INFINITE PRODUCTS, SERIES WITH LOGARITHMS, AND SERIES WITH ZETA VALUES

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Abstract. In this note, we point out an interesting connection between series with zeta values, series with logarithm values, and certain infinite products. Using this connection, we give a closed-form evaluation of various series with zeta values in the coefficients.

1. Introduction

In [3] the author studied the special constant
\[ M = \int_0^1 \frac{\psi(t + 1) + \gamma}{t} \, dt \approx 1.257746 \]  
(1.1)
and proved, among other things, the identity [7, p.142].
\[ M = \sum_{n=1}^{\infty} \frac{1}{n} \ln \left(1 + \frac{1}{n}\right) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} \zeta(n+1)}{n} \]  
(1.2)
where \( \psi(s) = \frac{d}{ds} \ln \Gamma(s) \) is the digamma function and \( \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \) (\( \text{Re} s > 1 \)) is Riemann’s zeta function.

In this note, we will extend equation (1.2) to the identity with parameters
\[ \sum_{n=1}^{\infty} \frac{1}{n^a} \ln \left(1 + \frac{\lambda}{n}\right) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} \lambda^{n-1} \zeta(nz + a)}{n} \]  
(1.3)
and provide several explicit evaluations of such series.

When \( \lambda = z = a = 1 \) equation (1.3) turns into (1.2).

The results in this paper complement those in [4].

2. Results and proofs

We start by considering series of the form
\[ \sum_{p=1}^{\infty} \frac{1}{p^a (\lambda + p^z)^b}, \quad \text{Re}(z) > 1, \ |\lambda| < 1, \ a \geq 0. \]
They will be related to series with zeta values.

Let $H_m^{(s)}$ be the generalized harmonic numbers
\[
H_m^{(s)} = 1 + \frac{1}{2^s} + \ldots + \frac{1}{m^s}, \quad H_0^{(s)} = 0
\]
which are partial sums of the Riemann zeta function $\zeta(s)$
\[
\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad Re > 1.
\]

We prove the theorem:

**Theorem 2.1.** For every integer $m \geq 1$, $|\lambda| < 1$, $a \geq 0$, $Re(z) > 1$
\[
\sum_{p=1}^{m} \frac{1}{p^a(\lambda + p^z)} = \sum_{n=1}^{\infty} (-1)^{n-1} \lambda^{n-1} H_m^{(n+z+a)}
\]
and also,
\[
\sum_{p=1}^{m} \frac{1}{p^a} \ln \left(1 + \frac{\lambda}{p^z}\right) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \lambda^n H_m^{(n+z+a)}.
\]

Changing $\lambda$ to $-\lambda$ we have as well
\[
\sum_{p=1}^{m} \frac{1}{p^a} \ln \left(1 - \frac{\lambda}{p^z}\right) = -\sum_{n=1}^{\infty} \frac{\lambda^n}{n} H_m^{(n+z+a)}.
\]

**Proof.** Using geometric series, we write
\[
\sum_{p=1}^{m} \frac{1}{p^a(\lambda + p^z)} = \sum_{p=1}^{m} \frac{1}{p^{a+z}} \left(1 - (-\lambda p^{-z})\right)^{-1} = \sum_{p=1}^{m} \frac{1}{p^{a+z}} \left\{ \sum_{k=0}^{\infty} \frac{(-1)^k \lambda^k}{p^k z} \right\}
\]
\[
= \sum_{p=1}^{m} \left\{ \sum_{k=0}^{\infty} \frac{(-1)^k \lambda^k}{p^k z} \right\} = \sum_{k=0}^{\infty} \frac{(-1)^k \lambda^k}{p^{k+1} z+a} \sum_{p=1}^{m} \frac{1}{p^{a+z}}.
\]
Changing the index in the last sum $k + 1 = n$, we obtain equation (2.1). Next, we integrate both sides in (2.1) with respect to $\lambda$. This gives
\[
\sum_{p=1}^{m} \frac{1}{p^a} \ln(\lambda + p^z) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \lambda^n H_m^{(n+z+a)} + C.
\]
Setting $\lambda = 0$ we find $C = \sum_{p=1}^{m} \frac{\ln(p^z)}{p^a}$, so that
\[
\sum_{p=1}^{m} \frac{1}{p^a} \ln(\lambda + p^z) - \sum_{p=1}^{m} \frac{1}{p^a} \ln(p^z) = \sum_{p=1}^{m} \frac{1}{p^a} \ln \left(1 + \frac{\lambda}{p^z}\right) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \lambda^n H_m^{(n+z+a)}
\]
and the theorem is proved. □

For example, for $a = 0$, $z = 1$ in (2.2) we have from [8]
\[
\prod_{p=1}^{m} \left(1 + \frac{\lambda}{p}\right) = \frac{\Gamma(m+\lambda+1)}{m!\Gamma(\lambda+1)}.
\]
This gives
\[
\sum_{p=1}^{m} \ln \left(1 + \frac{\lambda}{p}\right) = \ln \prod_{p=1}^{m} \left(1 + \frac{\lambda}{p}\right) = \ln \frac{\Gamma(m+\lambda+1)}{m!\Gamma(\lambda+1)} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \lambda^n H_m^{(n+z)}.
\]
Corollary 2.2. With \( a, z, \lambda \) as in Theorem 2.1,
\[
\sum_{p=1}^{\infty} \frac{1}{p^a(\lambda + p^2)} = \sum_{n=1}^{\infty} (-1)^{n-1} \lambda^{n-1} \zeta(nz + a) 
\]
(2.3)
\[
\sum_{p=1}^{\infty} \frac{1}{p^a} \ln \left( 1 + \frac{\lambda}{p^2} \right) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \lambda^n \zeta(nz + a) 
\]
(2.4)
\[
\sum_{p=1}^{\infty} \frac{1}{p^a} \ln \left( 1 - \frac{1}{p^2} \right) = - \sum_{n=1}^{\infty} \frac{\lambda^n}{n^a} \zeta(nz + a) \text{ (changing } \lambda \text{ to } -\lambda) 
\]

Proof. The result follows from Theorem 2.1 by letting \( m \to \infty \). The limit can go through the sum because the series is absolutely convergent. \( \Box \)

For \( a = \lambda = z = 1 \) in (2.4) we get equation (1.2).

With \( a = 1 \) we find from (2.4) the series identity
\[
\sum_{p=1}^{\infty} \frac{1}{p^a} \ln \left( 1 + \frac{\lambda}{p^2} \right) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \lambda^n \zeta(nz + 1). 
\]

The series are convergent also for \( \lambda = 1 \) (see argument below after equation (2.6)).

The case \( z = \lambda = 1 \) in (2.4) appeared in the papers [2, 5, 6]
\[
\sum_{p=1}^{\infty} \frac{1}{p^a} \ln \left( 1 + \frac{1}{p} \right) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \zeta(n + a). 
\]

When \( a > 1 \) we can write
\[
\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \zeta(n + a) = \sum_{p=1}^{\infty} \frac{1}{p^a} \ln \left( 1 + \frac{1}{p} \right) = \sum_{p=1}^{\infty} \frac{1}{p^a} \ln \left( \frac{p+1}{p} \right) = \sum_{p=1}^{\infty} \frac{\ln (p+1)}{p^a} - \sum_{p=1}^{\infty} \frac{\ln (p)}{p^a}
\]
and since \(- \sum_{p=1}^{\infty} \frac{\ln (p)}{p^a} = \zeta'(a) \) this becomes
\[
\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \zeta(n + a) = \sum_{p=1}^{\infty} \frac{1}{p^a} \ln \left( 1 + \frac{1}{p} \right) = \sum_{p=1}^{\infty} \frac{\ln (p+1)}{p^a} + \zeta'(a)
\]
(2 [Theorem 4] and [6, equation 4]).

The above series resist evaluation in closed form. Anyway, we want to mention one interesting identity from [5, Theorem 10] related to the above result. First, following the notations in [5], let
\[
\lambda_{1} = \frac{1}{2}, \lambda_{n+1} = \int_{0}^{1} x(1 - x) \cdots (n - x) dx
\]
be the non-alternating Cauchy numbers. Let also \( H_n^{(1)} = H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n} \) be the ordinary harmonic numbers. Then for integers \( a > 1 \), we have the representation
\[
\sum_{p=1}^{\infty} \frac{1}{p^a} \ln \left( 1 + \frac{1}{p} \right) = \zeta'(a) - \gamma \zeta(a) - \zeta(a + 1) + \sum_{n=1}^{\infty} \frac{H_n}{p^n} - \sum_{k=1}^{a-1} \sum_{n=1}^{\infty} \frac{1}{(n+1)^{n+k}} \times \frac{\zeta(k+n+a-k)}{n! n^2}
\]
\[
+ \sum_{n=1}^{\infty} \frac{\lambda_n}{n! n^2} P_{a-1}(H_n, H_n^{(2)}, \ldots H_n^{(a-1)})
\]
where \( P_m \) are the modified Bell polynomials defined by the generating function
\[
\exp \left( \sum_{k=1}^{\infty} x_k \frac{z^k}{k} \right) = \sum_{m=0}^{\infty} P_m(x_1, x_2, \ldots, x_m) z^m.
\]

**Corollary 2.3.** With \( a = 0 \) in (2.2) we have
\[
\sum_{p=1}^{m} \ln \left( 1 + \frac{\lambda}{p^2} \right) = \ln \prod_{p=1}^{m} \left( 1 + \frac{\lambda}{p^2} \right) = \sum_{n=1}^{m} \frac{(-1)^{n-1}}{n} \lambda^n H_m \tag{2.5}
\]
and with \( m \to \infty \)
\[
\sum_{p=1}^{\infty} \ln \left( 1 + \frac{\lambda}{p^2} \right) = \ln \prod_{p=1}^{\infty} \left( 1 + \frac{\lambda}{p^2} \right) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \lambda^n \zeta(nz). \tag{2.6}
\]

Note that the series with zeta values in (2.6) converges also for \( \lambda = 1 \), that is
\[
\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \zeta(nz) = \sum_{n=1}^{\infty} \ln \left( 1 + \frac{1}{p^2} \right) = \ln \prod_{p=1}^{\infty} \left( 1 + \frac{1}{p^2} \right)
\]
as \( \lim_{n \to \infty} |\zeta(nz)| = 1 \) and the series is alternating.

With \( \lambda = x^2 \) and \( z = 2 \) in (2.6) we come to the known identity
\[
\sum_{p=1}^{\infty} \ln \left( 1 + \frac{x^2}{p^2} \right) = \ln \prod_{p=1}^{\infty} \left( 1 + \frac{x^2}{p^2} \right) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^{2n} \zeta(2n) = \ln \frac{\sinh (\pi x)}{\pi x} \tag{2.7}
\]
by using the classical representation
\[
\frac{\sinh (\pi x)}{\pi x} = \prod_{p=1}^{\infty} \left( 1 + \frac{x^2}{p^2} \right).
\]

In particular, with \( x = 1 \)
\[
\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \zeta(2n) = \ln \frac{\sinh \pi}{\pi},
\]
(see also [10], p. 161) while the series \( \sum_{n=1}^{\infty} \zeta(2n) \) is divergent.

With \( x = 1/\mu, \mu > 1 \) identity (2.7) implies
\[
\ln \prod_{p=1}^{\infty} \left( 1 + \frac{1}{\mu^2 p^2} \right) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\mu^{2n} n} \zeta(2n) = \ln \frac{\mu \sinh (\pi/\mu)}{\pi} \tag{2.8}
\]
In particular, with \( \mu = 2 \),
\[
\ln \prod_{p=1}^{\infty} \left( 1 + \frac{1}{4 p^2} \right) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{4^n n} \zeta(2n) = \ln \frac{2 \sinh (\pi/2)}{\pi}
\]
From equation (2.6) and the above examples, we can make the following

**Conclusion.** When the infinite product \( \prod_{p=1}^{\infty} \left( 1 + \frac{\lambda}{p^2} \right) \) can be evaluated in explicit closed form, then the series \( \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \lambda^n \zeta(nz) \) can be evaluated in closed form.

We will show here some more examples following this observation. First, we will use a formula for infinite products from Hansen’s table [9] to evaluate explicitly certain series with zeta values.
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[9, Entry 89.6.8] reads (in corrected form)

\[ \prod_{p=1}^{\infty} \left(1 + \frac{x^3}{p^3}\right) = \frac{1}{\Gamma(1 + x) \Gamma(1 - \frac{x}{2} - \frac{x\sqrt{3}i}{2}) \Gamma(1 - \frac{x}{2} + \frac{x\sqrt{3}i}{2})}. \]

With \( \lambda = x^3, z = 3 \) in (2.6) we find

\[ \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^{3n} \zeta(3n) = -\ln \left( \frac{\Gamma(1 + x) \Gamma(1 - \frac{x}{2} - \frac{x\sqrt{3}i}{2}) \Gamma(1 - \frac{x}{2} + \frac{x\sqrt{3}i}{2})}{\Gamma(1 - \frac{x}{2} - \frac{x\sqrt{3}i}{2}) \Gamma(1 - \frac{x}{2} + \frac{x\sqrt{3}i}{2})} \right) \]

(this is the alternating variant of \[4\] equation (11)). For \( x = 1 \) this comes to

\[ \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \zeta(3n) = \ln \left( \frac{1}{\pi \cosh \left( \frac{\pi}{\sqrt{3}} \right)} \right) \]

(2.9)

(\[4\] equation (13)].

The case \( z = 4 \) was considered in \[4\]. For \( z = 5 \) we use \[12\] equation (33)

\[ \prod_{p=1}^{\infty} \left(1 + \frac{1}{p^6}\right) = |\Gamma(\exp(2\pi i/5)) \Gamma(\exp(6\pi i/5))|^{-2} \]

which provides the evaluation

\[ \sum_{n=1}^{\infty} \ln \left(1 + \frac{1}{p^6}\right) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \zeta(5n) = \ln \left( |\Gamma(\exp(2\pi i/5)) \Gamma(\exp(6\pi i/5))|^{-2} \right). \]

(2.10)

For \( z = 6 \) we use \[12\] equation 34] that says

\[ \prod_{p=1}^{\infty} \left(1 + \frac{1}{p^6}\right) = \frac{\sinh \pi (\cosh (\pi) - \cos (\pi \sqrt{3}))}{2\pi^3} \]

and it gives

\[ \sum_{n=1}^{\infty} \ln \left(1 + \frac{1}{p^6}\right) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \zeta(6n) = \ln \frac{\sinh \pi (\cosh (\pi) - \cos (\pi \sqrt{3}))}{2\pi^3}. \]

(2.11)

It is appropriate to mention here \[10\] Proposition 3.2] where it was shown by a different method that

\[ \sum_{n=1}^{\infty} \frac{(-1)^n}{n} [\zeta(kn) - 1] = \ln \prod_{j=1}^{k-1} \Gamma(2 - (-1)^{(2j+1)/k}) \]

(a result previously obtained by Adamchik and Srivastava \[11\] Proposition 1, p. 135]; see also \[11\] Proposition 3.5, p. 262)). The series on the left-hand side can be split into two series, the second one of which represents \( -\ln 2 \). This way equation (2.12) can be written in the form

\[ \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \zeta(kn) = \ln 2 + \ln \prod_{j=1}^{k-1} \Gamma(2 - (-1)^{(2j+1)/k}) = \ln 2 \prod_{j=1}^{k-1} \Gamma(2 - (-1)^{(2j+1)/k}), \]

(2.13)
that is,
\[
\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \zeta(kn) = -\ln 2 \prod_{j=1}^{k-1} \Gamma(2 - (-1)^{(2j+1)/k}).
\]

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References


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