NEW UPPER BOUNDS FOR MATHIEU–TYPE SERIES

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Abstract. The Mathieu’s series $S(r)$ was considered firstly by É.L. Mathieu in 1890, its alternating variant $\tilde{S}(r)$ has been recently introduced by Pogány et al. \cite[Appl. Math. Comput. 173 (2006), 69–108]{Pogány}, where various bounds have been established for $S, \tilde{S}$. In this note we obtain new upper bounds over $S(r), \tilde{S}(r)$ with the help of Hardy–Hilbert double integral inequality.

1. Introduction and preliminaries

The series

$$S(r) = \sum_{n=1}^{\infty} \frac{2n}{(n^2 + r^2)^2}$$

is named after Émile Léonard Mathieu (1835–1890), who investigated it in his 1890 book \cite{Mathieu} written on the elasticity of solid bodies. Bounds for this series are needed for the solution of boundary value problems for the biharmonic equations in a two–dimensional rectangular domain, see \cite[Eq. (54), p. 258]{Pogány}. The alternating version of $S(r)$, that is

$$\tilde{S}(r) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{2n}{(n^2 + r^2)^2}$$

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was introduced following certain Tomovski’s ideas and recently discussed by Pogany et al. in [12]. Applications of alternating Mathieu series \( \tilde{S}(r) \) concerning ODE which solution is the Butzer–Flocke–Hauss Omega function were studied in [3], [11]. The integral representations of \( S(r), \tilde{S}(r) \) [6], [12] respectively, reads as follows:

\[
S(r) = \frac{1}{r} \int_0^\infty \frac{x \sin(rx)}{e^x - 1} \, dx, \quad \tilde{S}(r) = \frac{1}{r} \int_0^\infty \frac{x \sin(rx)}{e^x + 1} \, dx.
\]  

These integral expressions will be the starting points in our study.

2. Results required

Let us consider a Hölder pair \((p, q)\), \( p^{-1} + q^{-1} = 1, p > 1 \), two non-negative functions \( f \in L^p(\mathbb{R}_+) \), \( g \in L^q(\mathbb{R}_+) \), and let us denote \( \| \cdot \|_{L_s(\mathbb{R}_+)} := \| \cdot \|_s \) the usual integral \( L_s \)-norm on the set of positive reals. The celebrated Hardy–Hilbert (or Hilbert) integral inequality [10] reads

\[
\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} \, dx \, dy \leq \frac{\pi}{\sin(\pi/p)} \| f \|_p \| g \|_q. \tag{2.1}
\]

The inequality is strict unless at least one of \( f, g \) is zero, and the constant on the right in (2.1) is the best possible [10].

Consider the scaled parametric integral

\[
I_p = \int_0^\infty \frac{\sin x}{x^p} \, dx, \quad (p > 1).
\]

We point out that in [2, p. 663] the following estimate has been proved:

\[
I_p \leq \frac{\pi}{2} \sqrt{\frac{p}{2}} \quad (p \geq 2).
\]

However, we shall give another estimate over \( I_p \) when \( p > 1 \).

**Lemma 2.1.** For all \( p > 1 \) the following estimate holds

\[
I_p \leq q \tag{2.2}
\]

where \( q \) is the conjugate Hölder pair to \( p \).

**Proof.** Let us write

\[
I_p := \int_0^1 \frac{\sin x}{x^p} \, dx + \int_1^\infty \frac{\sin x}{x^p} \, dx.
\]

Then, by the estimate \( \sin x \leq x, x \in [0, 1] \) and by the redundant \( |\sin x| \leq 1, x > 1 \) respectively, we easily deduce

\[
I_p \leq \int_0^1 \, dx + \int_1^\infty \frac{\, dx}{x^p} = 1 + \frac{1}{p-1} = q.
\]

This finishes the proof of the Lemma. \( \square \)
3. Main results

At first we establish an upper bound for both $S(r), \tilde{S}(r)$ of magnitude $O(r^{-1/2})$.

**Theorem 3.1.** Let $(p, q), p > 1$ be a Hölder pair. Then we have

$$\tilde{S}(r) \leq S(r) \leq \frac{16\sqrt{\pi} q^{1/(2p)} p^{1/(2q)}}{\sqrt{r} \sin^{1/2}(\pi/p)} =: C_p(r).$$

Moreover, the best/sharpest upper bound estimate

$$C_2(r) = \frac{16\sqrt{2\pi}}{\sqrt{r}}$$

is obtained if $p = q = 2$.

**Proof.** It is sufficient to prove the inequality on the left in (3.1) since the right one can be proved similarly. First, we give two elementary inequalities:

$$\frac{x}{e^x + 1} \leq \frac{x}{e^x - 1} \leq \frac{2}{e^{x/2}} \quad (x \geq 0) \quad (3.2)$$

$$\frac{xy(x+y)}{64} \leq \exp \left\{ \frac{x}{4} + \frac{y}{4} + \frac{x+y}{4} \right\} = \exp \left\{ \frac{x+y}{2} \right\} \quad (x, y \geq 0). \quad (3.3)$$

Thus, we have

$$\left( S(r) \right)^2 = \frac{1}{r^2} \int_0^\infty \int_0^\infty \frac{xy \sin(rx) \sin(ry)}{(e^x - 1)(e^y - 1)} \, dx \, dy$$

$$\leq \frac{4}{r^2} \int_0^\infty \int_0^\infty |\sin(rx)\sin(ry)| e^{-(x+y)/2} dx \, dy \quad \text{(by (3.2))}$$

$$\leq \frac{256}{r^2} \int_0^\infty \int_0^\infty \frac{|\sin(rx)\sin(ry)|}{xy(x+y)} \, dx \, dy. \quad \text{(by (3.3))}$$

Taking $f(x) = x^{-1} |\sin(rx)| = g(x)$ we apply the Hardy–Hilbert inequality to the last expression, such that one transforms into

$$\left( S(r) \right)^2 \leq \frac{256\pi}{r^2 \sin(\pi/p)} \left( \int_0^\infty \frac{|\sin(rx)|^p}{x^p} \, dx \right)^{1/p} \left( \int_0^\infty \frac{|\sin(ry)|^q}{y^q} \, dy \right)^{1/q}$$

$$= \frac{256\pi r^{(p-1)/p+(q-1)/q}}{r^2 \sin(\pi/p)} \left( \mathcal{I}_p \right)^{1/p} \left( \mathcal{I}_q \right)^{1/q}$$

$$\leq \frac{256\pi q^{1/p} \cdot p^{1/q}}{r \sin(\pi/p)} \quad \text{(by (2.2))}$$

This is equivalent to the asserted result (3.1), since the termwise comparation of defining formulæ shows that $\tilde{S}(r) \leq S(r)$.

Rewriting (3.1) with the notation $x := 1/p$, we deduce

$$C_{1/(x)}(r) = \frac{16\sqrt{\pi}}{\sqrt{r(1-x)^x}} x^{1-x} \sin(\pi x) \quad (0 < x < 1).$$

An easy computation shows that the denominator is maximal and hence the upper bound constant is minimal if $x = 1/2$, that is, when $p = q = 2$. \qed
Now, we will extend this result, scaling the exponent of $r$ in the upper bound (3.1). The achieved magnitude should be $O(r^{-1/(2p)})$, $p > 1$.

**Theorem 3.2.** Let $(p,q), p > 1$ be a Hölder pair. Then for all $r > 0, v > 1$ we have

$$\tilde{S}(r) \leq S(r) \leq \frac{C(p,v)}{r^{1/(2p)}}$$

(3.4)

where

$$C(p,v) := \frac{2^{(5q+1)/(2q)} \max\{2^{1/(2p)}, 2^{1/(2q)}\} (\pi p)^{1/(2p)} (\Gamma(q)\Gamma(2q))^{1/(2q)}}{q^{3/2}(\sin(\pi/p)(p-1/v)^{1/v}(p-1+1/v)^{1-1/v})^{1/(2p)}}.$$

**Proof.** For a given Hölder pair $(p,q), p > 1$ and for some $r > 0$ consider

$$(S(r))^{2} = \frac{1}{r^{2}} \int_{0}^{\infty} \int_{0}^{\infty} x y \sin(rx) \sin(ry) \left(\frac{e^{x} - 1}{e^{x} - 1}\right) \, dx \, dy$$

$$\quad = \frac{1}{r^{6}} \int_{0}^{\infty} \int_{0}^{\infty} \frac{\sin(x) \sin(y)}{xy(x+y)^{1/p}} \cdot \frac{x^{2}y^{2}(x+y)^{1/p}}{(e^{x/r} - 1)(e^{y/r} - 1)} \, dx \, dy.$$ 

By the Hölder inequality we conclude

$$(S(r))^{2} \leq \frac{1}{r^{6}} \left( \int_{0}^{\infty} \int_{0}^{\infty} |\sin(x) \sin(y)|^{p} \, \frac{x^{p}y^{p}(x+y)}{x^{p}y^{p}(x+y)} \, dx \, dy \right)^{1/p}$$

$$\quad \times \left( \int_{0}^{\infty} \int_{0}^{\infty} \frac{x^{2}y^{2}(x+y)^{q-1}}{(e^{x/r} - 1)^{q}(e^{y/r} - 1)^{q}} \, dx \, dy \right)^{1/q}.$$ 

(3.5)

Choosing this time $w$ as the Hölder conjugate pair to given $v > 1$ and specifying

$$f(x) = g(x) = x^{-p}|\sin(x)|^{p},$$

we evaluate by the Hardy–Hilbert inequality (2.1) the first integral from above:

$$\mathcal{J} = \int_{0}^{\infty} \int_{0}^{\infty} \frac{|\sin(x) \sin(y)|^{p}}{x^{p}y^{p}(x+y)} \, dx \, dy$$

$$\quad \leq \frac{\pi}{\sin(\pi/p)} \left( \int_{0}^{\infty} |\sin(x)|^{pv} \, dx \right)^{1/v} \left( \int_{0}^{\infty} |\sin(y)|^{pw} \, dy \right)^{1/w}.$$ 

(3.6)

Estimating (3.6) by (2.2) we deduce

$$\mathcal{J} \leq \frac{\pi}{\sin(\pi/p)} \frac{pv}{(pv-1)^{1/v}((v-1)v+1)^{1-1/v}}.$$
The second integral in (3.5) we evaluate in the following way:

$$K = \int_0^\infty \int_0^\infty \frac{x^{2q}y^{2q}(x+y)^{q-1}}{(e^{x/r}-1)^q(e^{y/r}-1)^q} \, dx \, dy = r^{5q+1} \int_0^\infty \int_0^\infty \frac{x^{2q}y^{2q}(x+y)^{q-1}}{(e^x-1)^q(e^y-1)^q} \, dx \, dy \leq r^{5q+1} \max\{2, 2^{q-1}\} \int_0^\infty x^{3q-1} \, dx \int_0^\infty y^{2q} \, dy \tag{3.7}$$

where in (3.7) we make use of the estimate (such that follows by (3.2)):

$$\int_0^\infty \frac{x^\alpha}{(e^x-1)^q} \, dx \leq 2\int_0^\infty x^{\alpha-q}e^{-qx/2} \, dx = \frac{2^{\alpha+1}}{q^{\alpha-q+1}} \Gamma(\alpha-q+1),$$

specified for $\alpha = 3q-1, 2q$ respectively. So, the upper bound over $S(r)$ in (3.4) is proved. Repeating the termwise comparation procedure for $S(r), \tilde{S}(r)$, we clearly deduce (3.4). □

4. Discussion

A. In this research note we derive upper bounds for $S(r), \tilde{S}(r)$, such that possess the form

$$S(r) \leq \frac{\Phi(\theta)}{r^\alpha} \quad (\alpha > 0).$$

Here $\Phi(\theta)$ is an absolute constant and $\theta$ denotes the vector of scaling parameters. We obtain our main results (3.1) and (3.4) via the Hardy–Hilbert integral inequality.

At first, we recall some ancestor results such that will be compared to our bounds for small $r$. In [9] Mathieu posed his famous conjecture $S(r) < r^{-2}, r > 0$. The conjecture was proved after more then 60 years by Berg [1] and by Makai [8]. Actually they showed more:

$$\frac{1}{r^2 + 1/2} < S(r) < \frac{1}{r^2} \quad (r > 0).$$

Another proof of this upper bound has been given by van der Corput and Heflinger [4]. Diananda [5] improved Mathieu’s bound to

$$S(r) \leq \frac{1}{r^2} - \frac{1}{(2r^2 + 2r + 1)(8r^2 + 3r + 3)} \quad (r > 0). \tag{4.1}$$

Here has to be mentioned Guo’s bound of magnitude $O(r^{-2}), [7, Eq. (10)]$.

B. We obtain easily an upper bound, such that is superior to Mathieu’s bound $r^{-2}$ for small $r$. Indeed, starting with the integral expressions for $S(r)$ and $\tilde{S}(r)$ in (1.1) we have

$$S(r) \leq \frac{1}{r} \int_0^\infty \frac{x \, dx}{e^x-1} = \frac{\pi^2}{6r} =: S^*(r) \quad \text{and} \quad \tilde{S}(r) \leq \frac{1}{r} \int_0^\infty \frac{x \, dx}{e^x+1} = \frac{\pi^2}{12r}.$$ 

So, when $r \in (0, 6/\pi]$, it follows $S^*(r) \leq r^{-2}$. 
C. Let us denote \( S_1(r) \) and \( S_2(r) \) the upper bounds in Theorems 3.1, 3.2 respectively. Comparing Mathieu’s bound with \( S_1(r) \), solving the equation \( S_1(r) = r^{-2} \) we find that

\[
S_1(r) \leq \frac{1}{r^2} \left( 0 < r \leq \frac{\sin^2(\pi/p)}{4\sqrt{4\pi p^{1/(3q)}q^{1/(3p)}}} := r_1(p) < 1 \right).
\]

Therefore, \( S_1(r) \) is obviously superior to bounds with magnitude \( O(r^{-2}) \), \( r \) small. Similar comparisons involving \( S_2(r) \) and/or Diananda’s (4.1) and Guo’s bounds one leaves to the interested reader. These analyses show that our bounds (3.1), (3.4) mainly improve the earlier ones.

D. Let us compare \( S_1(r) \) and \( S_2(r) \). It is not hard to see that

\[
r_0 := r_0(p,v) = \frac{2^{3q-1}\pi p^{2-q}q^{q-1}(p-1/v)^{1/v}(p-1+1/v)^{1-1/v}q^{-1}}{\sin(\pi/p) \max\{2,2^{q-1}\} \Gamma(q)\Gamma(2q)}
\]

is the unique positive solution of \( S_1(r) = S_2(r) \). Accordingly, it follows that

\[
S_2(r) < S_1(r) \quad (r \in (0, r_0)),
\]

while for \( r > r_0 \) the reversed conclusion holds. We point out that \( r_0 \) can easily skip 1; for instance \( r_0(2,2) = 512 \pi \).

E. Because the alternating Mathieu series has been introduced recently in [12], the here established bounds are unique until now. However, for \( r \) large the bounding inequalities presented also in [12] are sharper than the here presented ones.

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