EXISTENCE OF FIXED POINT BY USING F-CONTRACTION
AND F-SUZUKI CONTRACTION IN MODULAR FUNCTION
SPACES

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ABSTRACT. The purpose of this paper is to study the notions of F-contraction
and F-Suzuki contraction in context of modular function spaces and to prove
some fixed point results. Further we provide some examples to support our
main results.

1. INTRODUCTION

In 1950, Nakano [9] introduced the concept of the modular spaces that was
further generalized and redefined by Musielak and Orlicz [8] in 1959. Modular
function spaces are the generalization of some class of Banach spaces which attracts
many analysts to work in this field. The study of fixed point in modular function
spaces was initiated by Khamsi et al. [7] in 1990. On the basis of their results,
many work has been done in these spaces. Dhompongsa et al. [3] proved that every
$\rho$-contraction $T : C \to F_\rho(C)$ has a fixed point where $\rho$ is a convex function modular
satisfying $\Delta_2$-type condition, $C$ is a nonempty $\rho$-bounded, $\rho$-closed subset of $L_\rho$
and $F_\rho(C)$ is the collection of $\rho$-closed subset of $C$. In 2011, Khamsi and Kozlowski
[6] proved the existence of fixed points of asymptotic pointwise $\rho$-nonexpansive
mappings in modular function spaces.

In 2012, Wardowski [15] introduced a new type of contraction $F : \mathbb{R}^+ \to \mathbb{R}$ called
F-contraction and gave a fixed point result that generalized Banach contraction
Wardowski by applying some weaker conditions on the self map of a complete metric
space and on the mapping F, concerning the contraction defined by Wardowski and
with these weaker conditions, proved a fixed point result for F-Suzuki contraction
which generalizes the result of Wardowski. R. Jain [4] in 2018, proved the existence
of a fixed point for a nondecreasing mapping in partially ordered complete
b-metric space using sequential monotone property of the space. In 2020, R. Jain
[5] introduced the concept of generalized weak contraction mapping in setting of
generating space of b-dislocated metric space endowed with partial order and proved
some fixed-point theorems for the mappings in space satisfying the generalized weak contraction. Recently, Panwar and Pinki [10] transformed M iteration process in CAT(0) spaces to approximate fixed point of generalized \( \alpha \)-nonexpansive mappings.

In our paper, we study the concepts of F-contraction and F-Suzuki contraction in context of modular function spaces and establish some fixed point existence results in these spaces. Further we construct some examples to support our results.

2. Preliminaries

To finish our paper, we collect some basic definitions and important results.

Let \( \Omega \) be a nonempty set and \( \Sigma \) be a nontrivial \( \sigma \)-ring of subsets of \( \Omega \). Let \( \mathcal{P} \) be a nontrivial \( \delta \)-ring of subsets of \( \Omega \) which means that \( \mathcal{P} \) is closed under countable intersection, finite union and differences. Suppose that \( E \cap A \in \mathcal{P} \) for any \( E \in \mathcal{P} \) and \( A \in \Sigma \). Let us assume that there exists an increasing sequence of sets \( K_n \in \mathcal{P} \) such that \( \Omega = \bigcup K_n \). By \( \varepsilon \) we denote the linear space of all simple functions with support from \( \mathcal{P} \). Also \( \mathcal{M}_\infty \) denotes the space of all extended measurable functions, i.e., all functions \( f: \Omega \to [-\infty, \infty] \) such that there exists a sequence

\[
\{g_n\} \subset \varepsilon, \quad |g_n| \leq |f| \text{ and } g_n(w) \to f(w) \text{ for all } w \in \Omega.
\]

We define

\[
\mathcal{M} = \{ f \in \mathcal{M}_\infty : |f(w)| < \infty \text{ a.e.} \}.
\]

Now, we recall the definition of modular function.

**Definition 2.1.** [14] Let \( X \) be a vector space (R or C). A functional \( \rho : \mathcal{M} \to [0, \infty) \) is called a modular if for any arbitrary elements \( f, g \in X \), the following conditions hold:

(i) \( \rho(f) = 0 \iff f = 0 \)

(ii) \( \rho(\alpha f) = \rho(f) \) whenever \( |\alpha| = 1 \)

(iii) \( \rho(\alpha f + \beta g) \leq \rho(f) + \rho(g) \) whenever \( \alpha, \beta \geq 0, \ \alpha + \beta = 1 \).

If we replace (iii) by

(iv) \( \rho(\alpha f + \beta g) \leq \alpha \rho(f) + \beta \rho(g) \) whenever \( \alpha, \beta \geq 0, \ \alpha + \beta = 1 \).

Then modular \( \rho \) is called convex.

**Definition 2.2.** [14] If \( \rho \) is convex modular in \( X \), then the set defined by

\[
L_\rho = \{ f \in \mathcal{M} : \rho(\lambda f) \to 0 \text{ as } \lambda \to 0 \}
\]

is called modular function space.

**Definition 2.3.** [14] Let \( \rho : \mathcal{M}_\infty \to [0, \infty] \) be a nontrivial, convex and even function. Then \( \rho \) is a regular convex function pseudo modular if

(i) \( \rho(0) = 0 \);

(ii) \( \rho \) is monotone, i.e., \( |f(w)| \leq |g(w)| \) for any \( w \in \Omega \) implies \( \rho(f) \leq \rho(g) \), where \( f, g \in \mathcal{M}_\infty \);

(iii) \( \rho \) is orthogonally sub-additive, i.e., \( \rho(f\chi_{A\cup B}) \leq \rho(f\chi_A) + \rho(f\chi_B) \) for any \( A, B \in \sum \) such that \( A \cap B \neq \emptyset, \ f \in \mathcal{M}_\infty \);

(iv) \( \rho \) has Fatou property, i.e., \( |f_n(w)| \uparrow |f(w)| \) for \( w \in \Omega \) implies \( \rho(f_n) \uparrow \rho(f) \), where \( f \in \mathcal{M}_\infty \);

(v) \( \rho \) is order continuous in \( \varepsilon \), i.e., \( g_n \in \varepsilon \) and \( |g_n(w)| \downarrow 0 \) and \( \rho(g_n) \downarrow 0 \).

\( \rho \) is regular convex function modular if \( \rho(f) = 0 \) implies \( f = 0 \) a.e. The class of all nonzero regular convex function modular on \( \Omega \) is denoted by \( \mathcal{R} \).
Definition 2.4. [14] Let $\rho \in \mathfrak{B}$. Then $\rho$ satisfies $\Delta_2$-property if $\rho(2f_n) \to 0$ whenever $\rho(f_n) \to 0$ as $n \to \infty$.

Definition 2.5. [15] Let $F: \mathbb{R}^+ \to \mathbb{R}$ be a mapping satisfying:

(F1) $F$ is strictly increasing, i.e., for all $\alpha, \beta \in \mathbb{R}^+$ such that $\alpha < \beta$, $F(\alpha) < F(\beta)$;

(F2) For each sequence $\{\alpha_n\}_{n \in \mathbb{N}}$ of positive numbers $\lim_{n \to \infty} \alpha_n = 0$ if and only if $\lim_{n \to \infty} F(\alpha_n) = -\infty$;

(F3) There exists $k \in (0, 1)$ such that $\lim_{\alpha \to 0^+} \alpha k F(\alpha) = 0$.

The set of all functions satisfying the conditions (F1)-(F3) is denoted by $\mathcal{F}$.

3. Fixed point result for $F$-contraction

In the beginning of this section, we define $F$-contraction in modular function spaces and then some examples of $F$-contraction are provided. In the end, we prove a theorem for the existence of fixed point for $F$-contraction.

Definition 3.1. Let $\rho \in \mathfrak{B}$. Let $D_\rho$ be a nonempty, $\rho$-closed and $\rho$-bounded subset of $L_\rho$. Then a mapping $T: D_\rho \to D_\rho$ is said to be $F$-contraction if there exists $\tau > 0$ such that for all $f, g \in D_\rho$

$$\rho(Tf - Tg) > 0 \implies \tau + F(\rho(Tf - Tg)) \leq F(\rho(f - g)) \quad (3.1)$$

Now, we provide some examples of $F$-contraction.

Example 3.2. Let $F: \mathbb{R}^+ \to \mathbb{R}$ be defined by $F(\alpha) = \ln \alpha + \sqrt{\alpha}$. It can be easily shown that $F$ satisfies all the conditions of definition 2.5 for any $k \in (0, 1)$. Let $T: D_\rho \to D_\rho$ be a mapping defined as:

$$\frac{\rho(Tf - Tg)}{\rho(f - g)} e^{\sqrt{\rho(Tf - Tg) - \sqrt{\rho(f - g)}}} \leq e^{-\tau}$$

satisfying (3.1) is $F$-contraction.

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$$\frac{\rho(Tf - Tg)\left[1 + (\rho(Tf - Tg))^{-\frac{1}{2}}\right]}{\rho(f - g)\left[1 + (\rho(Tf - Tg))^{-\frac{1}{2}}\right]} \leq e^{-\tau}$$

satisfying (3.1) is $F$-contraction.

Example 3.4. Let $F: \mathbb{R}^+ \to \mathbb{R}$ be defined by $F(\alpha) = \frac{1}{3} \ln \alpha$. It can be easily shown that $F$ satisfies all the conditions of definition 2.5 for any $k \in (0, 1)$. Let $T: D_\rho \to D_\rho$ be a mapping defined as:

$$\frac{\rho(Tf - Tg)}{\rho(f - g)} \leq e^{-3\tau}$$

satisfying (3.1) is $F$-contraction.

Example 3.5. Let $F: \mathbb{R}^+ \to \mathbb{R}$ be defined by $F(\alpha) = \frac{1}{2} \ln \alpha + \alpha$. It can be easily shown that $F$ satisfies all the conditions of definition 2.5 for any $k \in (0, 1)$. Let $T: D_\rho \to D_\rho$ be a mapping defined as:

$$\frac{\rho(Tf - Tg)}{\rho(f - g)} e^{2[\rho(Tf - Tg) - \rho(f - g)]} \leq e^{-2\tau}$$
Theorem 3.6. Let $\rho \in R$ satisfying $\Delta_2$-type condition. If $D_\rho$ is a non-empty, $\rho$-closed and $\rho$-bounded subset of $L_\rho$ and $T : D_\rho \to D_\rho$ is an $F$-contraction then $T$ has a unique fixed point $f^*$ and for every $f_0 \in D_\rho$, the sequence $\{T^n f_0\}_{n \in N}$ converges to $f^*$.

Proof. We define a sequence $\{f_n\}_{n \in N} \subset D_\rho$, $f_{n+1} = T f_n$, $n = 1, 2, 3, ...$

Let $\alpha_n = \rho(f_{n+1} - f_n)$. If there exists $n_0 \in N$ for which $T f_{n_0} = f_{n_0}$, then nothing to prove. Suppose that $f_{n+1} \neq f_n$ for every $n \in N$. Then $\alpha_n > 0$ for all $n \in N$.

$$F(\rho(f_{n+1} - f_n)) = F(\rho(T f_n - T f_{n-1}))$$
$$\leq F(\rho(f_n - f_{n-1})) - \tau$$

or $F(\alpha_n) \leq F(\alpha_{n-1}) - \tau$

$$F(\alpha_n) \leq F(\alpha_{n-1}) - \tau \leq F(\alpha_{n-2}) - 2\tau \leq ... \leq F(\alpha_0) - n\tau \quad (3.2)$$

From inequality $(3.2)$, we get $\lim_{n \to \infty} F(\alpha_n) = -\infty$ that together with (F2) gives

$$\lim_{n \to \infty} \alpha_n = 0 \quad (3.3)$$

From (F3), there exists $k \in (0, 1)$ such that

$$\lim_{n \to \infty} \alpha_n^k F(\alpha_n) = 0 \quad (3.4)$$

By inequality $(3.2)$, the following inequality holds for all $n \in N$

$$\alpha_n^k F(\alpha_n) - \alpha_n^k F(\alpha_0) \leq \alpha_n^k (F(\alpha_0) - n\tau) - \alpha_n^k F(\alpha_0) = -n\alpha_n^k \tau \leq 0 \quad (3.5)$$

Letting $n \to \infty$ in inequality $(3.5)$, and using equations $(3.3)$ and $(3.4)$, we get

$$\lim_{n \to \infty} n\alpha_n^k = 0 \quad (3.6)$$

From equation $(3.6)$, there exists $n_1 \in N$ such that $n\alpha_n^k \leq 1$ for all $n \geq n_1$. Consequently, we have

$$\alpha_n \leq \frac{1}{n^k} \quad \text{for all } n \geq n_1 \quad (3.7)$$

We show that $\{f_n\}_{n \in N}$ is a Cauchy sequence. Consider $p, q \in N$ such that $p > q \geq n_1$.

We get

$$\rho(f_p - f_q) \leq \frac{\omega(p-q)}{p-q} \rho(f_p - f_{p-1}) + \rho(f_{p-1} - f_{p-2}) + ... + \rho(f_{q+1} - f_q)$$
$$\leq \omega(p-q) \rho(f_p - f_{p-1}) + \rho(f_{p-1} - f_{p-2}) + ... + \rho(f_{q+1} - f_q)$$
$$= \omega(p-q) [\alpha_{p-1} + \alpha_{p-2} + ... + \alpha_q]$$
$$= \omega(p-q) \sum_{i=q}^{p-1} \alpha_i < \sum_{i=q}^{\infty} \alpha_i$$
$$\leq \omega(p-q) \sum_{i=q}^{\infty} \frac{1}{n^k}$$.
Since \( \sum_{i=q}^{\infty} \frac{1}{i^2} \) is convergent, so \( \{f_n\}_{n \in \mathbb{N}} \) is a Cauchy sequence. By the completeness of \( D_\rho \), there exists \( f^* \in D_\rho \) such that \( \lim_{n \to \infty} f_n = f^* \).

\[
\rho(T f^* - f^*) = \lim_{n \to \infty} \rho(T f_n - f_n) \\
= \lim_{n \to \infty} \rho(f_{n+1} - f_n) = 0.
\]

This shows that \( f^* \) is the fixed point of \( T \). Now, we show that \( T \) has a unique fixed point. If \( f_1, f_2 \in D_\rho \) such that \( Tf_1 = f_1 \neq Tf_2 = f_2 \),

\[
\tau \leq F(\rho(f_1 - f_2)) - F(\rho(T f_1 - T f_2)) = 0
\]

\( \Rightarrow \tau \leq 0 \)

which contradicts to the fact that \( \tau > 0 \). Hence, \( T \) has a unique fixed point. \( \square \)

**Example 3.7.** Let the real number system \( \mathbb{R} \) be the space modulared as

\[
\rho(f) = |f|
\]

Consider the sequence \( \{S_n\}_{n \in \mathbb{N}} \) as defined below:

\[
\begin{align*}
S_1 &= 1 \\
S_2 &= 1 + 2 \\
&\vdots \\
S_n &= \frac{n(n+1)}{2}, \quad n \in \mathbb{N}
\end{align*}
\]

Let \( D_\rho = \{S_n : n \in \mathbb{N}\} \). Let \( T : D_\rho \to D_\rho \) be a mapping defined as:

\[
\begin{align*}
T(S_n) &= S_{n-1} \text{ for } n > 1 \\
T(S_1) &= S_1.
\end{align*}
\]

Consider the mappings \( F_1(\alpha) = \frac{1}{3} \ln \alpha \), \( F_2(\alpha) = \frac{1}{2} \ln \alpha + \alpha \) and \( F_3(\alpha) = \ln \alpha + \sqrt{\alpha} \).

Let us first consider \( F_1 \) defined in example 3.4, we have

\[
\lim_{n \to \infty} \frac{\rho(T S_n - T S_1)}{\rho(S_n - S_1)} = \lim_{n \to \infty} \frac{S_{n-1} - S_1}{S_n - S_1} = 1,
\]

which is a contradiction. So, \( T \) is not \( F_1 \)-contraction.

Now, we take \( F_2 \) defined in example 3.5, we observe that \( T \) is \( F_2 \)-contraction having \( \tau = 1 \). For all \( m, n \in \mathbb{N} \)

\[
T(S_n) \neq T(S_m) \iff m > 2 \text{ and } n = 1 \text{ or } m > n > 1.
\]

For all \( m > 2, m \in \mathbb{N} \) and \( n=1 \), we get

\[
\frac{\rho(T(S_m) - T(S_1))}{\rho(S_m - S_1)} e^{2[\rho(T(S_m) - T(S_1)) - \rho(S_m - S_1)]} = \frac{S_{m-1} - S_1}{S_m - S_1} e^{2[(S_{m-1} - S_1) - (S_m - S_1)]}
\]

\[
= \frac{m^2 - m - 2}{m^2 + m} e^{-2m} < e^{-2m} < e^{-2}
\]
For all $m, n \in \mathbb{N}, m > n > 1$, we have
\[
\frac{\rho(T(S_m) - T(S_n))}{\rho(S_m - S_n)} e^{2\rho(T(S_m) - T(S_n)) - \rho(S_m - S_n)} = \frac{S_{m-1} - S_{n-1}}{S_m - S_n} e^{2[(S_{m-1} - S_{n-1}) - (S_m - S_n)]} = \frac{m + n - 1}{m + n + 1} e^{2(n-m)} < e^{2(n-m)} \leq e^{-2}
\]

Now, taking $F_3$ defined in example 3.2, we observe that $T$ is $F_3$-contraction having $\tau = 0.37184$. For all $m, n \in \mathbb{N}$

$T(S_n) \neq T(S_m) \Leftrightarrow m > 2$ and $n = 1$ or $m > n > 1$.

For all $m > 2, m \in \mathbb{N}$ and $n = 1$, we get
\[
\frac{\rho(T(S_m) - T(S_1))}{\rho(S_m - S_1)} e^{\sqrt{\rho(T(S_m) - T(S_1))} - \rho(T(S_1))} = \frac{S_{m-1} - S_1}{S_m - S_1} e^{\sqrt{S_{m-1} - S_1} - \sqrt{S_m - S_1}} = \frac{m^2 - m - 2}{m^2 + m - 2} e^{\sqrt{\frac{m^2 - m - 2}{2} - \sqrt{\frac{m^2 + m - 2}{2}}}} \leq e^{\sqrt{\frac{m^2 - m - 2}{2} - \sqrt{\frac{m^2 + m - 2}{2}}}} \leq e^{\sqrt{2 - \sqrt{3}}} = e^{-0.82185},
\]

if we take $m=3$.

For all $m, n \in \mathbb{N}, m > n > 1$, we obtain the following calculation
\[
\frac{\rho(T(S_m) - T(S_n))}{\rho(S_m - S_n)} e^{\sqrt{\rho(T(S_m) - T(S_n))} - \rho(T(S_n))} = \frac{S_{m-1} - S_{n-1}}{S_m - S_n} e^{\sqrt{S_{m-1} - S_{n-1}} - \sqrt{S_m - S_n}} = \frac{m + n - 1}{m + n + 1} e^{\sqrt{\frac{(m-n)(m+n-1)}{2} - \sqrt{\frac{(m-n)(m+n+1)}{2}}}} \leq e^{\sqrt{\frac{(m-n)(m+n-1)}{2} - \sqrt{\frac{(m-n)(m+n+1)}{2}}}} \leq e^{\sqrt{2 - \sqrt{3}}} = e^{-0.37184},
\]

if we take $m=3, n=2$.

From this example, we conclude that $T$ is not $F_1$-contraction while it is $F_2$ and $F_3$-contraction. In the following table, we compare Banach contraction with $F$-contraction. The generated iteration start from a point $f_0 = S_{31} = 496$ and $C_F(S_n, S_1)$ denotes $F(\rho(S_n - S_1)) - F(\rho(T(S_n) - T(S_1)))$. From the table [3,7] we conclude that $S_1 = 1$ is the fixed point of $T$. 
4. Fixed point result for F-Suzuki contraction

In 2013, Secelean [12] proved the following lemma.

**Lemma 4.1.** Let $F : \mathbb{R}_+ \to \mathbb{R}$ be an increasing mapping and $\{\alpha_n\}_{n=1}^\infty$ be a sequence of positive real numbers. Then the following assertion.

1. (a) if $\lim_{n \to \infty} F(\alpha_n) = -\infty$, then $\lim_{n \to \infty} \alpha_n = 0$;
2. (b) if $\inf F = -\infty$, then $\lim_{n \to \infty} F(\alpha_n) = -\infty$.

The condition (F2) in definition 2.5 is replaced by Secelean [12] by an equivalent but a more simple condition with the help of lemma 4.1.

(F2') $\inf F = -\infty$

or also by

(F2'') there exists a sequence $\{\alpha_n\}_{n=1}^\infty$ of positive real numbers such that

$$\lim_{n \to \infty} F(\alpha_n) = -\infty.$$
The condition (F3) in definition 2.5 is replaced by Piri and Kumam [11] with the following condition:

\((F3')\) F is continuous on \((0, \infty)\).

The set of all functions satisfying the condition (F1), \((F2')\) and \((F3')\) is denoted by \(\mathcal{F}\).

**Example 4.2.** [11] Let \(F_1(\alpha) = -\frac{1}{\alpha}, F_2(\alpha) = -\frac{1}{\alpha} + \alpha, F_3(\alpha) = \frac{1}{1+\alpha},\)
\(F_4(\alpha) = \frac{1}{e\alpha - e\alpha}. \) Then \(F_1, F_2, F_3, F_4 \in \mathcal{F}\).

**Remark.** The condition (F3) and \((F3')\) are independent of each other. For example, \(F(\alpha) = -\frac{1}{\alpha}\) satisfies the conditions (F1), (F2) and \((F3')\) but it does not satisfy (F3). Therefore, \(\mathcal{F} \not\subseteq \mathcal{F}\). Also, \(F(\alpha) = -\frac{1}{\sqrt{\alpha+|\alpha|}}\), where \(|\alpha|\) denotes the integral part of \(\alpha\), satisfies conditions (F1), (F2) and (F3) for any \(k \in (\frac{1}{2}, 1)\) but it does not satisfy \((F3')\). Therefore, \(\mathcal{F} \not\subseteq \mathcal{F}\). But if we take \(F(\alpha) = \frac{1}{3}\ln \alpha\), then it satisfies conditions of both \(\mathcal{F}\) and \(\mathcal{F}\) and hence, \(F \in \mathcal{F} \cap \mathcal{F}\).

**Definition 4.3.** Let \(\rho \in \mathcal{R}\) and satisfy \(\Delta_2\)-condition. Then the growth function \(\omega : [0, \infty) \to [0, \infty)\) is defined as:

\[
\omega = \sup \left\{ \frac{\rho(tx)}{\rho(x)} : 0 < \rho(x) < \infty \right\}.
\]

Then, \(1 < \omega(2)\). In addition \(\rho(tx) \leq \omega(t)\rho(t), \forall t \geq 0, \forall x \in X_\rho\) and also that, for each positive integer \(l\) and for arbitrary \(x_1, x_2, ..., x_l \in X_\rho\)

\[
\rho(x_1 + x_2 + ... + x_l) \leq \omega(l) \frac{\rho(x_1) + \rho(x_2) + ... + \rho(x_l)}{l}.
\]

In 2008, Suzuki [13] introduced the condition (C). Motivated by his work, we transform this condition to modular structure resulting in the modular-(\(C_\rho\)) condition as follows:

**Definition 4.4.** Let \(\rho \in \mathcal{R}\). Assume that \(\rho\) satisfies \(\Delta_2\)-type condition and \(D_\rho\) be a nonempty subset of \(L_\rho\). A mapping \(T : D_\rho \to D_\rho\) is said to satisfy condition \((C_\rho)\) if

\[
\frac{1}{\omega(2)}\rho(f - Tf) \leq \rho(f - g) \implies \rho(Tf - Tg) \leq \rho(f - g), \forall f, g \in D_\rho.
\]

**Definition 4.5.** Let \(\rho \in \mathcal{R}\). Assume that \(\rho\) satisfies \(\Delta_2\)-type condition and \(D_\rho\) be a nonempty subset of \(L_\rho\). A mapping \(T : D_\rho \to D_\rho\) is said F-Suzuki contraction if there exists \(\tau > 0\) such that for all \(f, g \in D_\rho\) with \(Tf \neq Tg\)

\[
\frac{1}{\omega(2)}\rho(f - Tf) \leq \rho(f - g) \implies \tau + F(\rho(Tf - Tg)) \leq F(\rho(f - g)), \quad (4.1)
\]

where \(F \in \mathcal{F}\).

**Theorem 4.6.** Let \(\rho \in \mathcal{R}\). Assume that \(\rho\) satisfies \(\Delta_2\)-type condition and \(D_\rho\) be a nonempty bounded, closed subset of \(L_\rho\) and \(T : D_\rho \to D_\rho\) be an F-Suzuki contraction. Then \(T\) has a unique fixed point \(\mathcal{F} \in D_\rho\) and for every \(f_0 \in D_\rho\), the sequence \(\{T^n f_0\}\) converges to \(\mathcal{F}\).
Proof. We define a sequence \( \{f_n\}_{n \in \mathbb{N}} \subset D_\rho, f_n = T^{n-1}f_0, n \in \mathbb{N} \). If there exists \( n_0 \in \mathbb{N} \) for which \( T^{n_0}f_0 = f_{n_0} \), then nothing to prove. Suppose that \( f_{n+1} \neq f_n \) for every \( n \in \mathbb{N} \). As \( \rho(f_n - T^{n}f_n) > 0 \) for all \( n \in \mathbb{N} \), therefore

\[
\frac{1}{\omega(2)} \rho(f_n - T^{n}f_n) < \rho(f_n - T^{n}f_n), \forall n \in \mathbb{N} \quad (4.2)
\]

For any \( n \in \mathbb{N} \)

\[
F(\rho(f_{n+1} - T^{n+1}f_n)) = F(\rho(T^{n+1}f_n - T^{n+1}f_{n+1})) \\
\leq F(\rho(f_{n+1} - T^{n+1}f_{n+1})) - \tau.
\]

Continuing this process, we get

\[
F(\rho(f_n - T^{n}f_n)) \leq F(\rho(f_{n-1} - T^{n-1}f_{n-1})) - \tau \\
\leq F(\rho(f_{n-2} - T^{n-2}f_{n-2})) - 2\tau \\
dots \\
\leq F(\rho(f_0 - T^{0}f_0)) - n\tau \quad (4.3)
\]

From inequality \(4.3\), we get \( \lim_{n \to \infty} F(\rho(f_n - T^{n}f_n)) = -\infty \) that together with \((F2')\) gives

\[
\lim_{n \to \infty} \rho(f_n - T^{n}f_n) = 0 \quad (4.4)
\]

Now, we show that \( \{f_n\} \) is a Cauchy sequence. By contradiction, we suppose that there exists \( \epsilon > 0 \) and the sequences \( \{u(n)\}_{n=1}^{\infty} \) and \( \{v(n)\}_{n=1}^{\infty} \) of natural numbers such that

\[
u(n) > u(n) > v(n) > n, \rho(f_{u(n)} - f_{v(n)}) \geq \epsilon, \rho(f_{u(n)} - f_{v(n)}) < \frac{\epsilon}{\omega(2)}, \forall n \in \mathbb{N} \quad (4.5)
\]

so we have

\[
\epsilon \leq \rho(f_{u(n)} - f_{v(n)}) \\
\leq \omega(2)[\rho(f_{u(n)} - f_{u(n)-1}) + \rho(f_{u(n)-1} - f_{v(n)})] \\
\leq \omega(2)[\rho(f_{u(n)} - f_{u(n)-1}) + \frac{\epsilon}{\omega(2)}]
\]

Using equation \(4.4\) and above inequality, we get

\[
\lim_{n \to \infty} \rho(f_{u(n)} - f_{v(n)}) = \epsilon. \quad (4.6)
\]

From equation \(4.4\) and inequality \(4.5\), we can choose a positive integer \( n_1 \in \mathbb{N} \) such that

\[
\frac{1}{\omega(2)} \rho(f_{u(n)} - T^{n}f_{u(n)}) < \frac{\epsilon}{\omega(2)} \\
\leq \rho(f_{u(n)} - f_{v(n)}), \forall n \geq n_1.
\]

So, by definition of F-Suzuki contraction

\[
\tau + F(\rho(T^{n}f_{u(n)} - T^{n}f_{v(n)})) \leq F(\rho(f_{u(n)} - f_{v(n)})), \forall n \geq n_1 \\
\tau + F(\rho(f_{u(n)} - f_{v(n)})) \leq F(\rho(f_{u(n)} - f_{v(n)})), \forall n \geq n_1 \quad (4.7)
\]

From \((F3')\), inequalities \(4.6\) and \(4.7\), we get

\[
\tau + F(\epsilon) \leq F(\epsilon),
\]
which is a contradiction. Therefore, \( \{f_n\} \) is a Cauchy sequence. Since \( D_n \) is complete, so there exists \( \bar{f} \in D_p \) such that
\[
\lim_{n \to \infty} \rho(f_n - \bar{f}) = 0 \tag{4.8}
\]

We claim that
\[
\frac{1}{\omega(2)} \rho(f_n - T f_n) < \rho(f_n - \bar{f}) \quad \text{or} \quad \frac{1}{\omega(2)} \rho(T f_n - T^2 f_n) < \rho(T f_n - \bar{f}), \forall n \in \mathbb{N} \tag{4.9}
\]

But we suppose that there exists \( m \in \mathbb{N} \) such that
\[
\frac{1}{\omega(2)} \rho(f_m - T f_m) \geq \rho(f_m - \bar{f}) \quad \text{or} \quad \frac{1}{\omega(2)} \rho(T f_m - T^2 f_m) \geq \rho(T f_m - \bar{f}), \forall m \in \mathbb{N}.
\]

From first part of inequality (4.10),
\[
\rho(f_m - \bar{f}) \leq \frac{1}{\omega(2)} \rho(f_m - T f_m)
\]
\[
\leq \frac{1}{\omega(2)} \omega(2) \left[ \rho(f_m - \bar{f}) + \rho(\bar{f} - T f_m) \right]
\]
\[
\leq \frac{1}{2} \left[ \rho(f_m - \bar{f}) + \rho(\bar{f} - T f_m) \right]
\]
\[
\rho(f_m - \bar{f}) \leq \rho(\bar{f} - T f_m) \tag{4.11}
\]

From inequalities (4.10) and (4.11), we obtain
\[
\rho(f_m - \bar{f}) \leq \rho(\bar{f} - T f_m) \leq \frac{1}{\omega(2)} \rho(T f_m - T^2 f_m) \tag{4.12}
\]

Since, \( \frac{1}{\omega(2)} \rho(f_m - T f_m) < \rho(f_m - T f_m) \), therefore by definition 4.6,
\[
\tau + F(\rho(T f_m - T^2 f_m)) \leq F(\rho(f_m - T f_m))
\]

Since \( \tau > 0 \), \( F(\rho(T f_m - T^2 f_m)) < F(\rho(f_m - T f_m)) \). Using (F1), we get
\[
\rho(T f_m - T^2 f_m) < \rho(f_m - T f_m) \tag{4.13}
\]

From inequalities (4.10), (4.12) and (4.13), we get
\[
\rho(T f_m - T^2 f_m) < \rho(f_m - T f_m)
\]
\[
\leq \frac{\omega(2)}{2} \left[ \rho(f_m - \bar{f}) + \rho(\bar{f} - T f_m) \right]
\]
\[
\leq \frac{\omega(2)}{2} \left[ \frac{1}{\omega(2)} \rho(T f_m - T^2 f_m) + \frac{1}{\omega(2)} \rho(T f_m - T^2 f_m) \right]
\]
\[
= \rho(T f_m - T^2 f_m),
\]

which is a contradiction. Hence, the inequality (4.9) holds. So, from inequality (4.9) for all \( n \in \mathbb{N} \), we get
\[
either \tau + F(\rho(T f_n - T \bar{f})) \leq F(\rho(f_n - \bar{f}))
\]
\[
or \tau + F(\rho(T^2 f_n - T \bar{f})) \leq F(\rho(T f_n - \bar{f}))
\]
\[
or \tau + F(\rho(f_{n+2} - T \bar{f})) \leq F(\rho(f_{n+1} - \bar{f})).
\]

In first case, from inequality (4.9), (F2') and lemma [4.1], we obtain
\[
\lim_{n \to \infty} F(\rho(T f_n - T \bar{f})) = -\infty.
\]
From (F2') and lemma 4.1, \( \lim_{n \to \infty} \rho(T f_n - T \overline{T}) = 0 \), therefore
\[
\rho(\overline{T} - T \overline{T}) = \lim_{n \to \infty} \rho(f_{n+1} - T \overline{T}) = \lim_{n \to \infty} \rho(T f_n - T \overline{T}) = 0.
\]

In second case, from inequality (4.9), (F2') and lemma 4.1, we get
\[
\lim_{n \to \infty} F(\rho(T^2 f_n - T \overline{T})) = -\infty.
\]

From (F2') and lemma 4.1, \( \lim_{n \to \infty} \rho(T^2 f_n - T \overline{T}) = 0 \), therefore
\[
\rho(f - T f) = \lim_{n \to \infty} \rho(f_n + 1 - T f) = \lim_{n \to \infty} \rho(T^2 f_n - T \overline{T}) = 0.
\]

Hence, \( \overline{T} \) is a fixed point of \( T \). Now, we show that \( T \) has atmost one fixed point. If \( f_1, f_2 \in D_\rho \) such that \( Tf_1 = f_1 \neq f_2 = Tf_2 \), therefore \( \rho(T f_1 - T f_2) > 0 \), then we have
\[
\frac{1}{\omega(2)} \rho(f_1 - T f_1) < \rho(f_1 - f_2),
\]
therefore, \( \tau \leq F(\rho(f_1 - f_2)) - F(\rho(T f_1 - T f_2)) = 0 \) which implies that \( \tau \leq 0 \), which contradicts to the fact that \( \tau > 0 \). This shows that \( T \) has a unique fixed point. □

**Example 4.7.** Let the real number system \( \mathbb{R} \) be the space modulared as
\[
\rho(f) = |f|.
\]
The corresponding growth function \( \omega(t) = t, \forall t \geq 0 \). Consider the sequence \( \{S_n\}_{n \in \mathbb{N}} \) as defined below:
\[
\begin{align*}
S_1 &= 1^2 \\
S_2 &= 1^2 + 2^2 \\
&\vdots \\
S_n &= n(n+1)(2n+1)/6, \quad n \in \mathbb{N}
\end{align*}
\]
Let \( D_\rho = \{S_n : n \in \mathbb{N}\} \). Let \( T : D_\rho \to D_\rho \) be a mapping defined as:
\[
T(S_n) = S_{n-1} \text{ for } n > 1 \text{ and } T(S_1) = S_1.
\]
Since
\[
\lim_{n \to \infty} \rho(T S_n - T S_1) = \lim_{n \to \infty} \rho(S_{n-1} - S_1)/\rho(S_n - S_1) = 1
\]
\( T \) is neither Banach contraction nor Suzuki contraction. Taking \( F(\alpha) = -\frac{1}{\alpha} + \alpha \in \mathbb{R} \), we observe that \( T \) is an F-Suzuki contraction with \( \tau = 4 \). To see this, let us consider the following calculations. We observe that
\[
\frac{1}{\omega(2)} \rho(S_n - T S_n) < \rho(S_n - S_m) \iff [(1 = n < m) \lor (1 \leq m < n) \lor (1 < m < n)].
\]
For $1 = n < m$, we get
\[
| TS_m - TS_1 | = | S_{m-1} - S_1 |
= 2^2 + 3^2 + \ldots + (m - 1)^2
\]
\[
| S_m - S_1 | = 2^2 + 3^2 + \ldots + m^2
\]
Since $m > 1$ and
\[
4 - \frac{1}{2^2 + 3^2 + \ldots + (m - 1)^2} < \frac{1}{2^2 + 3^2 + \ldots + m^2}
\]
\[
4 - \frac{1}{2^2 + 3^2 + \ldots + (m - 1)^2} + [2^2 + 3^2 + \ldots + (m - 1)^2] < 4 - \frac{1}{2^2 + 3^2 + \ldots + m^2}
\]
\[
4 - \frac{1}{2^2 + 3^2 + \ldots + (m - 1)^2} + [2^2 + 3^2 + \ldots + (m - 1)^2] < 4 - \frac{1}{2^2 + 3^2 + \ldots + m^2}
\]
for $1 \leq m < n$, similar to $1 = n < m$.
And now, for $1 < m < n$, we have
\[
| TS_m - TS_n | = | S_{m-1} - S_{n-1} |
= n^2 + (n + 1)^2 + \ldots + (m - 1)^2
\]
\[
| S_m - S_n | = | S_m - S_n |
= (n + 1)^2 + (n + 1)^2 + \ldots + m^2.
\]
Since $m > 1$ and
\[
4 - \frac{1}{n^2 + (n + 1)^2 + \ldots + (m - 1)^2} < \frac{1}{(n + 1)^2 + (n + 2)^2 + \ldots + m^2}
\]
\[
4 - \frac{1}{n^2 + (n + 1)^2 + \ldots + (m - 1)^2} < 4 - \frac{1}{2^2 + 3^2 + \ldots + m^2}
\]
\[
4 - \frac{1}{n^2 + (n + 1)^2 + \ldots + (m - 1)^2} + [2^2 + (n + 1)^2 + \ldots + (m - 1)^2]
\]
\[
< 4 - \frac{1}{(n + 1)^2 + (n + 2)^2 + \ldots + m^2} + [2^2 + (n + 1)^2 + \ldots + (m - 1)^2]
\]
\[
4 - \frac{1}{n^2 + (n + 1)^2 + \ldots + (m - 1)^2} + [2^2 + (n + 1)^2 + \ldots + (m - 1)^2]
\]
\[
< \frac{1}{(n + 1)^2 + (n + 2)^2 + \ldots + m^2} + [(n + 1)^2 + (n + 2)^2 + \ldots + (m - 1)^2]
\]
\[
4 - \frac{1}{| TS_m - TS_n |} + | TS_m - TS_n | < -\frac{1}{| S_m - S_n |} + | S_m - S_n |
\]
Therefore, $\tau + F(\rho(TS_m - TS_n)) \leq F(\rho(S_m - S_n))$, for all $m, n \in \mathbb{N}$. Hence $T$ is an $F$-Suzuki contraction. The following table shows the comparison of Banach contraction with $F$-contraction for $F_1(\alpha) = \ln \alpha, F_2(\alpha) = \ln \alpha + \sqrt{\alpha}, F_3(\alpha) = -\frac{1}{\sqrt{\alpha}} + \ldots$
\[ \beta = \frac{1}{\sqrt[2]{\alpha + |\alpha|}} \quad \text{where} \quad F_1, F_2 \in \mathbb{F} \cap \Phi, F_3 \in \Phi - F \quad \text{and} \quad F_3 \in F - \Phi. \]

The generated iteration starts from a point \( f_0 = S_{29} = 8555 \) and \( C_F(S_n, S_1) \) denotes \( F(\rho(S_n - S_1)) - F(\rho(T(S_n) - T(S_1))) \). From the table, we conclude that \( S_1 = 1 \) is the fixed point of \( T \).

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