A GENERALIZATION OF THE WEAK AMENABILITY OF BANACH ALGEBRAS

A. BODAGHI¹, M. ESHAGHI GORDJI² and A. R. MEDGHALCHI³

Abstract. Let $A$ be a Banach algebra and let $\varphi$ and $\psi$ be continuous homomorphisms on $A$. We consider the following module actions on $A$,

\begin{align*}
  a \cdot x &= \varphi(a)x, \\
  x \cdot a &= x\psi(a)
\end{align*}

$(a, x \in A)$. We denote by $A_{(\varphi, \psi)}$ the above $A$-module. We call the Banach algebra $A$, $(\varphi, \psi)$-weakly amenable if every derivation from $A$ into $(A_{(\varphi, \psi)})^*$ is inner. In this paper among many other things we investigate the relations between weak amenability and $(\varphi, \psi)$-weak amenability of $A$. Some conditions can be imposed on $A$ such that the $(\varphi'', \psi'')$-weak amenability of $A^{**}$ implies the $(\varphi, \psi)$-weak amenability of $A$.

1. Introduction and preliminaries

Let $A$ be a Banach algebra and let $X$ be a Banach $A$-module. Then a derivation from $A$ into $X$ is a (bounded) linear map $D : A \rightarrow X$ such that for every $a, b \in A$, $D(ab) = D(a) \cdot b + a \cdot D(b)$. If $x \in X$, the map $a \mapsto a \cdot x - x \cdot a$, $(a \in A)$ is a derivation. A derivation of this form is called an inner derivation. The set of all bounded linear operators from $A$ into $X$ is denoted by $B(A, X)$. The set of all derivations from $A$ into $X$ is denoted by $Z^1(A, X)$, and the set of all inner derivations from $A$ into $X$ is denoted by $B^1(A, X)$. Then $H^1(A, X) = \frac{Z^1(A, X)}{B^1(A, X)}$ is the first Hochschild cohomology group of $A$ with coefficients in $X$.

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* Corresponding author.
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Let $A$ be a Banach algebra and $X$ be a Banach $A$-module. Then $X^*$ is the dual of Banach $A$-module $X$, and is also a Banach $A$-module as well, if for each $a \in A$, $x \in X$ and $x^* \in X^*$ we define

$$\langle a \cdot x^*, x \rangle = \langle x^*, x \cdot a \rangle, \quad \langle x^* \cdot a, x \rangle = \langle x^*, a \cdot x \rangle.$$  

A Banach algebra $A$ is amenable if every derivation from $A$ into every dual Banach $A$-module is inner, equivalently if $H^1(A, X^*) = \{0\}$ for every Banach $A$-module $X$, this definition was introduced by Johnson in [12]. $A$ is weakly amenable if $H^1(A, A^*) = \{0\}$; this definition generalizes that introduced by Bade, Curtis and Dales in [1]. We introduce the following new definition of amenability which is related to homomorphisms of Banach algebras.

Let $A$ be a Banach algebra and let $\varphi$ and $\psi$ be continuous homomorphisms on $A$. We consider the following module actions on $A$,

$$a \cdot x := \varphi(a)x, \quad x \cdot a := x\psi(a) \quad (a, x \in A).$$

We denote the above $A$-module by $A_{(\varphi, \psi)}$.

Let $X$ be an $A$-module. A bounded linear mapping $d : A \rightarrow X$ is called a $(\varphi, \psi)$-derivation if

$$d(ab) = d(a) \cdot \varphi(b) + \psi(a) \cdot d(b) \quad (a, b \in A).$$

A bounded linear mapping $d : A \rightarrow X$ is called a $(\varphi, \psi)$-inner derivation if there exists $x \in X$ such that

$$d(a) = x \cdot \varphi(a) - \psi(a) \cdot x \quad (a \in A).$$

Derivations of this form are studied in [14, 15, 16].

**Definition 1.1.** Let $A$ be a Banach algebra and let $\varphi$ and $\psi$ be continuous homomorphisms on $A$. Then $A$ is called $(\varphi, \psi)$-weakly amenable if $H^1(A, (A_{(\varphi, \psi)})^*) = \{0\}$.

Let $A$ and $B$ be Banach algebras. We denote by $Hom(A, B)$ the metric space of all bounded homomorphisms from $A$ into $B$, with the metric derived from the usual linear operator norm $\| \cdot \|$ on $B(A, B)$ and denote $Hom(A, A)$ by $Hom(A)$.

The following assertions hold for any Banach algebra $A$.

(a) If $A$ is amenable then $A$ is an $(\varphi, \psi)$-weakly amenable for each $\varphi$ and $\psi$ in $Hom(A)$.

(b) $A$ is weakly amenable if and only if $A$ is an $(id, id)$-weakly amenable ($id = \text{identity homomorphism}$).

(c) Let $A$ be a commutative weakly amenable Banach algebra. Then $Z^1(A, X) = \{0\}$ for each Banach $A$-module $X$ [3, Theorem 2.8.63]. Therefore $A$ is $(\varphi, \psi)$-weakly amenable for all $\varphi, \psi \in Hom(A)$ if and only if $A$ is commutative and weakly amenable.

**Definition 1.2.** Let $A$ be a Banach algebra, $X$ be a Banach $A$-module and let $\varphi, \psi \in Hom(A)$. A derivation $D : A \rightarrow X$ is called approximately $(\varphi, \psi)$-inner if there exists a net $(x_\alpha)$ in $X$ such that, for all $a \in A$, $D(a) = \lim_\alpha x_\alpha \cdot \varphi(a) - \psi(a) \cdot x_\alpha$ in norm.
Definition 1.3. A Banach algebra $A$ is approximately $(\varphi, \psi)$-weakly amenable if every derivation $D : A \to (A_{(\varphi, \psi)})^*$ is approximately $(\varphi, \psi)$-inner.

Whenever $\varphi = \psi = \text{id}$, this is just the definition of approximate weak amenability developed by Ghahramani and Loy in [9].

Definition 1.4. Let $A$ be an algebra, and let $\varphi, \psi \in \text{Hom}(A)$ (where $\Phi_A$ be the character space of $A$). A linear functional $d$ on $A$ is a point derivation at $\varphi$ if

$$d(ab) = \varphi(a)d(b) + \varphi(b)d(a) \quad (a, b \in A).$$

Throughout this paper $A$ denotes a Banach algebra and $A^{**}$ is the second dual of $A$ equipped with the first Arens product. This product can be characterized as the extension to $A^{**} \times A^{**}$ of the bilinear map $A \times A \to A : (a, b) \to ab$ with the following properties:

i) for fixed $b'' \in A^{**}$, $a'' \mapsto a''b''$ is $w^*$-continuous on $A^{**}$.

ii) for fixed $b \in A$, $a'' \mapsto ba''$ is $w^*$-continuous on $A^{**}$.

The image of $A$ in $A^{**}$ under the canonical embedding is denoted by $\hat{A}$.

In section 2 we prove the main results for this new concept of amenability. In section 3, we develop the relation between the $(\varphi, \psi)$-weak amenability of a Banach algebra $A$ and $A^{**}$. Finally in section 4 we give some examples to show that the new concept of amenability is different from amenability and weak amenability.

2. $(\varphi, \psi)$-WEAK AMENABILITY

Let $A$ be a Banach algebra, and let $A^2 = \text{span}\{ab : a, b \in A\}$.

Proposition 2.1. Let $A$ be Banach algebra and let $\varphi, \psi \in \text{Hom}(A)$ such that $\varphi(a)b = a\psi(b)$ for all $a, b \in A$. If $A$ is $(\varphi, \psi)$-weakly amenable, then $\overline{A^2} = A$, where $\overline{A^2}$ is the closure of $A^2$ in $A$.

Proof. Suppose $\overline{A^2} \neq A$. Take $a_0 \in A \setminus \overline{A^2}$ and $f \in A^*$ such that $f|_{A^2} = 0$ and $\langle f, a_0 \rangle = 1$. Define $d : A \to (A_{(\varphi, \psi)})^*$ by $d(a) = \langle f, a \rangle f$. It is easy check that $d$ is a $(\varphi, \psi)$-derivation. Since $A$ is $(\varphi, \psi)$-weakly amenable, $d$ is $(\varphi, \psi)$-inner, so that there is a $g \in (A_{(\varphi, \psi)})^*$ such that $d(a) = g \cdot \varphi(a) - \psi(a) \cdot g$, for all $a \in A$. So we have $\langle da_0, a_0 \rangle = 1$. On the other hand

$$\langle da_0, a_0 \rangle = \langle g, \varphi(a_0)a_0 \rangle - \langle g, a_0 \psi(a_0) \rangle = 0.$$

This is a contradiction. \[ \square \]

Corollary 2.2. Let $A$ be a Banach algebra. Then $A$ is $(0, 0)$-weakly amenable if and only if $\overline{A^2} = A$.

Proof. Let $A$ be $(0, 0)$-weakly amenable. Then by the above theorem, $\overline{A^2} = A$. For the converse let $d : A \to (A_{(0, 0)})^*$ be a $(0, 0)$-derivation. Then we have $d(A^2) = \{0\}$. Since $d$ is continuous, we have $d = 0$. So $d$ is $(0, 0)$-inner. \[ \square \]

Let $A$ be a weakly amenable Banach algebra or $A$ be a Banach algebra with a bounded left (right) approximate identity. Then $A^2$ is dense in $A$. Thus $A$ is $(0, 0)$-weakly amenable.
Proposition 2.3. Let $A$ be a Banach algebra and $\psi, \varphi$ and $\lambda$ are continuous homomorphisms from $A$ into $A$. If $\varphi$ is an epimorphism and $A$ is $(\psi \circ \varphi, \lambda \circ \varphi)$-weakly amenable, then $A$ is $(\psi, \lambda)$-weakly amenable.

Proof. Let $d : A \rightarrow (A_{(\psi, \lambda)})^*$ be a continuous $(\psi, \lambda)$-derivation, and $D = d \circ \varphi$. We see that $D$ is a $(\psi \circ \varphi, \lambda \circ \varphi)$-derivation. So there exists a $f \in (A_{(\psi \circ \varphi, \lambda \circ \varphi)})^*$ such that for each $a \in A$, $D(a) = f \cdot (\psi \circ \varphi)(a) - (\lambda \circ \varphi)(a) \cdot f$. Let $b \in A$. Then there exists a $a \in A$ such that $\varphi(a) = b$ and so

$$d(b) = d(\varphi(a)) = D(a) = f \cdot \psi(\varphi(a)) - \lambda(\varphi(a)) \cdot f = f \cdot \psi(b) - \lambda(b) \cdot f.$$ 

Thus $d$ is an $(\psi, \lambda)$-inner. 

□

Corollary 2.4. Let $A$ be a Banach algebra and let $\varphi \in \text{Hom}(A)$. If $\varphi$ is an epimorphism and $A$ is $(\varphi^n, \varphi^n)$-weakly amenable for some $n \in \mathbb{N}$. Then $A$ is weakly amenable.

There are Banach algebras which are $(\varphi, \varphi)$-weakly amenable where $\varphi$ is not an epimorphism, and $A$ is weakly amenable. This will be presented in Examples 4.3 and 4.4. The converse of the Corollary 2.4 is true when $\varphi^2 = 1_A$ or $\varphi$ is an epimorphism such that $\varphi^2[A] = 1_A$ where $[A] = \{ab - ba | a, b \in A\}$. In the following theorems and corollaries we prove the above claims.

Theorem 2.5. Let $A$ be a Banach algebra and let $\psi, \lambda, \varphi \in \text{Hom}(A)$ and $\varphi^2 = 1_A$. If $A$ is $(\psi, \lambda)$-weakly amenable, then $A$ is $(\psi \circ \varphi, \lambda \circ \varphi)$-weakly amenable.

Proof. Let $D : A \rightarrow (A_{(\psi \circ \varphi, \lambda \circ \varphi)})^*$ be a $(\psi \circ \varphi, \lambda \circ \varphi)$-derivation and let $d = D \circ \varphi^{-1}$. It can be shown that $d$ is a $(\psi, \lambda)$-derivation. Thus there exist a $f \in (A_{(\psi, \lambda)})^*$ such that for all $a \in A$, $d(a) = f \cdot \psi(a) - \lambda(a) \cdot f$ and so we have $D(a) = D(\varphi^{-1}(\varphi(a))) = d(\varphi(a)) = f \cdot \psi(\varphi(a)) - \lambda(\varphi(a)) \cdot f$, i.e., $D$ is an $(\psi \circ \varphi, \lambda \circ \varphi)$-inner derivation. 

□

Corollary 2.6. If $A$ is weakly amenable and $\varphi \in \text{Hom}(A)$ such that $\varphi^2 = 1_A$, then $A$ is $(\varphi^n, \varphi^n)$-weakly amenable for all $n \in \mathbb{N}$.

Theorem 2.7. Let $\varphi, \psi \in \text{Hom}(A)$ and let $A$ be $(\psi, \psi)$-weakly amenable. If $\varphi|_{[A]} = \text{id}$ and $\varphi$ is an epimorphism, then $A$ is $(\varphi \circ \psi, \varphi \circ \psi)$-weakly amenable.

Proof. Suppose that $D : A \rightarrow (A_{(\varphi \circ \psi, \varphi \circ \psi)})^*$ is an $(\varphi \circ \psi, \varphi \circ \psi)$-derivation. Set $d : A \rightarrow (A_{(\psi, \psi)})^*$ as follows

$$\langle d(a), b \rangle := \langle D(a), \varphi(b) \rangle.$$ 

Then $d$ is a $(\psi, \psi)$-derivation. Thus there exists a $f \in (A_{(\psi, \psi)})^*$ such that for every $a \in A$, $d(a) = f \cdot \psi(a) - \psi(a) \cdot f$. Let $b \in A$. Since $\varphi$ is onto, there exists $b_1 \in A$ such that $b = \varphi(b_1)$. So

$$\langle D(a), b \rangle = \langle d(a), b_1 \rangle$$
$$= \langle f \cdot \psi(a) - \psi(a) \cdot f, b_1 \rangle$$
$$= \langle f, \varphi(\psi(a)b_1 - b_1\psi(a)) \rangle$$
$$= \langle f \cdot \varphi \circ \psi(a) - \varphi \circ \psi(a) \cdot f, b \rangle.$$ 

Therefore $D$ is an $(\varphi \circ \psi, \varphi \circ \psi)$-inner. 

□
Corollary 2.8. Let \( A \) be a Banach algebra and let \( \varphi \in \text{Hom}(A) \). Suppose that \( A \) is weakly amenable and \( \varphi \) is an epimorphism such that \( \varphi|_1 = \text{id} \). Then \( A \) is \((\varphi^n, \varphi^n)\)-weakly amenable for all \( n \in \mathbb{N} \).

**Proposition 2.9.** Let \( A \) be a Banach algebra and \( \varphi \in \text{Hom}(A) \). Suppose that \( A \) is \((\varphi^n, \varphi^n)\)-weakly amenable for all \( n \in \mathbb{N} \), and \( \varphi^n \to 1_A \) in norm. Then \( A \) is approximately weakly amenable.

**Proof.** Let \( D : A \to A^* \) be a derivation. For every \( n \in \mathbb{N} \) set \( D_n : A \to (A_{(\varphi^n, \varphi^n)})^* \), \( D_n(a) = D(\varphi^n(a)) \). It is clear that \( D_n \) is an \((\varphi^n, \varphi^n)\)-derivation. So there exists a sequence \((f_n)\) in \( A^* \) such that \( D_n(a) = f_n \cdot \varphi^n(a) - \varphi^n(a) \cdot f_n \). Since \( \varphi^n(a) \to a, D_n(a) \to D(a) \). Therefore \( D(a) = \lim_n(f_n \cdot a - a \cdot f_n) \).

In the proof of the Proposition 2.9, if the sequence \((f_n)\) has an accumulation point then \( A \) is weakly amenable.

**Theorem 2.10.** Let \( A \) be a Banach algebra, \( \varphi \in \text{Hom}(A) \) and \( 0 \neq \psi \in \Phi_A \). Let \( A \) be \((\varphi, \varphi)\)-weakly amenable and \( \text{Im} \varphi \not\subseteq \ker \psi \). Then there are no non-zero continuous point derivations at \( \psi \circ \varphi \).

**Proof.** Let \( \psi \in \Phi_A \) and let \( \varphi \in \text{Hom}(A) \). Then \( \psi \circ \varphi \in \Phi_A \). Suppose that \( d = d_{\psi \circ \varphi} : A \to \mathbb{C} \) is a point derivation at \( \psi \circ \varphi \). We define \( D : A \to (A_{(\varphi, \varphi)})^* \) by \( D(a) := d(a)\psi \). Then clearly \( D \) is a \((\varphi, \varphi)\)-derivation. Since \( A \) is \((\varphi, \varphi)\)-weakly amenable, there exists a \( f \in (A_{(\varphi, \varphi)})^* \) such that \( D(a) = f \cdot \varphi(a) - \varphi(a) \cdot f \). On the other hand since \( \text{Im} \varphi \not\subseteq \ker \psi \), there exist \( a_1 \in A \) such that \( \psi(\varphi(a_1)) = 1 \). If \( d_{\psi \circ \varphi} \) is a non-zero point derivation, then \( \ker \psi \circ \varphi \not\subseteq \ker d_{\psi \circ \varphi} \). In fact if \( \ker \psi \circ \varphi \subset \ker d_{\psi \circ \varphi} \), then there is an \( \alpha \in \mathbb{C} \) such that \( d_{\psi \circ \varphi} = \alpha(\psi \circ \varphi) \). So

\[
2\alpha = 2\alpha((\psi \circ \varphi)(a_1)) = 2d_{\psi \circ \varphi}(a_1)
\]

\[
= 2d_{\psi \circ \varphi}(a_1)\psi \circ \varphi(a_1)
\]

\[
= d_{\psi \circ \varphi}(a_1)\psi \circ \varphi(a_1) + \psi \circ \varphi(a_1)d_{\psi \circ \varphi}(a_1)
\]

\[
= d_{\psi \circ \varphi}(a_1^2) = \alpha(\psi \circ \varphi)(a_1^2) = \alpha.
\]

Thus \( \alpha = 0 \), i.e. \( d = 0 \) which is a contradiction.

Therefore there exist \( a_2 \in \ker \psi \circ \varphi \) such that \( d(a_2) = 1 \). Put \( a_0 = a_1 + (1 - d(a_1))a_2 \), then

\[
\psi \circ \varphi(a_0) = \psi \circ \varphi(a_1) + (1 - d(a_1))\psi \circ \varphi(a_2) = 1,
\]

and

\[
d(a_0) = d(a_1) + (1 - d(a_1))d(a_2) = d(a_1) + 1 - d(a_1) = 1.
\]

Therefore

\[
1 = d(a_0)\psi(\varphi(a_0))
\]

\[
= \langle D(a_0), \varphi(a_0) \rangle = \langle f \cdot \varphi(a_0) - \varphi(a_0) \cdot f, \varphi(a_0) \rangle
\]

\[
= \langle f, (\varphi(a_0))^2 \rangle - \langle f, (\varphi(a_0))^2 \rangle = 0.
\]

Which is a contradiction. \( \square \)
By using the Theorem 2.10, if \( A \) is a weakly amenable Banach algebra, then there is no non-zero continuous point derivation on \( A \). Therefore Theorem 2.10 could be considered as a generalization of [3, Theorem 2.8.63]. If \( A \) is approximately \((\varphi, \varphi)\)-weakly amenable, then the Theorem 2.10 is also true.

**Theorem 2.11.** Let \( \varphi \in \text{Hom}(A) \) and \( \varphi^2 = \varphi \). Suppose that \( A \) and \( \ker \varphi \) are weakly amenable, \( \text{Im} \varphi \) is an ideal of \( A \). Then \( A \) is \((\varphi, \varphi)\)-weakly amenable.

**Proof.** Let \( D : A \to (A_{\varphi, \varphi})^* \) be a \((\varphi, \varphi)\)-derivation. Take \( d : A \to A^* \) with \( \langle d(a), b \rangle := \langle D(a), \varphi(b) \rangle \), and so \( d \) is a derivation. Then there exists a \( f \in A^* \) such that \( d(a) = f \cdot a - a \cdot f \), \( (a \in A) \).

Since \( \varphi : A \to \text{Im} \varphi \) is a projection, \( A = \text{Im} \varphi \oplus \ker \varphi \) where \( \text{Im} \varphi \) and \( \ker \varphi \) are closed ideals of \( A \). Let \( a \in A \). Then there exist \( a_1, a_2 \in A \) such that \( a = a_1 + a_2 \) where \( a_1 \in \text{Im} \varphi \) and \( a_2 \in \ker \varphi \). Since \( \ker \varphi \) is weakly amenable, \( (\ker \varphi)^2 = \ker \varphi \). So, there is a net \( (t_\alpha s_\alpha) \alpha \subset (\ker \varphi)^2 \) such that \( t_\alpha s_\alpha \to a_2 \), and \( D(a_2) = \lim_\alpha D(t_\alpha s_\alpha) = \lim_\alpha (D(t_\alpha) \cdot \varphi(s_\alpha) - \varphi(t_\alpha) \cdot D(s_\alpha)) = 0 \).

Therefore

\[
D(a) = D(a_1) = D(\varphi(a_1)) = D(\varphi(a)),
\]

so we have

\[
\langle D(a), \varphi(b) \rangle = \langle D(\varphi(a)), \varphi(b) \rangle = \langle d(\varphi(a)), b \rangle \\
= \langle f \cdot \varphi(a) - \varphi(a) \cdot f, b \rangle = \langle f, \varphi(a)b - b\varphi(a) \rangle \\
= \langle d(-b), \varphi(a) \rangle = \langle D(-\varphi(b)), \varphi^2(a) \rangle \\
= \langle d(-\varphi(b)), \varphi(a) \rangle = (f \cdot \varphi(a) - \varphi(a) \cdot f, \varphi(b)).
\]

On the other hand \( b = b_1 + b_2 \) such that \( b_1 \in \text{Im} \varphi, b_2 \in \ker \varphi \) we have \( \varphi(b) = \varphi(b_1) = b_1 \), hence

\[
\langle D(a), b_2 \rangle = \lim_\alpha \lim_\beta \langle D(s_1\beta s_2\beta), t_1\alpha t_2\alpha \rangle \\
= \lim_\alpha \lim_\beta \langle D(s_1\beta) \cdot \varphi(s_2\beta) + \varphi(s_1\beta) \cdot D(s_2\beta), t_1\alpha t_2\alpha \rangle \\
= \lim_\alpha \lim_\beta \langle D(s_1\beta), \varphi(s_2\beta)t_1\alpha t_2\alpha \rangle + \lim_\alpha \lim_\beta \langle D(s_2\beta), t_1\alpha t_2\alpha \varphi(s_1\beta) \rangle \\
= 0,
\]

since \( s_1\beta, s_2\beta \in \ker \varphi. \varphi(s_2\beta)t_1\alpha t_2\alpha, t_1\alpha t_2\alpha \varphi(s_1\beta) \in \text{Im} \varphi \cap \ker \varphi = \{0\} \). Therefore \( A \) is \((\varphi, \varphi)\)-weakly amenable.

\[\square\]

**3. \((\varphi, \psi)\)-Weak Amenability of the Second Dual**

Let \( A \) be a Banach algebra. We consider \( A^{**} \) the second dual of \( A \). It is known that the Banach algebra \( A \) inherits amenability from \( A^{**} \) [11]. No example is yet known whether this fails if one considers the weak amenability instead, but the property is known to hold for the Banach algebras \( A \) which are left ideals in \( A^{**} \) [10], the dual Banach algebras [8], the Banach algebras \( A \) which are Arens regular and every derivation from \( A \) into \( A^* \) is weakly compact [5], Banach algebras for which the second adjoint of each derivation \( D : A \to A^* \) satisfies \( D''(A^{**}) \subseteq \text{WAP}(A) \), and the Banach algebras \( A \) which are right ideals
in $A^{**}$ and satisfy $A^{**}A = A^{**}$ [7]. Now let $A$ be a Banach algebra and let $\varphi, \psi \in \text{Hom}(A)$ such that $\varphi(a)b = a\psi(b)$ for all $a, b \in A$. If $A^{**}$ is $(\varphi'', \psi'')$-weakly amenable, then by Proposition 2.1, $A^{**2} = A^{**}$. Thus we can show that $A^{**}$ is not (id,id)-weakly amenable, then by Proposition 2.1, $A^{**2} = A$ [8, Proposition 2.1]. So by Corollary 2.2, $A$ is $(0, 0)$-weakly amenable. A question remains whether the $(\varphi'', \psi'')$-weak amenability of $A^{**}$ implies the $(\varphi, \psi)$-weak amenability of $A$ and vice versa?

$L^1(\mathbb{R})$ is (id,id)-weakly amenable but $L^1(\mathbb{R})$ is not (id,id)-weakly amenable. In general if $G$ is a nondiscrete locally compact group then $L^1(G)$ is not (id,id)-weakly amenable [4], but $L^1(G)$ is (id,id)-weakly amenable [13].

For an infinite compact metric space $X$, $\text{lip}_\alpha(X)$ is (id,id)-weakly amenable, for $0 < \alpha < 1/2$, but $\text{lip}_\alpha(X)$ is not (id,id)-weakly amenable [1].

**Proposition 3.1.** Let $A$ be Banach algebra and $\varphi \in \text{Hom}(A)$, if $A^{**}$ is $(\varphi'', 0)$-weakly amenable, then $A$ is $(\varphi, 0)$-weakly amenable.

**Proof.** Suppose that $D : A \to (A(\varphi, 0))^*$ is a continuous $(\varphi, 0)$-derivation. Take $a'', b'' \in A^{**}$ and take bounded nets $(a_\alpha)$ and $(b_\beta)$ in $A$ with $\hat{a}_\alpha \to a''$, $\hat{b}_\beta \to b''$ in the $w^*$-topology of $A^{**}$. Then $D'' : A^{**} \to (A(\varphi'', 0))^*$ is an $(\varphi'', 0)$-derivation because

$$D''(a'b'') = w^* - \lim_{\alpha, \beta} D''(\hat{a}_\alpha \hat{b}_\beta) = w^* - \lim_{\alpha, \beta} (D(a_\alpha) \cdot \varphi(b_\beta)) = D''(a'') \cdot \varphi''(b'').$$

Therefore there exists $a'''_0 \in A^{***}$ such that $D''(a'') = a'''_0 \varphi''(a'')$ for all $a'' \in A^{**}$. We obtain $D(a) = a_0 \varphi(a)$ for all $a \in A$, where $a'_0$ is the restriction of $a'''_0$ to $A$. Thus $A$ is an $(\varphi, 0)$-weakly amenable.

If $A^{**}$ is $(0, \psi'')$-weakly amenable, then $A$ is $(0, \psi)$-weakly amenable if and only if $D''(a'b'') = \psi''(a'') \cdot D''(b'')$ if and only if $\psi''(a'') \cdot D''(b'') = w^* - \lim_{\alpha} \psi''(\hat{a}_\alpha) \cdot D''(b'')$ [9]. The last equality is true if $A^{**}$ is a Banach algebra under the second Arens product. Let $A$ be a Banach algebra with a bounded approximate identity, then $A$ is $(\varphi, 0)$ and $(0, \psi)$-weakly amenable (see Example 4.2).

For a Banach algebra $A$, let $A^{op}$ be the Banach algebra with underlying Banach space $A$ and with product $\circ$ given by $a \circ b = ba$. We have the following simple observation.

**Proposition 3.2.** Let $A$ be Banach algebra and $\varphi, \psi \in \text{Hom}(A)$. Then $A$ is $(\varphi, \psi)$-weakly amenable if and only if $A^{op}$ is $(\psi, \varphi)$-weakly amenable.

For a Banach algebra $A$, $A^{**}$ is $(0, 0)$-weakly amenable with the first Arens product if and only if $A^{**}$ is $(0, 0)$-weakly amenable with the second Arens product. We immediately observe that the $(0, 0)$-weak amenability of $A^{**}$ implies that $A$ is $(0, 0)$-weakly amenable. However, the $(0, 0)$-weak amenability of $A$ does not imply that $A^{**}$ is $(0, 0)$-weakly amenable unless every derivation from $A^{**}$ to $A^{***}$ is $w^*$-continuous. Some conditions can be imposed on $A$ such that the $(\varphi'', \psi'')$-weak amenability of $A^{**}$ implies the $(\varphi, \psi)$-weak amenability of $A$ where $\varphi, \psi \neq 0$. 
**Theorem 3.3.** Let $A$ be a Banach algebra and $\varphi, \psi \in \text{Hom}(A)$. Let $A^{**}$ be $(\varphi'', \psi'')$-weakly amenable, and suppose $\hat{A}$ is a left ideal in $A^{**}$. Then $A$ is $(\varphi, \psi)$-weakly amenable.

**Proof.** It is known that $\varphi''(\hat{a}) = \hat{\varphi(a)}$ and $\psi''(\hat{a}) = \hat{\psi(a)}$, for all $a \in A$. The proof of the Theorem is similar to [10, Theorem 2.3].

A Banach algebra $A$ is said to be dual if there is a closed submodule $A_*$ of $A^*$ such that $A = A_*^*$. Let $i : A_* \to A^*$ be the canonical embedding and $i'$ be the adjoint of $i$. If $a \in A$, we have $i'(\hat{a}) = \hat{a}$. Obviously $i$ is norm-continuous, hence $i'$ is $w^*$-continuous. Let $a'', b'' \in A^{**}$ and take nets $(a_\alpha)$ and $(b_\beta)$ in $A$ such that $\alpha A_\alpha \xrightarrow{w^*} a''$ and $\beta b_\beta \xrightarrow{w^*} b''$. Then

$$i'(a''b'') = i'(w^* - \lim \alpha \beta \hat{a}_\alpha \hat{b}_\beta) = w^* - \lim \alpha \beta i'(\hat{a}_\alpha \hat{b}_\beta)$$

$$= w^* - \lim \alpha \beta (\hat{a}_\alpha \hat{b}_\beta) = (w^* - \lim \alpha \hat{a}_\alpha)(w^* - \lim \beta \hat{b}_\beta)$$

$$= i'(w^* - \lim \alpha \hat{a}_\alpha)i'(w^* - \lim \beta \hat{b}_\beta) = i'(a'')i'(b'').$$

Hence $i'$ is an algebra homomorphism from $A^{**}$ onto $A$. Let $\varphi : A \to A$ be a continuous homomorphism. Then the second conjugate $\varphi''$ is $w^*$-continuous and

$$\langle \varphi'(\varphi''(\hat{a})), b \rangle = \langle \varphi''(\varphi''(\hat{a})), b \rangle = \langle \varphi''(i'(\hat{a})), b \rangle.$$  

Hence $\varphi'(\varphi''(\hat{a})) = \varphi''(i'(\hat{a}))$. We know that $\varphi''|_A = \varphi$, if $a'' \in A^{**}$, $(a_\alpha) \subset A$, $\alpha \xrightarrow{w^*} a''$, then

$$\varphi(\varphi'(a'')) = \varphi''(i'(\hat{a})) = \varphi''(w^* - \lim \alpha \hat{a}_\alpha) = w^* - \lim \alpha \varphi''(i'(\hat{a}_\alpha))$$

$$= w^* - \lim \alpha i'(\varphi''(\hat{a}_\alpha)) = i'(\varphi''(w^* - \lim \alpha \hat{a}_\alpha)) = i'(\varphi''(a'')).$$

**Theorem 3.4.** Let $A$ be a dual Banach algebra and let $\varphi, \psi \in \text{Hom}(A)$. If $A^{**}$ is $(\varphi'', \psi'')$-weakly amenable then $A$ is $(\varphi, \psi)$-weakly amenable.

**Proof.** Let $i$ be as above. Suppose that $d : A \to (A(\varphi, \psi))^*$ is an $(\varphi, \psi)$-derivation. Set $D = i'' \circ d \circ i' : A^{**} \to (A^{**}(\varphi'', \psi''))^*$, then for every $a'', b'', c'' \in A^{**}$ we have

$$\langle D(a''b''), c'' \rangle = \langle d(i''(a''))i'(b''), c'' \rangle$$

$$= \langle d(i''(a'')) \cdot \varphi(i'(b'')), c'' \rangle + \langle d(i''(a'')) \cdot \varphi(i'(b'')), c'' \rangle$$

$$= \langle d(i''(a'')) \cdot \varphi''(i'(b''))i'(c''), c'' \rangle + \langle d(i''(a'')) \cdot \varphi''(i'(b''))i'(c''), c'' \rangle$$

$$= \langle (i'' \circ d \circ i'(a'')), \varphi''(b'')c'' \rangle + \langle (i'' \circ d \circ i'(b'')), c'' \varphi''(a'') \rangle$$

$$= \langle D(a'') \cdot \varphi''(b'') + \psi''(a''), b'' \rangle.$$

Therefore $D$ is an $(\varphi'', \psi'')$-derivation. Since $A^{**}$ is $(\varphi'', \psi'')$-weakly amenable, there exists $a'''_0 \in A^{***}$ such that

$$D(a'') = a'''_0 \cdot \varphi''(a'') - \psi''(a'') \cdot a'''_0, \quad a'' \in A^{**}.$$
Now let $R : A^{***} \to A^*$ be the restriction map. Set $a'_0 = R(a''_0)$. For every $a, b \in A$ we have

$$
\langle d(a), b \rangle = \langle d(i'(\widehat{a}), i'(\widehat{b})) = \langle i'' \circ d \circ i'(\widehat{a}), \widehat{b} \rangle = \langle D(\widehat{a}), \widehat{b} \rangle = \langle a''_0 \cdot \varphi''(\widehat{a}), \widehat{b} \rangle - \langle \psi''(\widehat{a}) \cdot a''_0, \widehat{b} \rangle = \langle a''_0, \varphi''(\widehat{a}) \rangle - \langle a''_0, b \varphi''(\widehat{a}) \rangle = \langle a''_0, \varphi(a) \rangle - \langle a''_0, b \varphi(a) \rangle = \langle a_0', \varphi(a) \rangle - \langle a_0', b \varphi(a) \rangle.
$$

So $d(a) = a_0' \cdot \varphi(a) - \psi(a) \cdot a'_0$. Therefore $d$ is an $(\varphi, \psi)$-inner.

\[ \square \]

4. Examples

Example 4.1. Let $A$ be a commutative weakly amenable Banach algebra. A Banach $A$-module $X$ is called symmetric if $a x = x a$, for $a \in A$ and $x \in X$. Then for every symmetric Banach $A$-module $X$ we have $H^1(A, X) = \{0\}$ [1]. On the other hand for every $\varphi \in \text{Hom}(A)$, $(A_{(\varphi, \varphi)})^*$ is a symmetric Banach $A$-module. Thus $A$ is $(\varphi, \varphi)$-weakly amenable.

Example 4.2. Let $A$ be a Banach algebra with a bounded right approximate identity $(e_a)$. Let $D : A \to (A_{(\varphi, \varphi)})^*$ be a derivation. Then for every $a, b \in A$, we have $D(ab) = \psi(a) \cdot D(b)$. Since $D$ is bounded, $(D(e_a))$ is a bounded net in $(A_{(\varphi, \varphi)})^*$. Let $f \in (A_{(\varphi, \varphi)})^*$ be a cluster point of $(D(e_a))$. We can suppose that $w^* - \lim_a D(e_a) = f$ in $(A_{(\varphi, \varphi)})^*$. Then for every $a \in A$, we have $w^* - \lim_a a D(e_a) = af$ in $(A_{(\varphi, \varphi)})^*$. Thus we have

$$
D(a) = \lim_a D(\alpha e_a) = \lim \psi(a) \cdot D(e_a) = \psi(a) \cdot f.
$$

This means that $D$ is $(0, \psi)$-inner. So $A$ is $(0, \psi)$-weakly amenable. Similarly every Banach algebra with a bounded left approximate identity is $(\varphi, 0)$-weakly amenable. So every group algebra and $C^*$-algebra are $(\varphi, 0)$ and $(0, \psi)$-weakly amenable.

Example 4.3. Let $A = l^1(\mathbb{N})$ with the product $\alpha \beta := \alpha(1) \beta(1)$ $(\alpha, \beta \in l^1(\mathbb{N}))$. For every $\varphi, \psi \in \text{Hom}(A)$, $A$ is $(\varphi, \psi)$-weakly amenable [16]. It is easy to check that $A$ does not have a bounded right approximate identity, thus $A$ is not amenable.

Example 4.4. Let $S = \{x_1, x_2, x_3, x_4, x_5\}$ be a semigroup with $x_1^2 = x_1$, $x_1 x_2 = x_2$, $x_3 x_1 = x_3$, $x_3 x_2 = x_4$ and all other products equal to $x_5$. We identify the elements of $S$ with the point masses on $S(\delta_x := x)$. We know that, for any semigroup $S$,

$$
l^1(S) = \left\{ \sum_{s \in S} \alpha_s \delta_s : \sum_{s \in S} | \alpha_s | < \infty, s \in S, \alpha_s \in \mathbb{C} \right\}
$$

is a Banach algebra with the norm $\| \sum_{s \in S} \alpha_s \delta_s \| = \sum_{s \in S} | \alpha_s |$ and the convolution reduced to $\delta_s * \delta_t = \delta_{st}$, for $s, t \in S$ (see [citeDal] for details). In our case

$$
l^1(S) = \left\{ \lambda = \sum_{n=1}^5 \alpha_n x_n; \ \{\alpha_n\}_{n=1}^5 \subset \mathbb{C}, \ \{x_n\}_{n=1}^5 \subset S, \ \| \lambda \| = \sum_{n=1}^5 | \alpha_n | \right\},
$$
and \(l^1(S)\) is weakly amenable [2]. Since \(S\) is not regular semigroup, \(l^1(S)\) is not amenable [6]. Let \(\varphi, \psi : l^1(S) \to l^1(S)\) be continuous homomorphisms and \(D : l^1(S) \to (l^1(S)(\varphi, \psi))^*\) be a \((\varphi, \psi)\)-derivation we show that \(D = 0\). Therefore \(D\) is an \((\varphi, \psi)\)-inner derivation for each \(\varphi, \psi \in Hom(l^1(S))\). If \(x, y \in S\) we show that \(\langle Dx, y \rangle = 0\).

Suppose that \(\varphi(x_j) = \sum_{k=1}^{5} \alpha_{jk}x_k, \psi(x_j) = \sum_{k=1}^{5} \beta_{jk}x_k, (1 \leq j \leq 5, \alpha_{jk}, \beta_{jk} \in \mathbb{C})\). Since \(\varphi(x_1^2) = \varphi(x), \alpha_{11} = \alpha_{11}, \alpha_{11}\alpha_{12} = \alpha_{12}, \alpha_{13}\alpha_{11} = \alpha_{13}, \alpha_{13}\alpha_{12} = \alpha_{14}\).

I) If \(\alpha_{11} = 0\), then \(\alpha_{12} = \alpha_{13} = \alpha_{14} = 0, \alpha_{15}^2 = \alpha_{15}\). It is easy to show that above \((\varphi, \psi)\)-derivation is zero.

II) If \(\alpha_{11} = \beta_{11} = 1\) then

\[
\varphi(x_j) = x_1 + \sum_{k=2}^{5} \alpha_{jk}x_k, \quad \psi(x_j) = x_1 + \sum_{k=2}^{5} \beta_{jk}x_k \quad (2 \leq j \leq 5). \quad (4.1)
\]

We put \(\langle Dx_i, x_j \rangle = t_{ij}, (i, j \in \{1, 2, 3, 4, 5\}, x_i, x_j \in S, t_{ij} \in \mathbb{C})\). Also

\[
\langle Dx_i, x_k \varphi(x_k) + \langle Dx_k, x_j \psi(x_i) \rangle \quad (4.2)
\]

for all \(i, j, k \in \{1, 2, 3, 4, 5\}\).

Since \(x_1^2 = x_1, t_{1j} = \langle Dx_1, \varphi(x_1)x_j \rangle + \langle Dx_1, x_j \psi(x_1) \rangle\). Therefore

\[
t_{14} = t_{15} = (2 + \alpha_{12} + \alpha_{13} + \alpha_{14} + \alpha_{15} + \beta_{12} + \beta_{13} + \beta_{14} + \beta_{15})t_{15}. \quad (4.3)
\]

If

\[
\alpha_{12} + \alpha_{13} + \alpha_{14} + \alpha_{15} + \beta_{12} + \beta_{13} + \beta_{14} + \beta_{15} + 1 = 0 \quad (4.4)
\]

since \(\varphi(x_1^2) = \varphi(x_1)\) and \(\psi(x_1^2) = \psi(x_1)\), we have \(\alpha_{14} = \alpha_{12}\alpha_{13}, \beta_{14} = \beta_{12}\beta_{13}\) and

\[
\sum_{i=2}^{5} \alpha_{i2}^2 + \alpha_{i2}^2 + \alpha_{i2}^2\alpha_{13}^2 + \alpha_{i2}^2 + 2\alpha_{i2}\alpha_{13} + 2\alpha_{i2}\alpha_{12}^2 + 2\alpha_{i2}\alpha_{13}\alpha_{15} + 3\alpha_{i2}\alpha_{13} + 2\alpha_{i2}\alpha_{15} + 2\alpha_{i2}C + \alpha_{i2}C + \alpha_{i2}C + \alpha_{i2}C = 0, \quad (4.5)
\]

From (4.4) and (4.5) the following relation is obtained

\[
\begin{align*}
\alpha_{15} &= -\alpha_{12}\alpha_{13} - \alpha_{12} - \alpha_{13} + 1 \\
\text{or} \quad \alpha_{15} &= -\alpha_{12}\alpha_{13} - \alpha_{12} - \alpha_{13}
\end{align*} \quad (4.6)
\]

and

\[
\begin{align*}
\beta_{15} &= -\beta_{12}\beta_{13} - \beta_{12} - \beta_{13} + 1 \\
\text{or} \quad \beta_{15} &= -\beta_{12}\beta_{13} - \beta_{12} - \beta_{13}
\end{align*} \quad (4.7)
\]

by inserting these solutions in (4.3), we get the contradictions: \(3 = 0, 2 = 0, 1 = 0\). Therefore

\[
t_{14} = t_{15} = 0. \quad (4.8)
\]

Since \(\varphi(x_5^2) = \varphi(x_5)\) and \(\psi(x_5^2) = \psi(x_5)\), similar to the above

\[
t_{54} = t_{55} = 0. \quad (4.9)
\]
From (4.1), (4.2) and (4.8), we deduce that
\[ t_{24} = t_{25} = (1 + \beta_{12} + \beta_{13} + \beta_{14} + \beta_{15})t_{25}. \]

Since \( D_x = D_{x_2}x_1 \),
\[ t_{5j} = \langle D_{x_2}, \varphi(x_1)x_j \rangle + \langle D_{x_1}, x_j \psi(x_2) \rangle. \] (4.10)

From (4.8), (4.9) and (4.10), we have
\[ (1 + \alpha_{12} + \alpha_{13} + \alpha_{14} + \alpha_{15})t_{25} = 0. \]

If \( \beta_{12} + \beta_{13} + \beta_{14} + \beta_{15} = 0 \) and \( 1 + \alpha_{12} + \alpha_{13} + \alpha_{14} + \alpha_{15} = 0 \), from the relations
\( \alpha_{14} = \alpha_{12}\alpha_{13}, \beta_{14} = \beta_{12}\beta_{13} \), (4.6) and (4.7), we conclude the contradictions:
\( 1 = -1 \) and \( 1 = 0 \). Therefore
\[ t_{24} = t_{25} = 0. \] (4.11)

From (4.1), (4.2) and (4.8), we have \( t_{34} = t_{35} = (1 + \alpha_{12} + \alpha_{13} + \alpha_{14} + \alpha_{15})t_{35} \).
Since \( 1 + \alpha_{12} + \alpha_{13} + \alpha_{14} + \alpha_{15} \neq 0 \),
\[ t_{34} = t_{35} = 0. \] (4.12)

Since \( D_x = D_{x_3}x_2 \),
\[ t_{4j} = \langle D_{x_3}, \varphi(x_2)x_j \rangle + \langle D_{x_2}, x_j \psi(x_3) \rangle. \] (4.13)

From (4.1) and (4.13), we have
\[ t_{44} = t_{45} = (1 + \sum_{k=2}^{5} \alpha_{2k})t_{25} + (1 + \sum_{k=2}^{5} \beta_{3k})t_{35}. \] (4.14)

From (4.11), (4.12) and (4.14), we conclude that
\[ t_{44} = t_{45} = 0. \]

If \( t_{11} \neq 0 \), we can conclude that \( \alpha_{43} = 0 \) and \( \alpha_{43} = 2 \), which is a contradiction. Hence \( t_{11} = 0 \). By using of the above relations \( t_{ij} = 0 \) for every \( i \) and \( j \). Therefore
\( D = 0 \).

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**References**


1Science and Research Branch, Islamic Azad University (IAU), Tehran, Iran. E-mail address: abasalt_bodaghi@yahoo.com

2Department of Mathematics, Semnan University, P. O. Box 35195-363, Semnan, Iran. E-mail address: madjid.eshaghi@gmail.com, maj_ess@yahoo.com

3Department of Mathematics, Tarbiat Moallem University, Tehran, Iran. E-mail address: a_medghalchi@saba.tmu.ac.ir