 2-tensors $\varphi \in C^\infty S^2_0 M$. Based on this fact, Bourguignon defined all three operators $D_1$, $D_2$ and $D_3$ and proved that the kernel of the operator $D_1$ consists of the divergence-free 2-tensors $\varphi \in C^\infty S^2_0 M$. He argued that the kernels of $D_2$ and $D_3$ do not have a simple geometric interpretation. In [25], these arguments were applied to a pseudo-Riemannian manifold $(M, g)$, all three Stein-Weiss operators were redefined on $C^\infty$-sections of $S^2_0 M$, and a geometric interpretation of traceless symmetric 2-tensors lying in the kernel of each of them was given. It was proved that the kernel of $D_1$ consists of (traceless) **symmetric conformal Killing 2-tensors** (see [26, p. 559]), and the kernel of $D_2$ consists of traceless **conformal**
Codazzi 2-tensors defined in [24]. The main difference of these tensors from well-known Codazzi 2-tensors (e.g., [3, pp. 434; 436-440]) is their conformal invariance.

Consider a bundle $S^p_0 M$ ($p \geq 2$) of traceless symmetric $p$-tensors on $M$. For each $\varphi \in S^p_0 M$, the equality $\text{trace}_g \varphi = 0$ is valid.

**Lemma 4.1.** Let $(M, g)$ be a Riemannian manifold of dimension $n \geq 2$. Then the following pointwise $O(n, \mathbb{R})$-irreducible decomposition is valid:

$$T^* M \otimes S^p_0 M = \Pr_{S^{p+1}_0} (T^* M \otimes S^p_0 M) \oplus \Pr_{S^{p-1}_0} (T^* M \otimes S^p_0 M) \oplus 
\Pr_{\ker S^{p+1}_0, \ker \text{trace}_g} (T^* M \otimes S^p_0 M).$$

**Proof.** The fiber of $T^*_x M \otimes S^p_0 (T^*_x M)$ at any point $x \in M$ is an $n$-dimensional ($n > 1$) cotangent vector space $T^*_x M$. We will consider this tensor space as the space of representations $V^* \otimes S^p_0 V$ of $O(n, \mathbb{R})$.

There are three orthogonal subspaces $\ker S^{p+1}_0 \cap \ker \text{trace}_g$, $S^{p+1}_0 V$ and $S^{p-1}_0 V$ of $V^* \otimes S^p_0 V$ such that (see [1])

$$V^* \otimes S^p_0 V = \Pr_{S^{p+1}_0} (V^* \otimes S^p_0 V) \oplus \Pr_{S^{p-1}_0} (V^* \otimes S^p_0 V) \oplus \Pr_{\ker S^{p+1}_0, \ker \text{trace}_g} (V^* \otimes S^p_0 V).$$

The irreducibility of the components of the decomposition of $V^* \otimes S^p_0 V$ under the action of $O(n, \mathbb{R})$ follows from Theorem by G. Weyl on quadratic $O(n, \mathbb{R})$-invariant forms (see [6, pp. 313–314]).

There are three such independent invariant quadratic forms, which are specified using components $\phi_{i_1i_2\ldots i_p} = \phi(e_{i_1}, e_{i_2}, \ldots, e_{i_p})$ of $\phi \in V^* \otimes S^p_0 V$ in the orthonormal basis $e_1, \ldots, e_n$ of $V$, and have the form

$$\Psi_1(\phi) = \sum_{i_0, i_1, \ldots, i_p=1}^n (\phi_{i_0i_1\ldots i_p})^2, \quad \Psi_2(\phi) = \sum_{i_1, i_2, \ldots, i_p=1}^n (\phi_{ii_1\ldots i_p})^2, \quad \Psi_3(\phi) = \sum_{i_0, i_1, i_2, \ldots, i_p=1}^n \phi_{i_0i_1i_2\ldots i_p} \phi_{i_0i_1i_2\ldots i_p}.$$

They represent all possible traces of the $\phi \otimes \phi$-form. Since there are three such forms, the decomposition $V^* \otimes S^p_0 V$, which also has three tensor components, is $O(n, \mathbb{R})$-irreducible according to result of H. Weil (see [6, pp. 313–314]). \hfill \Box

Let $\text{Diff}(S^p_0 M, T^* M \otimes S^p_0 M)$ denote the $C^\infty M$-module of first-order linear differential operators $D : C^\infty S^p_0 M \rightarrow C^\infty (T^* M \otimes S^p_0 M)$ on the space of smooth sections $C^\infty S^p_0 M$ of the bundle $S^p_0 M$. Due to the pointwise orthogonal decomposition of the bundle $T^* M \otimes S^p_0 M$ from [1], we get the pointwise $O(n, \mathbb{R})$-irreducible decomposition (4.1) of the covariant derivative of any tensor field $\varphi \in C^\infty S^p_0 M$. Then certain $D_1$, $D_2$ and $D_3$ are Stein-Weiss operators on $C^\infty S^p_0 M$. The Stein-Weiss operator $D_1$, whose symbol is the projector onto the pointwise irreducible component $S^p_0 M$, is

$$D_1 \varphi = \frac{1}{p+1} \left( \delta^* \varphi + \frac{p(p+1)}{n+2(p-1)} g \odot \delta \varphi \right)$$

for any $\varphi \in C^\infty S^p_0 M$ and an algebraic operator $g \odot : S^{p-1} M \rightarrow S^{p+1} M$ defined pointwise by $g \odot := (2p-1)S^{p+1}(g \odot)$ (see [1]). In local coordinates $x^1, \ldots, x^n$ on $(M, g)$, the expression (4.2) appears as

$$D_1 \varphi)_{i_0i_1i_2\ldots i_p} = \frac{1}{p+1} \left( \delta^* \varphi_{i_0i_1i_2\ldots i_p} + \frac{p(p+1)}{n+2(p-1)} g_{(i_0i_1} \delta \varphi_{i_2\ldots i_p)}.\right.$$
On kernels of second-order elliptic operators

Using the identity \( g(\iota_0, \delta \varphi_{i_2 \ldots i_p}) = g(\iota_0(\iota_1 \delta \varphi_{i_2 \ldots i_p})) \) for the pointwise symmetrization operator \( S^{p+1}(g \otimes \delta \varphi)_{i_0 i_1 \ldots i_p} = g(\iota_1 \delta \varphi_{i_2 \ldots i_p}) \), we rewrite (4.3) in the form

\[
(D_1 \varphi)_{i_0 i_1 \ldots i_p} = \frac{1}{p+1} \left( \delta^* (g_{i_0 i_1 i_2 i_3 \ldots i_{p-1} i_p} + \frac{1}{n+2(p-1)} (g_{i_0 i_1 \delta \varphi_{i_2 i_3 \ldots i_{p-1} i_p} + \ldots + g_{i_0 i_p} \delta \varphi_{i_1 i_2 i_3 \ldots i_{p-2} i_p})
\right),
\]

Based on (4.4), we get \( D_1 \varphi \in C^{\infty}S_0^{p+1}M \). We call \( \varphi \in C^{\infty}S_0^pM \) a symmetric conformal Killing p-tensor, if \( D_1 \varphi = 0 \), which coincides with the notion of a conformal Killing p-tensor, e.g., [7, 8]. For \( p = 1 \) condition \( D_1 \varphi = 0 \) takes the form of well-known equations of a conformal Killing vector (see [26, pp. 559]). Formally conjugate to (4.2) operator \( D_1^*: C^{\infty}S_0^{p+1}M \to C^{\infty}S_0^pM \) is given for any \( \tilde{\varphi} \in C^{\infty}S_0^{p+1}M \) by

\[
(D_1^* \tilde{\varphi})_{i_0 i_1 \ldots i_p} = \frac{1}{p+1} \left( \delta \tilde{\varphi} + \frac{p(p+1)}{n+2(p-1)} (g \otimes \delta)^* \tilde{\varphi} \right) = \frac{1}{p+1} \delta \tilde{\varphi},
\]

because \((g \otimes \delta)^* = \text{trace}_g \). Therefore, \((g \otimes \delta)^* \tilde{\varphi} = (2p-1) \delta^*(\text{trace}_g \tilde{\varphi}) = 0 \) for any traceless tensor \( \tilde{\varphi} \in C^{\infty}S_0^{p+1}M \). Based on the operators \( D_1 \) and \( D_1^* \), we define a second-order differential operator of the form \( D_1 D_1: C^{\infty}S_0^pM \to C^{\infty}S_0^pM \), which according to (4.4) and (4.5) is given by the following equality:

\[
(D_1 D_1 \varphi)_{i_0 i_1 \ldots i_p} = \frac{1}{(p+1)^2} \left( \delta \delta^* \varphi + \frac{1}{n+2(p-1)} (-2 \delta^* \delta \varphi + p(p-1) g \otimes \delta \delta \varphi) \right).
\]

For the Sampson Laplacian operator \( \Delta_S = \delta \delta^* - \delta^* \delta \), (4.6) can be rewritten as

\[
(D_1 D_1 \varphi)_{i_0 i_1 \ldots i_p} = \frac{1}{(p+1)^2} \left( \delta \delta^* \varphi + \frac{1}{n+2(p-2)} \delta \delta^* \varphi + p(p-1) g \otimes \delta \delta \varphi \right).
\]

Let us prove the ellipticity of the operator \( D_1 D_1 \). First, note that at each point \( x \in M \) for any \( \varphi \in C^{\infty}S_0^pM \) and \( \xi \in T^*_x M \setminus \{0\} \) the inequality \( g(\sigma(\delta \delta^* \varphi)(\xi, \varphi_x), \varphi_x) = 0 \) holds, which is a consequence of the tracelessness of the tensor field \( \varphi \). Second, for any nonzero \( \varphi \in C^{\infty}S_0^pM \) the inequality \( -g(\sigma(\Delta_S \varphi)(\xi, \varphi_x), \varphi_x) = g(\xi, \xi) \| \varphi_x \|^2 > 0 \) holds (see [15]). By (3.3), \( -g(\sigma(\delta \delta^*) \varphi)(\xi, \varphi_x), \varphi_x) > 0 \) holds. Thus, the inequality \( -g(\sigma(\delta \delta^*) \varphi)(\xi, \varphi_x), \varphi_x) > 0 \) takes place; hence, \( D_1 D_1 \) is elliptic. Then its kernel on a compact \( (M, g) \) is finite-dimensional. Moreover, \( \int_M g(D_1^* D_1 \varphi, \varphi) dV_g = \int_M g(D_1 \varphi, D_1 \varphi) dV_g \geq 0 \), thus, this vector space consists of symmetric conformal Killing p-tensors \( \varphi \in C^{\infty}S_0^pM \). The following orthogonal decomposition takes place:

\[
C^{\infty}S^{p+1}_0M = \ker D_1^* \oplus \text{Im } D_1
\]

for the \( L^2 \)-global scalar product on the compact \( (M, g) \). Summing up, we formulate

**Proposition 4.2.** The pointwise O(n, \( \mathbb{R} \))-irreducible decomposition (4.1) of the covariant derivative of any tensor field \( \varphi \in C^{\infty}S_0^pM \) holds. On a compact \( (M, g) \), a second-order differential operator \( D_1^* D_1 \) for the Stein-Weiss operator

\[
D_1 \varphi = (p+1)^{-1} (\delta^* \varphi + (n+2(p-1))^{-1}(g \otimes \delta \varphi)),
\]
and its formally conjugate \( D^*_1 \), is a nonnegative elliptic operator, whose kernel is a finite-dimensional vector space over \( \mathbb{R} \) and consists of symmetric conformal Killing \( p \)-tensors. Moreover, the orthogonal decomposition (4.8) is valid.

The second Stein-Weiss differential operator \( D_2 \), whose symbol is the projector onto the second pointwise irreducible component of the decomposition \( TM^* \otimes S^p_0 M \) is

\[
(D_2 \varphi)_{i_0 i_1 i_2 \ldots i_{p-2} i_{p-1} i_p} = -p(n + p - 1)^{-1} g_{i_0 (i_1 \delta \varphi_{i_2 \ldots i_p})}
\]

(see [1]), and its kernel consists of traceless divergence-free \( p \)-tensors.

The third Stein-Weiss differential operator \( D_3 \), whose symbol is the projector onto the third pointwise irreducible component of the decomposition \( TM^* \otimes S^p_0 M \), is

\[
(D_3 \varphi)_{i_0 i_1 i_2 \ldots i_{p-2} i_{p-1} i_p} = \nabla_{i_0} \varphi_{i_1 i_2 \ldots i_p} + \frac{p}{n+p-1} g_{i_0 (i_1 \delta \varphi_{i_2 \ldots i_p})} - \frac{1}{p+1} \left( \delta^* \varphi_{i_0 i_1 i_2 \ldots i_p} + \frac{p(p+1)}{n+2(p-1)} g_{i_0 i_1 \delta \varphi_{i_2 \ldots i_p}} \right)
\]

for any \( \varphi \in C^\infty S^p_0 M \) (see [1]). For any \( \varphi \in \ker D_3 \), the following equations hold:

\[
(4.9) \quad \nabla_{i_0 \varphi_{i_1 i_2 \ldots i_p}} - \nabla_{i_1} \varphi_{i_0 i_2 \ldots i_p} = \frac{p}{n+p-1} \left( g_{i_0 (i_1 \delta \varphi_{i_2 \ldots i_p})} - g_{i_1 (i_0 \delta \varphi_{i_2 \ldots i_p})} \right).
\]

5 Global Riemannian geometry of conformal Killing tensors

The kernel of \( D_1 \) consists of \( p \)-tensors \( \varphi \in C^\infty S^p_0 M \) for \( p \geq 2 \) that satisfy

\[
(5.1) \quad \delta^* \varphi = -\frac{p(p+1)}{n-2(p-1)} g \otimes \delta \varphi.
\]

Each such \( p \)-tensor is a symmetric conformal Killing \( p \)-tensor (e.g., [7, 8]). Note that the requirement of tracelessness is included here in the definition of the conformal Killing \( p \)-tensor (\( p \geq 2 \)) as well as in [26, p. 559] for the case \( p = 2 \). The condition \( \varphi \in \ker D_1 \cap \ker \delta \) defines a symmetric Killing \( p \)-tensor \( \varphi \in C^\infty S^p_0 M \), because (5.1) implies that \( \delta^* \varphi = 0 \). Taking into account (4.7), we find

\[
(5.2) \quad g(\Delta_S \varphi, \varphi) = -2^{-1}(n + 2(p - 2)) g(\delta \delta^* \varphi, \varphi)
\]

for Sampson Laplacian \( \Delta_S = \delta \delta^* - \delta^* \delta \) and conformal Killing tensors \( \varphi \in C^\infty S^p_0 M \). From (5.2) we conclude that the symmetric divergence-free (traceless) conformal Killing tensor, or, equivalently, the symmetric traceless \( p \)-Killing tensor belongs to the kernel of \( \Delta_S \). For a compact manifold \((M, g)\), it follows from (5.2) that

\[
\int_M g(\Delta_S \varphi, \varphi) dV_g = -2^{-1}(n + 2(p - 2)) \int_M g(\delta \varphi, \delta \varphi) dV_g.
\]

Thus, any traceless conformal Killing \( p \)-tensor belonging to the kernel of the Sampson Laplacian is divergence-free, thus it is a Killing \( p \)-tensor. We get the following

**Proposition 5.1.** On a compact Riemannian manifold, a symmetric (traceless) conformal Killing \( p \)-tensor belongs to the kernel of the Sampson Laplacian if and only if it is a traceless \( p \)-Killing tensor.
For any Killing $p$-tensor ($p \geq 2$), direct calculations lead to the following formula: $2 \delta \varphi = \delta^* (\text{trace}_g \varphi)$. Thus, on a compact Riemannian manifold of negative Ricci curvature, every symmetric Killing tensor of rank 3 is traceless. The Sampson Laplacian $\Delta_S : C^\infty S^p M \to C^\infty S^p M$ admits the Weitzenböck decomposition (see [15])

$$\Delta_S \varphi = \bar{\Delta} \varphi - \Re(\varphi).$$

The formula (5.3) indicates that $\Delta_S$ is a particular form of Lichnerovich’s Laplacian (see [3, p. 79] and [9]). Here, $\Re$ is linearly expressed in terms of the Riemannian curvature tensor and the Ricci tensor of the Levi-Civita connection and satisfies $g(\Re(\varphi), \varphi') = g(\Re(\varphi'), \varphi)$ for any $\varphi, \varphi' \in C^\infty S^p M$ (see [15]). Thus, $\Phi_p(\varphi, \varphi) = g(\Re(\varphi), \varphi)$ is a quadratic form for any $\varphi \in S^p(T_x^* M)$ and $x \in M$. Since $\Delta_S$ is an elliptic operator, by [3, p. 632], the $L^2(M)$-orthogonal decomposition $C^\infty S^p M = \ker \Delta_S \oplus \text{Im} \Delta_S$ is valid. The symmetric tensor $\varphi \in C^\infty S^p M$ such that $\varphi \in \ker \Delta_S$ is called $\Delta_S$-harmonic section (see [16, p. 104]), and the space of such tensors on a compact Riemannian manifold $(M, g)$ is finite-dimensional. The following is valid.

**Proposition 5.2.** On a compact Riemannian manifold $(M, g)$ the space of $\Delta_S$-harmonic sections is finite-dimensional.

Using Proposition 5.2 and (5.3), we can formulate the following

**Corollary 5.3.** On a Riemannian manifold $(M, g)$, any divergence-free or, e.g., traceless Killing $p$-tensor is a $\Delta_S$-harmonic section.

From (5.3) we deduce the Bochner-Weitzenböck formula (e.g., [15] and [16, p. 106])

$$\frac{1}{2} \Delta \| \varphi \|^2 = -g(\Delta_S \varphi, \varphi) - g(\Re(\varphi), \varphi) + \| \nabla \varphi \|^2,$$

where for $\nabla \varphi$ the pointwise $O(n, \Re)$-irreducible decomposition (4.1) holds. Thus,

$$\frac{1}{2} \Delta \| \varphi \|^2 = -g(\Delta_S \varphi, \varphi) - g(\Re(\varphi), \varphi) + \| D_1 \varphi \|^2 + \| D_2 \varphi \|^2 + \| D_3 \varphi \|^2.$$  

For a symmetric conformal Killing $p$-tensor, the formula (5.4) takes the form

$$\frac{1}{2} \Delta \| \varphi \|^2 = 2^{-1}(n + 2(p - 2)) g(\delta \delta^* \varphi, \varphi) - g(\Re(\varphi), \varphi) + \| D_2 \varphi \|^2 + \| D_3 \varphi \|^2.$$  

Suppose that $M$ is compact, then integrating (5.5) we obtain

$$\int_M g(\Re(\varphi), \varphi) dV_g = 2^{-1}(n + 2(p - 2)) \int_M \| \delta^* \varphi \|^2 dV_g + \int_M (\| D_2 \varphi \|^2 + \| D_3 \varphi \|^2) dV_g \geq 0,$$

because $\int_M g(\delta \delta^* \varphi, \varphi) dV_g = \int_M \| \delta^* \varphi \|^2 dV_g \geq 0$. On $(M, g)$ of nonpositive curvature $\Phi_p(\varphi, \varphi) = g(\Re(\varphi), \varphi) \leq 0$ holds for any $\varphi \in S^p_0 M$ (see [8, 7]). If there is a point at which the sectional curvature is negative, then $\Phi_p(\varphi, \varphi) = g(\Re(\varphi), \varphi) < 0$ for any symmetric $p$-form $\varphi \in S^p_0 M$. Based on the above equality, we get the following

**Proposition 5.4.** On a compact Riemannian manifold $(M, g)$ of nonpositive sectional curvature sec, each symmetric conformal Killing tensor $\varphi \in C^\infty S^p_0 M$ is parallel, i.e., $\nabla \varphi = 0$. Moreover, if there is a point at which $\text{sec} < 0$, then on $(M, g)$ there are no nonzero symmetric conformal Killing $p$-tensors $\varphi \in C^\infty S^p_0 M$. 

One can show $\frac{1}{2} \Delta ||\varphi||^2 = ||\varphi|| \Delta ||\varphi|| + ||d \varphi||^2$, where $||\nabla \varphi||^2 \geq ||d \varphi||^2$ by Kato’s inequality (e.g., [16, p. 105]). Thus, the above equality takes the following form:

$$||\varphi|| \Delta ||\varphi|| = \frac{1}{2} \Delta ||\varphi||^2 - ||d \varphi||^2 \geq \frac{1}{2} \Delta ||\varphi||^2 - ||\nabla \varphi||^2,$$

where $\Delta ||\varphi||^2$ due to (5.4) satisfies the inequality

$$\frac{1}{2} \Delta ||\varphi||^2 \geq -g(\Delta S \varphi, \varphi) - g(\Re(\varphi), \varphi).$$

Summing up, we get the following inequality:

$$\|\varphi\| \Delta \|\varphi\| \geq -g(\Delta S \varphi, \varphi) - \Phi_p(\varphi, \varphi). \tag{5.6}$$

Let further $\varphi \in C^\infty S^0_0 M$ be a Killing $p$-tensor, for which, as was proved above, $\Delta S \varphi = 0$, then the inequality (5.6) can be rewritten as

$$\|\varphi\| \Delta \|\varphi\| \geq -\Phi_p(\varphi, \varphi). \tag{5.7}$$

For $(M, g)$ of nonpositive curvature, from (5.7) we find $\Delta \|\varphi\| \geq 0$, thus, $\|\varphi\|$ is a nonnegative subharmonic function for any Killing $p$-tensor $\varphi \in S^0_0 M$. There is a well-known theorem (see [14, p. 288]): On a complete simply connected Riemannian manifold $(M, g)$ of nonpositive curvature, any nonnegative subharmonic function $f \in C^2(M)$ satisfying $\int_M f^q \, dV_g < \infty$ for some $q \in (0, \infty)$, is constant. Setting $f = \|\varphi\|$, we find $\|\varphi\| = C$ for some real constant $C$, thus, $\nabla \varphi = 0$. On the other hand, in this case

$$\int_M \|\varphi\|^q \, dV_g = C^q \int_M dV_g = C^q \text{Vol}(M, g).$$

Since we assume $\|\varphi\| \in L^q(M)$ for some $0 < q < \infty$, then for $C \neq 0$ the volume of $(M, g)$ must be finite. If the volume of $(M, g)$ is infinite, then necessarily $\varphi \equiv 0$. The following has been proven.

**Theorem 5.5.** If a simply connected complete $(M, g)$ has nonpositive sectional curvature, then the symmetric Killing $p$-tensor $(p \geq 2) \varphi \in S^0_0 M$ such that

$$\int_M \|\varphi\|^q \, dV_g < \infty \tag{5.8}$$

for some $q \in (0, \infty)$ is parallel; and if $(M, g)$ has infinite volume, then $\varphi \equiv 0$.

A Riemannian manifold $(M, g)$ with $\delta^* \text{Ric} = 0$ was popular [3, pp. 450-451]. In this case, $\Delta_S \text{Ric} = 0$, thus, by Theorem 5.5, $\text{Ric} = 0$ (for a compact $M$, see [3, p. 451]).

Let $M = G/H$ be a Riemannian symmetric space of noncompact type with a $G$-invariant metric $g$. Then $(M, g)$ is a complete Riemannian manifold of nonpositive sectional curvature and negative definite Ricci tensor, thus, it is irreducible (see [12, pp. 226, 236]). Therefore, it is true the following

**Corollary 5.6.** On a Riemannian symmetric space $(M, g)$ of noncompact type, each symmetric Killing $p$-tensor $(p \geq 2) \varphi \in S^0_0 M$ such that (5.8) holds for some $q \in (0, \infty)$, is parallel. Moreover, if $p = 2$, then $\varphi \equiv 0$. 

6 Global Riemannian geometry of rank $p \geq 2$ Codazzi tensors

For a Codazzi $p$-tensor ($p > 3$) $\varphi \in C^\infty S^p M$, from $\nabla \varphi \in C^\infty S^{p+1} M$ we conclude that $\nabla (\text{trace}_g \varphi) \in C^\infty S^{p-2} M$. From the condition (also defining the Codazzi $p$-tensor)

(6.1) $P_2 \varphi = \nabla \varphi - \frac{1}{p+1} \delta^* \varphi = 0$,

it follows that $\delta \varphi = -\nabla (\text{trace}_g \varphi)$ for any $p \geq 2$. Therefore, the following is true.

**Proposition 6.1.** For any Codazzi $p$-tensor $\varphi \in S^p M$, where $p > 3$, on the Riemannian manifold $(M, g)$ the symmetric form $\text{trace}_g \varphi$ is a Codazzi ($p - 2$)-tensor. For $p \geq 2$, each traceless Codazzi $p$-tensor $\varphi$ has zero divergence.

Based on (6.1) for the divergence-free Codazzi tensor $\varphi \in S^p M$, we obtain

$$
\bar{\Delta} \varphi = \frac{1}{p+1} P_1^* P_1 \varphi = \frac{1}{p+1} \Delta S \varphi.
$$

Thus, it follows from the Weitzenböck expansion (5.3) that

(6.2) $\bar{\Delta} \varphi = -\frac{1}{p+1} \mathcal{R}(\varphi)$.

Therefore, we can formulate the following

**Proposition 6.2.** Any divergence-free Codazzi $p$-tensor $\varphi$ on a Riemannian manifold $(M, g)$ belongs to the kernel of the Lichnerovich Laplacian $\Delta_L = \Delta + \frac{1}{p+1} \mathcal{R}$.

From (6.2) we get the Bochner-Weitzenböck formula

(6.3) $\frac{1}{2} \Delta \| \varphi \|^2 = \frac{1}{p+1} \Phi_p(\varphi, \varphi) + \| \nabla P_1 \|^2$.

Using (6.3), we obtain the inequality

(6.4) $\| \varphi \| \Delta \| \varphi \| \geq \frac{1}{p+1} \Phi_p(\varphi, \varphi)$.

On $(M, g)$ of nonnegative sectional curvature, we have the inequality $\Phi_p(\varphi, \varphi) \geq 0$ for any $\varphi \in S^p M$ (see [4]). If this assumption is true, then from (6.4) we get $\Delta \| \varphi \| \geq 0$. As a result, $\| \varphi \|$ becomes a nonnegative subharmonic function for any divergence-free Codazzi $p$-tensor $\varphi \in S^p M$. Due to S.T. Yau (see [16, p. 262] and [28]), on a complete $(M, g)$ of infinite volume the only nonnegative subharmonic function $f$ satisfying $f \in L^q(M)$ for some $1 < q < \infty$, is $f \equiv 0$. Since a complete noncompact Riemannian manifold of nonnegative sectional curvature has infinite volume (see [14]), we get $\varphi \equiv 0$. The following theorem is proved.

**Theorem 6.3.** On a complete noncompact Riemannian manifold $(M, g)$ of nonnegative sectional curvature there is no nonzero divergence-free Codazzi tensor $\varphi \in S^p M$ ($p \geq 2$) such that (5.8) holds for some $q > 1$. 

**Remark 6.1.** There are no complete noncompact conformally flat \((M, g)\) of nonnegative sectional curvature and constant scalar curvature such that Ric satisfies (5.8) for some \(q > 1\), since, in this case, Ric is a Codazzi divergence-free tensor, [3, p. 432].

Let \(M = G/H\) be a Riemannian symmetric space of compact type with a \(G\)-invariant metric \(g\). Then \((M, g)\) is compact with nonnegative sectional curvature and positive definite Ricci tensor, thus, it is irreducible (see [12, p. 256]).

The following theorem generalizes the result from [10].

**Corollary 6.4.** On a Riemannian symmetric space \((M, g)\) of compact type, any divergence-free Codazzi \(p\)-tensor \(\varphi \in \mathbb{S}^p M\) for \(p \geq 2\) has a constant length. In particular, if \(p = 2\), then \(\varphi = C g\) for some real constant \(C\).

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**References**


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