Results on quasi-\(*\)-Einstein metric

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Abstract. We study the quasi-\(*\)-Einstein metric on Sasakian and \((\kappa, \mu)\)-manifolds. We show that on Sasakian manifolds the \(*\)-Ricci operator commutes with tensor field \(\phi\) and quasi-\(*\)-Einstein Sasakian metric is \(*\)-flat. Further, we study \((\kappa < 1, \mu)\)-manifolds with quasi-\(*\)-Einstein metric and obtain that such manifold is \(*\)-flat or locally isometric to \(E^{n+1} \times S^n(4)\) or \(*\)-Einstein.

Key words: quasi-\(*\)Einstein metric; contact metric; Sasakian manifold; \((\kappa, \mu)\)-manifold.

1 Introduction

A Riemannian manifold \((M, g)\) is called \(*\)Einstein with \(S^* = \nu g\), where \(S^*\) denotes \(*\)-Ricci tensor and \(\nu\) is a constant. A \(*\)-Ricci soliton is a generalization of \(*\)Einstein metric which is given by [13]

\[
\frac{1}{2} L_U g + S^* = \nu g,
\]

where \(\nu\) is a constant and \(U\) is the potential field. If for a smooth function \(F\), \(U = \nabla F\), then (1.1) is called gradient \(*\)-Ricci soliton.

The \(*\)-Ricci tensor on an almost contact metric manifold \(M\) is defined as follows [12]:

\[
S^*(Z, V) = \frac{1}{2} \text{trace}(X \mapsto R(Z, \phi V)\phi X), \quad \forall Z, V, X \in TM,
\]

where \(\phi\) is a \((1, 1)\)-tensor field and \(R\) is a Riemann curvature tensor.

The Einstein condition \(S = \nu g\) and its generalizations have been studied extensively in contact geometry (see [8, 11, 16]). A generalization of Einstein metric emerged from the m-Bakry-Emery Ricci tensor \(S^F_m\) defined as:

\[
S^F_m = S + \nabla^2 F - \frac{1}{m} dF \otimes dF,
\]

has been studied in [4, 5], where $0 < m \leq \infty$, $S$ is the Ricci tensor and $\nabla^2 F$ denotes the Hessian form of $F$.

A Riemannian manifold $(M, g, F, m)$ is called $m$–quasi-Einstein [7] if it satisfies

\begin{equation}
S + \text{Hess} F - \frac{1}{m} dF \otimes dF = \nu g,
\end{equation}

for some $m \in \mathbb{Z}^+$. Similarly, we call $(M, g, F, m)$, $m$–quasi-$^*$Einstein if it satisfies

\begin{equation}
S^* + \text{Hess} F - \frac{1}{m} dF \otimes dF = \nu g.
\end{equation}

If $m = \infty$, then (1.5) gives the gradient $^*$-Ricci soliton. A quasi-$^*$Einstein metric is $^*$Einstein if $F = \text{constant}$. We call a quasi-$^*$Einstein metric steady, expanding, or shrinking, respectively, when $\nu = 0$, $< 0$ or $> 0$.

Sharma [15] proved that a complete $K$-contact metric with gradient Ricci soliton is a compact Einstein Sasakian manifold and gradient soliton is expanding. As every Sasakian manifold is a $K$-contact manifold, this result is also true for Sasakian manifolds. Ghosh et al. extended this result for $(\kappa, \mu)$-spaces [11]. Quasi-Einstein metrics have been studied in extent for a general manifold, and gap results and rigid properties were obtained in (cf. [4, 17]). Further, Ghosh [9] studied quasi-Einstein contact metric on Sasakian manifolds and on $(\kappa, \mu)$-spaces. Recently, Chen [6] studied the quasi-Einstein metric on almost cosymplectic manifolds.

However, only very little literature is available on the study of $^*$Einstein and its generalization. This inspired us to study quasi-$^*$Einstein structure associated with contact metric manifolds.

## 2 Preliminaries and some basic results

An odd-dimensional Riemannian manifold $M^{2n+1}$ is called an almost contact metric manifold if it admits a $(1, 1)$ tensor field $\phi$, a vector field $\xi$, a 1-form $\eta$ and a Riemannian metric $g$ satisfying [1]

\begin{align}
\phi^2 Z &= -Z + \eta(Z) \xi, \quad \eta(\xi) = 1, \\
\eta \circ \phi &= 0, \quad \phi \xi = 0, \\
g(\phi Z, \phi V) &= g(Z, V) - \eta(Z)\eta(V), \quad \eta(Z) = g(Z, \xi).
\end{align}

where $Z, V \in TM$. A contact manifold is a Riemannian manifold $M^{2n+1}$ with a global 1-form $\eta$ called a contact 1-form such that $\eta \wedge (d\eta)^n \neq 0$ everywhere on $M$ and $d\eta(Z, V) = g(Z, \phi V)$.

A contact metric manifold is called Sasakian [1] if

\begin{equation}
(\nabla_Z \phi)V = g(Z, V)\xi - \eta(V)Z,
\end{equation}

where $Z, V \in TM$.

The Riemann curvature tensor $R$ on a Sasakian manifold is given by [1]

\begin{equation}
R(Z, V)\xi = \eta(V)Z - \eta(Z)V.
\end{equation}
On a $K$-contact manifold [1], we have
\begin{align}
\nabla_Z \xi &= -\phi Z, \\
QZ &= 2n\xi, \\
R(Z, \xi) \xi &= Z - \eta(Z)\xi.
\end{align}

Now, we define self-adjoint operators $h = \frac{1}{2} L_\xi \phi$ and $l = -R(\xi, \cdot)\xi$, which satisfy [1]:
\begin{align}
l\xi = 0 = h\xi, &\quad \phi h = -h\phi, &\quad g(hX, Y) = g(hY, X), \\
tr h = tr h\phi = 0.
\end{align}

A contact metric manifold $M^{2n+1}$ is said to be $(\kappa, \mu)$-space if Riemann curvature tensor $R$ satisfies [2]
\begin{equation}
R(Z, V)\xi = \kappa\{\eta(V)Z - \eta(Z)V\} + \mu\{\eta(V)hZ - \eta(Z)hV\},
\end{equation}
for some $(\kappa, \mu) \in \mathbb{R}^2$. If $\kappa = 1$ and $h = 0$, then $(\kappa, \mu)$-spaces reduce to the Sasakian manifolds. The non-Sasakian manifolds have proven to be more interesting as there exists the unit tangent sphere bundle of a flat Riemannian manifold with the usual contact metric structure as an example of non-Sasakian spaces of this type. Moreover, this type of space is invariant under $D$-homothetic transformations. These are the driving factor for the study of this type of manifold. Boeckx proved that non-Sasakian contact metric manifold satisfying (2.11) is completely determined locally by its dimension for the constant values of $\kappa, \mu$ [3].

3 Quasi-*Einstein metric on a Sasakian manifold

In this section, we study the quasi-*Einstein metric on a Sasakian manifold. On a Sasakian manifold $M^{2n+1}$ [10]
\begin{align}
Q^*Z &= QZ - (2n - 1)Z - \eta(Z)\xi, \\
r^* &= r - 4n^2,
\end{align}
where $Q^*$, $Q$, $r^*$ and $r$ are $*$-Ricci operator, Ricci operator, $*$-scalar curvature and scalar curvature respectively.

**Lemma 3.1.** [14] The curvature tensor $R$ of a Riemannian manifold $M^{2n+1}$ with quasi-*Einstein metric satisfies
\begin{align}
R(Z, V)\nabla F &= (\nabla_V Q^*)Z - (\nabla_Z Q^*)V - \frac{\nu}{m}\{(ZF)V - (VF)Z\} \\
&\quad + \frac{1}{m}\{(ZF)Q^*V - (VF)Q^*Z\},
\end{align}
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\[ S(V, \nabla F) = \frac{1}{2} V r^* + \frac{2\nu}{m} (VF) + \frac{1}{m} \{(Q^*V)F - r^*(VF)\} , \]

for any \( Z, V \in TM \), where \( \nabla F \) is gradient of \( F \).

**Proof.** From (1.5), we have

\[ Q^*V + \nabla V \nabla F - \frac{1}{m} (VF) \nabla F = \nu V. \quad (3.5) \]

We know that

\[ R(Z, V)X = \nabla_Z \nabla_V X - \nabla_V \nabla_Z X - \nabla_{[Z,V]} X. \quad (3.6) \]

Using (3.5) in (3.6), we obtain (3.3). Further, contracting (3.3) over \( Z \), we get (3.4). \( \square \)

**Lemma 3.2.** [9] If \( F \in C^\infty \) on a contact metric manifold \( M \) such that \( dF = (\xi F)\eta \), where \( d \) denotes the exterior differentiation. Then \( F \) is constant on \( M \).

Next,

**Lemma 3.3.** On a Sasakian manifold \( M^{2n+1} \)

\[ Q^* \phi = \phi Q^*. \quad (3.7) \]

**Proof.** Putting \( Z = \xi \) in (3.1) and using (2.6), we obtain

\[ Q^* \xi = 0. \quad (3.8) \]

Differentiating (3.8) with respect to \( Z \), we get

\[ (\nabla_Z Q^*) \xi = Q^* \phi Z. \quad (3.9) \]

If \( V, Z \in \{ \xi^\perp \} \), then with respect to \( \phi \) basis \( \{ Z_i, Z_{n+i} = \phi Z_i, \xi \} \), we have

\[
g(Q^*\phi Z, \phi V) = \frac{1}{2} \sum_{i=1}^{2n} g(R(\phi Z, \phi^2 V)\phi Z_i, Z_i) + \frac{1}{2} g(R(\phi Z, \phi^2 V)\phi \xi, \xi) \]

\[ = \frac{1}{2} \sum_{i=1}^{2n} g(R(V, \phi Z)\phi Z_i, Z_i), \]

which gives

\[ g(Q^*\phi Z, \phi V) = g(Q^*V, Z). \quad (3.10) \]

From (3.1) and using the fact that \( Q \) is symmetric, we get

\[ g(Q^*V, Z) = g(Q^*Z, V). \quad (3.11) \]

From (3.10) and (3.11), we get

\[ -\phi Q^* \phi Z = Q^* Z, \]

wherein multiplying with ‘\( \phi \)’ on both sides and using (2.1), we find

\[ Q^* \phi Z = \phi Q^* Z, \]

whereby proof is complete. \( \square \)
Theorem 3.4. Let $M^{2n+1}$ be a Sasakian manifold, then $\mathcal{L}_\xi S^* = 0$ and $\nabla_\xi Q^* = 0$.

Proof. It is well-known that $\mathcal{L}_\xi g = 0$ on a Sasakian manifold. Also, we have

\begin{align*}
(3.12) \quad & \mathcal{L}_\xi (R_{Z,\phi V} \phi X) = \mathcal{L}_\xi (\nabla_Z \phi V \phi X - \nabla_{\phi V} \nabla_Z \phi X - \nabla_{\{Z,\phi V\}} \phi X), \\
(3.13) \quad & R_{\mathcal{L}_\xi Z,\phi V} \phi X = \nabla_{\mathcal{L}_\xi Z} \phi V \phi X - \nabla_{\phi V} \nabla_{\mathcal{L}_\xi Z} \phi X - \nabla_{\{\mathcal{L}_\xi Z,\phi V\}} \phi X, \\
(3.14) \quad & R_{Z,\mathcal{L}_\xi \phi V} \phi X = \nabla_Z \nabla_{\mathcal{L}_\xi \phi V} \phi X - \nabla_{\mathcal{L}_\xi \phi V} \nabla_Z \phi X - \nabla_{\{Z,\mathcal{L}_\xi \phi V\}} \phi X, \\
(3.15) \quad & R_{Z,\phi V} \mathcal{L}_\xi \phi X = \nabla_Z \nabla_{\phi V} (\mathcal{L}_\xi \phi X) - \nabla_{\phi V} \nabla_Z (\mathcal{L}_\xi \phi X) - \nabla_{\{Z,\phi V\}} \mathcal{L}_\xi \phi X,
\end{align*}

\forall Z, V, X \in TM.

From (3.12)∼(3.15), we obtain

\[
\mathcal{L}_\xi (R_{Z,\phi V} \phi X) = R_{\mathcal{L}_\xi Z,\phi V} \phi X + R_{Z,\mathcal{L}_\xi \phi V} \phi X + R_{Z,\phi V} \mathcal{L}_\xi \phi X,
\]

which gives $(\mathcal{L}_\xi R)_{Z,\phi V} \phi X = 0$ and contracting it over $X$, we get $\mathcal{L}_\xi S^* = 0$. This implies

\[
\mathcal{L}_\xi (g(Q^* Z, V)) - S^* (\mathcal{L}_\xi Z, V) - S^* (Z, \mathcal{L}_\xi V) = 0.
\]

(3.16)

Simplifying (3.16), we find

\[
[\xi, Q^* Z] - Q^* ([\xi, Z]) = 0.
\]

(3.17)

Using (2.5) in (3.17), we get

\[
\nabla_\xi Q^* = Q^* \phi - \phi Q^*.
\]

(3.18)

Using Lemma 3.3 in (3.18), we obtain the result. □

Theorem 3.5. Let $M^{2n+1}$ be a Sasakian manifold satisfying (1.5), then $F$ is constant and quasi-*Einstein soliton is steady.

Proof. Taking inner product of (3.3) with $\xi$ and using (3.8) and (3.9), we get

\[
g (R(Z, V) \nabla F, \xi) = -2g(Q^* \phi Z, V) + \frac{\nu}{m} \{(VF)\eta(Z) - (ZF)\eta(V)\}.
\]

(3.19)

Putting $V = \xi$ in (3.19) and using (2.4), we get

\[
(\frac{\nu}{m} - 1)(\xi F)\eta(Z) - (ZF)\eta(V) = 0.
\]

(3.20)

Now, we claim that $\nu \neq m$. Infact, if $\nu = m$, then putting $Z = \xi$ in (3.3), we get

\[
R(\xi, V) \nabla F = (\nabla_V Q^*)\xi - (\nabla_\xi Q^*)V - \{\xi F\}V - (\xi F)V
\]

\[+ \frac{1}{m} \{\xi F\}Q^* V - (VF)Q^* \xi.
\]

(3.21)

From (2.4), we obtain

\[
R(\xi, V) \nabla F = (VF)\xi - (\xi F)V.
\]

(3.22)
Using (3.22) in (3.21), we get

\[(3.23) \quad (\nabla_V Q^*) \xi - (\nabla_\xi Q^*) V + \frac{1}{m} \{ (\xi F) Q^* V - (VF) Q^* \xi \} = 0.\]

Using Theorem 3.4, (3.8) and (3.9) in (3.23), we obtain

\[(3.24) \quad Q^* \phi V + \frac{\xi F}{m} Q^* V = 0.\]

Taking inner product of (3.24) with \(Z\), we find

\[(3.25) \quad g(Q^* \phi V, Z) + \frac{\xi F}{m} g(Q^* V, Z) = 0.\]

Interchanging \(V\) and \(Z\) in (3.25), we get

\[(3.26) \quad g(Q^* \phi Z, V) + \frac{\xi F}{m} g(Q^* Z, V) = 0.\]

Subtracting (3.26) from (3.25), we obtain

\[(3.27) \quad (Q^* \phi + \phi Q^*) V = 0,\]

\(\forall V \in TM\). Using Lemma 3.3 in (3.27), we get \(\phi Q^* V = 0\). Which further using (2.1) gives \(Q^* V = 0\). Hence \(r^* = 0\). Using this in (3.4), we find

\[(3.28) \quad S(V, \nabla F) = 2n(V F).\]

On the other hand from (3.1), we get

\[(3.29) \quad QV = (2n - 1)V + \eta(V) \xi.\]

Using (3.29) in (3.28), we obtain \(VF = (\xi F) \eta(V)\). Which gives \(F\) constant by use of Lemma 3.2. Therefore, from (1.5), we obtain \(\nu = 0\), that gives \(m = 0\), a contradiction of the fact that \(m > 0\).

Hence \(\nu \neq m\) and from (3.20), we have \(ZF = (\xi F) \eta(Z)\). By using Lemma 3.2 we find that

\[(3.30) \quad F = \text{constant}.\]

From (3.8), we see that

\[(3.31) \quad S^*(\xi, \xi) = 0.\]

Using (3.30) and (3.31) in (1.5), we get \(\nu = 0\). Hence quasi-*Einstein soliton is steady.

\[\square\]

**Corollary 3.6.** Let \(M^{2n+1}\) be a Sasakian manifold satisfying (1.5), then \(M\) is *-Ricci flat, \(\eta\)-Einstein and scalar curvature is constant.
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**Proof.** Using \( \nu = 0 \) and (3.30) in (1.5), we see that

\[
(3.32) \quad Q^* = 0.
\]

Using (3.32) in (3.1), we find

\[
(3.33) \quad QZ = (2n - 1)Z + \eta(Z)\xi.
\]

Hence \( M \) is \( \eta \)-Einstein. Further, from (3.33) we obtain \( r = 4n^2 \). Hence scalar curvature is constant. \( \square \)

Now, we give an example

**Example 3.1.** Consider the manifold \( M^3 = \{ (x, y, z) \in \mathbb{R}^3 : x, y \neq 0 \} \) endowed with the structure \( \{ \phi, \xi, \eta, g \} \)

\[
(3.34) \quad \begin{cases}
\phi(e_1) = e_2, \phi(e_2) = -e_1, \phi(e_3) = 0, \eta = \frac{4}{3x^2 + 3y^2} (ydx - xdy) + dz, \\
e_3 = \xi = \frac{\partial}{\partial y}, \quad g = \frac{1}{(1 + 2x^2 + 2y^2)^2} (dx \otimes dx + dy \otimes dy) + \eta \otimes \eta, \\
e_1 = (1 + \frac{3x^2 + 3y^2}{4}) \frac{\partial}{\partial x} - y \frac{\partial}{\partial z}, \quad e_2 = (1 + \frac{3x^2 + 3y^2}{4}) \frac{\partial}{\partial y} + x \frac{\partial}{\partial z}.
\end{cases}
\]

We subsequently have

\[
(3.35) \quad [e_1, e_3] = 0 \text{ for } l = 1, 2; \quad [e_1, e_2] = -\frac{3y}{2} e_1 + \frac{3x}{2} e_2 + 2e_3.
\]

\[
(3.36) \quad \begin{cases}
\nabla e_1 e_1 = \frac{3y}{2} e_2, \quad \nabla e_1 e_2 = -\frac{3x}{2} e_2 - e_3, \quad \nabla e_3 e_1 = -e_2, \quad \nabla e_1 e_2 = -\frac{3y}{2} e_1 + e_3, \\
\nabla e_2 e_1 = \frac{3y}{2} e_1, \quad \nabla e_3 e_2 = e_1, \quad \nabla e_1 e_3 = -e_2, \quad \nabla e_2 e_3 = e_1, \quad \nabla e_3 e_3 = 0.
\end{cases}
\]

\[
(3.37) \quad \begin{cases}
R(e_1, e_p)e_1 = 0, \quad l \neq p \neq s, \quad l, p, s = 1, 2, 3, \quad R(e_1, e_3)e_3 = e_1, \quad l = 1, 2, \\
R(e_1, e_3)e_1 = -e_3, \quad l = 1, 2, \quad R(e_1, e_p)e_p = 0, \quad l \neq p, \quad l, p = 1, 2, \\
S^*(e_1, e_p) = 0, \quad l, p = 1, 2, 3.
\end{cases}
\]

Further, by using (3.37) in (1.5), we infer the following system of differential equations:

\[
(3.38) \quad \begin{cases}
\alpha^2 F_{xx} - 2y\alpha F_{xx} + y^2 F_{zz} - \frac{3\alpha}{2} F_y - \frac{3y}{2} F_z - \alpha^2 F_x^2 + 2\frac{\alpha}{m} F_x F_z \\
+ \frac{3\alpha}{2} F_x = \frac{y^2}{m} F_y^2 = \nu, \quad F_{zz} - \frac{1}{m} F_z^2 = \nu, \\
\alpha^2 F_{yy} + 2x\alpha F_{yy} + x^2 F_{zz} - \frac{3\alpha}{2} F_x - \frac{3x}{2} F_y + \frac{3\alpha}{2} F_y - \frac{\alpha}{m} F_y^2 \\
- \frac{2\alpha}{m} F_y F_z - \frac{x}{m} F_z^2 = \nu, \\
\frac{3\alpha}{2} F_y + \alpha^2 F_{xy} - y\alpha F_{xy} + \alpha F_z + x\alpha F_{xz} - yx F_{zz} + \frac{3\alpha}{2} F_x - \frac{3y}{2} F_z - F_z \\
- \frac{x^2}{m} F_x F_y + \frac{3\alpha}{2} F_y F_z - \frac{\alpha}{m} F_z^2 = 0, \quad m = 0, \\
\alpha F_{xx} - y F_{zz} + \alpha F_y + x F_z - \frac{\alpha}{m} F_x F_z + \frac{x}{m} F_z^2 = 0, \\
\alpha F_{yy} - x F_{zz} - \alpha F_x + y F_x - \frac{\alpha}{m} F_y F_z - \frac{x}{m} F_z^2 = 0,
\end{cases}
\]
where \( \alpha = 1 + \frac{3x^2 + 3y^2}{4} \) and indices denote the derivative with respect to \( x, y \) and \( z \).

**Case A:** Assume that the potential function \( F \) depends only on \( x \), so (3.38) reduces to

\[
\nu = 0, \quad F_x = 0.
\]

Therefore, quasi-*Einstein* soliton is steady and \( F \) is constant.

**Case B:** Now, assume that the potential function \( F \) depends only on \( y \), so (3.38) reduces to

\[
\nu = 0, \quad F_y = 0.
\]

That is, quasi-*Einstein* soliton is steady and \( F \) is constant.

**Case C:** Further, assume that the potential function \( F \) depends only on \( z \), so (3.38) reduces to

\[
\nu = 0, \quad F_z = 0.
\]

Which implies quasi-*Einstein* soliton is steady and \( F \) is constant.

### 4 Quasi-*Einstein* \((\kappa < 1, \mu)\) spaces

In this section, we study quasi-*Einstein* \((\kappa < 1, \mu)\)-spaces.

For a \((\kappa, \mu)\)-space [2]

\[
h^2 = -(1 - \kappa)\phi^2,
\]

where \( \kappa \leq 1 \).

**Theorem 4.1.** [3] Let \((M^{2n+1}, \xi, \eta, \phi, g)\) be a non-Sasakian \((\kappa, \mu)\)-space\((\kappa < 1)\). Then its Riemann curvature tensor \( R \) is given by

\[
g(R(Z, V)Y, W) = (1 - \frac{\mu}{2})R_1 + R_2 + \left( \frac{1 - \frac{\mu}{2}}{1 - \kappa} \right) R_3 - \frac{\mu}{2} R_4 + \left( \frac{\kappa - \frac{\mu}{2}}{1 - \kappa} \right) R_5 \\
+ \mu g(\phi Z, V)g(\phi Y, W) + \eta(Z)\eta(W)R_6 - \eta(Z)\eta(Y)R_7 \\
+ \eta(V)\eta(Y)R_8 - \eta(V)\eta(W)R_9,
\]

where

\[
R_1 = g(V, Y)g(Z, W) - g(Z, Y)g(V, W), \\
R_2 = g(V, Y)g(hZ, W) - g(Z, Y)g(hV, W) \\
- g(V, W)g(hZ, Y) + g(Z, W)g(hV, Y), \\
R_3 = g(hV, Y)g(hZ, W) - g(hZ, Y)g(hV, W), \\
R_4 = g(\phi V, Y)g(\phi Z, W) - g(\phi Z, Y)g(\phi V, W), \\
R_5 = g(\phi hV, Y)g(\phi hZ, W) - g(\phi hV, W)g(\phi hZ, Y), \\
R_6 = (-1 + \kappa + \frac{\mu}{2})g(V, Y) + (-1 + \mu)g(hV, Y), \\
R_7 = (-1 + \kappa + \frac{\mu}{2})g(V, W) + (-1 + \mu)g(hV, W), \\
R_8 = (-1 + \kappa + \frac{\mu}{2})g(Z, W) + (-1 + \mu)g(hZ, W), \\
R_9 = (-1 + \kappa + \frac{\mu}{2})g(Z, Y) + (-1 + \mu)g(hZ, Y),
\]
∀ Z, V, Y, W ∈ TM.

Using (1.2) and (4.2) we find that

\[ Q^*Z = (\kappa + n\mu)\phi^2Z, \]

and \( \ast \)-scalar curvature is given by

\[ r^* = -2n(n\mu + \kappa), \]

which is constant.

**Theorem 4.2.** Let \( M^{2n+1} \) be a \( (\kappa < 1, \mu) \)-space satisfying (1.5). Then, \( M \) is a \( \ast \)-flat or locally isometric to \( E^{n+1} \times S^n(4) \) or \( \ast \)Einstein.

**Proof.** From (4.3), we have

\[ Q^*\xi = 0. \]

Differentiating (4.5) along \( Z \in TM \) and using (2.9), we have

\[ (\nabla_Z Q^*)\xi = Q^*\phi Z + Q^*\phi hZ. \]

Taking inner product of (3.3) with \( \xi \) and using (4.5), we obtain

\[ g(R(Z, V)\nabla F, \xi) = g((\nabla_V Q^*)Z - (\nabla_Z Q^*)V, \xi) \]

\[ -\frac{\nu}{m}\{(ZF)\eta(V) - (VF)\eta(Z)\}. \]

Using (2.11) and (4.6) in (4.7) and then replacing \( Z \) by \( \phi Z \) and \( V \) by \( \phi V \), we get

\[ (Q^*\phi + \phi Q^*)Z - hQ^*\phi Z - \phi Q^*hZ = 0, \]

∀ \( Z \in TM \). Further, using (2.1) and (4.3) in (4.8), we find that

\[ \kappa + n\mu = 0. \]

Putting \( V = \xi \) in (4.7), using (2.11), \( \langle (\nabla_Z Q^*)\xi, \xi \rangle = 0 \), \( \langle (\nabla_\xi Q^*)Z, \xi \rangle = 0 \), we get

\[ h\nabla F = \sigma(\nabla F - (\xi F)\xi), \]

where \( \sigma = \frac{\nu - \kappa m}{m\mu} \) is constant. Differentiating (4.10) along \( Z \in TM \) and using (2.9), (3.5) and (4.10), we obtain

\[ (\nabla_Z h)\nabla F - hQ^*Z - \frac{\sigma(\xi F)}{m}(ZF)\xi + \nu hZ \]

\[ = \sigma[\nu Z - Q^*Z - (Z(\xi F))\xi + (\xi F)(\phi Z + \phi hZ)]. \]

Using (4.5) in (3.5), we get

\[ \xi(\xi F) - \frac{1}{m}(\xi F)^2 = \nu. \]
On the other hand, using (2.11), we get \( l = \mu h - \kappa \phi^2 \). Using this and (4.1) in (2.10), we find

\[
\nabla_{\xi} h = -\mu \phi h.
\]

(4.13)

Putting \( \xi \) in place of \( Z \) in (4.11) and using (4.12) and (4.13), we find

\[
\mu h \nabla F = 0.
\]

(4.14)

From (4.14), we have either \( \mu = 0 \) or \( \mu \neq 0 \).

**Case A:** Let \( \mu = 0 \). Then using this in (4.9), we obtain \( \kappa = 0 \). Thus \( R(Z,V)\xi = 0 \) and hence in dimension 3, \( M \) is \( * \)-flat and in higher dimension \( M \) is locally isometric to \( E^{n+1} \times S^n(4) \).[1]

**Case B:** In this case, we have \( h \nabla F = 0 \). Using \( h^2 = (\kappa - 1)\phi^2 \), we get

\[
0 = h^2 \nabla F = (\kappa - 1)\phi^2 \nabla F.
\]

(4.15)

Since \( \kappa < 1 \), we have \( \nabla F = (\xi F)\xi \). Hence from Lemma 3.2, we get \( F \) is constant. Consequently from (3.5), \( Q^* Z = \nu Z \). Hence \( M \) is \( * \)-Einstein. \( \Box \)

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**References**


Results on quasi-*Einstein metric


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