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ON UNIVALENCY AND CONVEXITY OF INTEGRAL OPERATORS INVOLVING NORMALIZED RABOTNOV FUNCTION

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Abstract. The main objective of this paper is to obtain sufficiency conditions for some families of integral operators involving the normalized form of Rabotnov functions to satisfy basic characteristics in Geometric Function Theory of univalency and convexity in the open unit disk. We consider several corollaries and special cases of our main results. In particular, we mention cases which give useful applications by including also some examples yielding corresponding simpler conditions for integral operators involving the exponential and hyperbolic functions. Relevance with known results are also pointed out.

1. Introduction and preliminaries

Yu. N. Rabotnov [33] in his work on viscoelasticity introduced a function of time t which depends on two parameters α and β, where $\alpha \in (-1, 0]$ is related to the type of viscoelasticity and $\beta \in \mathbb{R}$(set of real numbers). This function is defined by

$$R_{\alpha,\beta}(t) = t^{\alpha} \sum_{k=0}^{\infty} \frac{\beta^k}{\Gamma((k+1)(1+\alpha))} t^{k(1+\alpha)},\ t \geq 0. \tag{1.1}$$

For more details about the function (1.1), one may refer to [36]. Rabotnov called the function (1.1) as the fractional exponential function mainly because of the fact that for $\alpha = 0$, this function reduces to the standard exponential function $\exp(\beta t)$. The relation of this function with the Mittag-Leffler function in two parameters is quite obvious. Indeed, we have the relation that

$$R_{\alpha,\beta}(z) = z^{\alpha} E_{1+\alpha,1+\alpha}(\beta z^{1+\alpha}),$$

where $\alpha, \beta, z \in \mathbb{C}$, and the function E is the Mittag-Leffler function [24].

Let $\mathbb{U} := \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk in the complex plane $\mathbb{C}$, and let $\mathcal{A}$ denote the class of functions f that are analytic in $\mathbb{U}$ and normalized by the

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conditions $f(0) = f'(0) - 1 = 0$. Thus, each function $f \in \mathcal{A}$ has the following series representation:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad (z \in \mathbb{U}).$$

We denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of functions f that are univalent in $\mathbb{U}$. A function $f \in \mathcal{A}$ that maps $\mathbb{U}$ onto a convex domain (i.e. the line segment joining any two points in $f(\mathbb{U})$ lies completely inside $f(\mathbb{U})$) is called a convex function. We denote by $\mathcal{K}$ the class of all functions $f \in \mathcal{A}$ that are convex. The analytic characterization of $\mathcal{K}$ is

$$\mathcal{K} = \left\{ f : f \in \mathcal{A}, \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0, z \in \mathbb{U} \right\}.$$

It is clear that the Rabotnov function $R_{\alpha,\beta}(z)$ does not belong to the family $\mathcal{A}$. Therefore, for the purpose of this paper, we consider the following normalization of Rabotnov functions [16]:

$$\begin{aligned} \mathbb{R}_{\alpha,\beta}(z) &= \Gamma(1+\alpha) z^{1/(1+\alpha)} R_{\alpha,\beta}(z^{1/(1+\alpha)}) \\ &= z + \sum_{k=2}^{\infty} \frac{\beta^{k-1}\Gamma(1+\alpha)}{\Gamma((1+\alpha)k)} z^k \qquad (1.2) \\ &= \sum_{k=1}^{\infty} \frac{\beta^{k-1}\Gamma(1+\alpha)}{\Gamma((1+\alpha)k)} z^k. \end{aligned}$$

We mention below some special cases of $\mathbb{R}_{\alpha,\beta}(z)$ for particular values of the parameters α and β.

(1) By putting $\alpha = 0$ in (1.2), we have

$$\mathbb{R}_{0,\beta}(z) = \sum_{k=1}^{\infty} \frac{\beta^{k-1}}{\Gamma(k)} z^k = z \sum_{k=1}^{\infty} \frac{(\beta z)^{k-1}}{(k-1)!} = z e^{\beta z}. \qquad (1.3)$$

(2) Next, if we set $\alpha = 1$ in (1.2), we have then

$$\begin{aligned} \mathbb{R}_{1,\beta}(z) &= \sum_{k=1}^{\infty} \frac{\Gamma(2)\beta^{k-1}}{\Gamma(2k)} z^k = z \sum_{k=1}^{\infty} \frac{(\beta z)^{k-1}}{(2k-1)!} \\ &= z \sum_{k=1}^{\infty} \frac{(\sqrt{\beta z})^{2k-2}}{(2k-1)!} = \frac{z}{\sqrt{\beta z}} \sum_{k=1}^{\infty} \frac{(\sqrt{\beta z})^{2k-1}}{(2k-1)!} \\ &= \sqrt{\frac{z}{\beta}} \sinh(\sqrt{\beta z}). \qquad (1.4) \end{aligned}$$

The geometric properties of various special functions were studied in many earlier works. Such investigations appeared for hypergeometric function, incomplete beta function, Laguerre polynomials, Bessel function, Lommel and Sturve functions, Mittag-Leffler function and Wright function. See, the related works in [2], [3],[4],[5],[6],[7],[12],[13],[16] [22] and [34].

An important field in the geometric function theory is also the field of integral operators on spaces of analytic functions. Several integral operators have been investigated by many authors. One may refer to the works of Bernardi [7], Libera [19], Causey [14, 15], Kim and Merkes [18], Merkes and Wright [20], Miller, Mocanu and Reade [21, 22, 23], Owa and Srivastava [26], Pascu [27], Pfaltzgraff [32], Pescar [28, 29] and many others.

Our aim in this paper is to determine some sufficient conditions for certain families of integral operators defined in terms of the normalized form of the Rabotnov functions (1.2) (see below) to be univalent and convex in the open unit disk $\mathbb{U}$. We also consider various special cases of our main results and point out specific applications of some of the deduced corollaries.

We mention now necessary details of various families of integral operators which have been studied in recent past in Geometric Function Theory.

Definition 1.1. *The first family of integral operators studied by Seenivasagan and Breaz* [37] *is defined as follows (see also the recent investigations on this subject by Baricz and Frasin* [4] *and Srivastava et al.* [39]*):*

$$\mathcal{F}_{\lambda_1,\ldots,\lambda_n,\mu}(z) = \left[\mu \int_0^z t^{\mu-1} \prod_{j=1}^{n} \left(\frac{f_j(t)}{t}\right)^{1/\lambda_j} dt\right]^{1/\mu}, \tag{1.5}$$

where each of the functions f_j $(j = 1, \cdots, n)$ belongs to the class $\mathcal{A}$ and the parameters $\lambda_j \in \mathbb{C}/\{0\}$ $(j = 1, ..., n)$ and $\mu \in \mathbb{C}$ are so constrained such that the integral operators in (1.5) *exist. We note that if $\lambda_j = \lambda$ $(j = 1, ..., n)$, then the integral operator $\mathcal{F}_{\lambda_1,\ldots,\lambda_n,\mu}(z)$ reduces to the operator $\mathcal{F}_{\lambda,\mu}(z)$ which is related closely to some known integral operators investigated earlier in Geometric Functions Theory (see, for details,* [38]*). The operators $\mathcal{F}_{\lambda,\mu}(z)$ and $\mathcal{F}_{\lambda,\lambda}(z)$ were studied by Breaz and Breaz* [10] *and Pescar* [29]*, respectively. Upon setting $\mu = 1$ and also $\lambda = 1$ in $\mathcal{F}_{\lambda,\mu}(z)$, we obtain the operators $\mathcal{F}_{\lambda,1}(z)$ and $\mathcal{F}_{1,1}(z)$, which were studied by Breaz and Breaz* [9] *and Alexander* [2]*, respectively.*
Furthermore, in their special cases when

$$n = \mu = 1 \qquad \text{and} \qquad \lambda_j = \frac{1}{\lambda} \qquad (j = 1, , n),$$

then the integral operator in (1.5) *would obviously reduce to the operator $\mathcal{F}_{1/\lambda,1}(z)$, which was studied earlier by Pescar and Owa* [30]*. In particular, for $\lambda \in [0, 1]$, a special case of the operator $\mathcal{F}_{1/\lambda,1}(z)$ was also studied by Miller et al.* [22]*.*

Definition 1.2. *The second family of integral operators was introduced by Breaz and Breaz* [11] *and it has the following form (see also a recent investigation on this subject by Breaz et al.* [12]*):*

$$\mathcal{G}_{n,\gamma}(z) = \left[(n\gamma + 1) \int_0^z \prod_{j=1}^{n} [g_j(t)]^{\gamma} dt\right]^{1/(n\gamma+1)}, \tag{1.6}$$

where the functions $g_j \in \mathcal{A}$ $(j = 1, ..., n)$ and the parameter $\gamma \in \mathbb{C}$ is so constrained such that the integral operators in (1.6) *exist. In particular, for $n = 1$, the integral operator $\mathcal{G}_{1,\gamma}(z)$ was earlier studied by Moldoveanu and Pascu* [25]*.*

Definition 1.3. *The third family of integral operators was introduced by Breaz and Breaz* [8] *and it has the following form:*

$$\mathcal{H}_{\nu_1,\ldots,\nu_n,\xi}(z) = \left[\xi \int_0^z t^{\xi-1} \prod_{j=1}^{n} \left[h_j'(t)\right]^{\nu_j} dt\right]^{1/\xi}, \tag{1.7}$$

where the functions $h_j \in \mathcal{A}$ $(j = 1, ..., n)$ *and the parameters* $\xi \in \mathbb{C}$ *and* $\nu_j \in \mathbb{C}$ $(j = 1, ..., n)$ *are so constrained that the integral operators in* (1.7) *exist. In particular, for* $\xi = 1$ *in* (1.7)*, the integral operator* $\mathcal{H}_{\nu_1,...,\nu_n,\xi}(z)$ *reduces to the operator* $\mathcal{H}_{\nu_1,...,\nu_n}(z)$ *which was studied by Breaz et al.* [13]*. We observe also that for* $n = \xi = 1$*, the integral operator* $\mathcal{H}(z)$ *was introduced and studied by Kim and Merkes* [18]*; see also Pfaltzgraff* [32]*.*

Definition 1.4. *The fourth family of integral operators was introduced by Pescar* [31] *as follows:*

$$\mathcal{Q}_\delta(z) = \left[\delta \int_0^z t^{\delta-1} \left(e^{q(t)}\right)^\delta dt\right]^{1/\delta}, \tag{1.8}$$

where the function $q \in \mathcal{A}$ *and the parameter* $\delta \in \mathbb{C}$ *are so constrained that the integral operators in* (1.8) *exist.*

Two of the most important and known univalence criteria for analytic functions defined in the open unit disk $\mathbb{U}$ were obtained by Ahlfors [1] and Becker [5]. Some extensions of these two univalence criteria were given by Pescar [28] involving a parameter μ (which for $\mu = 1$ yields the Ahlfors-Becker univalence criterion) and another criteria by Pascu [27] involving two parameters λ and μ (which for $\mu = \lambda = 1$, yields the Beckers univalence criterion).
Further, in this paper we have mentioned sufficient conditions in terms of the Lambert function $W(x)$, also called omega function or product logarithm is a multivalued function. The Lambert function $W(x)$ is defined by

$$W(x) = f^{-1}(x) \text{ where } f(x) = xe^x. \tag{1.9}$$

Evidentaly, then $W(xe^x) = x$ and $W(e) = 1$. One can use ProductLog function of Wolfram Mathematica to evaluate value of Lambert function at any point of its domain. We shall make use of the following lemmas in order to prove our main results.

Lemma 1.1. ([28]). *Let* μ *and* c *be complex numbers such that*

$$\Re(\mu) > 0 \qquad \text{and} \qquad |c| \le 1 \quad (|c| \ne -1).$$

If the function $f \in \mathcal{A}$ *satisfies the following inequality:*

$$\left| c|z|^{2\mu} + \left(1 - |z|^{2\mu}\right) \frac{zf''(z)}{\mu f'(z)} \right| \le 1 \ \ (z \in \mathbb{U}),$$

then the function F_μ *defined by*

$$F_\mu(z) = \left(\mu \int_0^z t^{\mu-1} f'(t) dt\right)^{1/\mu} \tag{1.10}$$

is in the class $\mathcal{S}$ *of normalized univalent functions in* $\mathbb{U}$*.*

Lemma 1.2. ([27]). *If* $f \in \mathcal{A}$ *satisfies the following inequality:*

$$\left(\frac{1 - |z|^{2\Re(\gamma)}}{\Re(\gamma)}\right) \left|\frac{zf''(z)}{f'(z)}\right| \le 1 \ \ (z \in \mathbb{U}, \Re(\gamma) > 0),$$

then for all $\mu \in \mathbb{C}$ *such that* $\Re(\mu) \ge \Re(\gamma)$*, the function* F_μ *defined by* (1.10) *is in the class* $\mathcal{S}$ *of normalized univalent functions in* $\mathbb{U}$*.*

Lemma 1.3. ([30]). *Let the parameters $\delta \in \mathbb{C}$ and $\theta \in \mathbb{R}$ be so constrained that*

$$\Re(\delta) \geq 1, \quad \theta > 1 \qquad \textit{and} \qquad 2\theta|\delta| \leq 3\sqrt{3}.$$

If the function $q \in \mathcal{A}$ satisfies the following inequality:

$$|zq'(z)| \leq \theta \qquad (z \in \mathbb{U}),$$

then the function $\mathcal{Q}_\delta : \mathbb{U} \to \mathbb{C}$ defined by

$$\mathcal{Q}_\delta(z) = \left[\delta \int_0^z t^{\delta-1} \left(e^{q(t)}\right)^\delta dt\right]^{1/\delta}$$

is in the class $\mathcal{S}$ of normalized univalent functions in $\mathbb{U}$.

Lemma 1.3 follows from the Beckers univalence criterion [6, 27] and the well-known Schwarz lemma [17, p.25]. We shall further require the following known results proved recently in Theorem 2.4 and Theorem 2.5 of [16].

Lemma 1.4. ([16]). *If the parameters $\alpha \geq 0$ and $\beta > 0$, then the function $\mathbb{R}_{\alpha,\beta}(z) : \mathbb{U} \to \mathbb{C}$ defined by* (1.2) *satisfies the following inequalities:*

$$\left|\mathbb{R}'_{\alpha,\beta}(z) - \frac{\mathbb{R}_{\alpha,\beta}(z)}{z}\right| \leq \frac{\beta}{(1+\alpha)} e^{\frac{\beta}{1+\alpha}}, \tag{1.11}$$

$$\left|\frac{z\mathbb{R}'_{\alpha,\beta}(z)}{\mathbb{R}_{\alpha,\beta}(z)} - 1\right| \leq \frac{\frac{\beta}{(1+\alpha)} e^{\frac{\beta}{1+\alpha}}}{2 - e^{\frac{\beta}{1+\alpha}}} \quad \left(\text{provided that } \frac{\beta}{1+\alpha} < ln(2)\right) \tag{1.12}$$

and

$$\left|\frac{z\mathbb{R}''_{\alpha,\beta}(z)}{\mathbb{R}'_{\alpha,\beta}(z)}\right| \leq \frac{\beta(2\alpha+\beta+2)e^{\frac{\beta}{1+\alpha}}}{(1+\alpha)\left(2(1+\alpha) - (1+\alpha+\beta)e^{\frac{\beta}{1+\alpha}}\right)} \quad \left(\text{provided that } \frac{\beta}{1+\alpha} < W(2e) - 1\right), \tag{1.13}$$

where W is a Lambert function defined by (1.9).

We also prove the following lemma in order to establish our main results.

Lemma 1.5. *If $\alpha \geq 0$ and $\beta > 0$, then for $z \in \mathbb{U}$:*

$$2 - \left(1 + \frac{\beta}{1+\alpha}\right) e^{\frac{\beta}{1+\alpha}} \leq |z\mathbb{R}'_{\alpha,\beta}(z)| \leq \left(1 + \frac{\beta}{1+\alpha}\right) e^{\frac{\beta}{1+\alpha}}. \tag{1.14}$$

The lower bound of (1.14) *exists only when $\alpha \geq \frac{\beta}{W(2e)-1} - 1$, where the function W is defined by* (1.9). *Further,*

$$\left|z^2\mathbb{R}''_{\alpha,\beta}(z)\right| \leq \frac{\beta(2+2\alpha+\beta)}{(1+\alpha)^2} e^{\frac{\beta}{1+\alpha}}. \tag{1.15}$$

Proof. We begin with the inequality

$$\Gamma(1+\alpha)(1+\alpha)^{k-1}(k-1)! \leq \Gamma((1+\alpha)k) \ \ \forall k \in \mathbb{N} \text{ and } \alpha \geq 0, \tag{1.16}$$

which is easy to verify (see [16, p. 1253, Lemma 2.5]). In view of (1.2), we have

$$\begin{aligned}
|z\mathbb{R}'_{\alpha,\beta}(z)| &= \left| z + \sum_{k=2}^{\infty} \frac{k\beta^{k-1}\Gamma(1+\alpha)}{\Gamma((1+\alpha)k)} z^k \right| \leq |z| + \sum_{k=2}^{\infty} \frac{k\beta^{k-1}\Gamma(1+\alpha)}{\Gamma((1+\alpha)k)} |z|^k \\
&\leq 1 + \sum_{k=2}^{\infty} \frac{k\beta^{k-1}\Gamma(1+\alpha)}{\Gamma((1+\alpha)k)} \leq 1 + \sum_{k=2}^{\infty} \frac{k\beta^{k-1}}{(1+\alpha)^{k-1}(k-1)!} \quad \text{(on using (1.16))} \\
&= 1 + \sum_{k=2}^{\infty} \frac{\beta^{k-1}}{(1+\alpha)^{k-1}(k-2)!} + \sum_{k=2}^{\infty} \frac{\beta^{k-1}}{(1+\alpha)^{k-1}(k-1)!} \\
&= \left(1 + \frac{\beta}{1+\alpha}\right) e^{\frac{\beta}{1+\alpha}}.
\end{aligned}$$

Similarly, it follows from (1.2) that

$$\begin{aligned}
|z\mathbb{R}'_{\alpha,\beta}(z)| &= \left| z + \sum_{k=2}^{\infty} \frac{k\beta^{k-1}\Gamma(1+\alpha)}{\Gamma((1+\alpha)k)} z^k \right| \geq 1 - \sum_{k=2}^{\infty} \frac{k\beta^{k-1}\Gamma(1+\alpha)}{\Gamma((1+\alpha)k)} \\
&\geq 1 - \sum_{k=2}^{\infty} \frac{k\beta^{k-1}}{(1+\alpha)^{k-1}(k-1)!} = 1 - \sum_{k=2}^{\infty} \frac{((k-1)+1)\beta^{k-1}}{(1+\alpha)^{k-1}(k-1)!} \quad \text{(on using (1.16))} \\
&= 1 - \frac{\beta}{1+\alpha} \sum_{k=2}^{\infty} \frac{\beta^{k-2}}{(1+\alpha)^{k-2}(k-2)!} - \sum_{k=2}^{\infty} \frac{\beta^{k-1}}{(1+\alpha)^{k-1}(k-1)!} \\
&= 2 - \left(1 + \frac{\beta}{1+\alpha}\right) e^{\frac{\beta}{1+\alpha}}.
\end{aligned}$$

This completes the proof of (1.14). Further to establish (1.15), we obtain from (1.2) that

$$\begin{aligned}
\left|z^2\mathbb{R}''_{\alpha,\beta}(z)\right| &= \left| \sum_{k=1}^{\infty} \frac{k(k-1)\beta^{k-1}\Gamma(1+\alpha)}{\Gamma((1+\alpha)k)} z^k \right| \leq \sum_{k=1}^{\infty} \frac{k(k-1)\beta^{k-1}\Gamma(1+\alpha)}{\Gamma((1+\alpha)k)} |z|^k \\
&\leq \sum_{k=1}^{\infty} \frac{k(k-1)\beta^{k-1}\Gamma(1+\alpha)}{\Gamma((1+\alpha)k)} \leq \sum_{k=1}^{\infty} \frac{k(k-1)\beta^{k-1}}{(1+\alpha)^{k-1}(k-1)!} \quad \text{(on using (1.16))} \\
&= \sum_{k=1}^{\infty} \frac{(k-2)(k-1) + 2(k-1)}{(k-1)!} \frac{\beta^{k-1}}{(1+\alpha)^{k-1}} \\
&= \left(\frac{\beta}{1+\alpha}\right)^2 \sum_{k=3}^{\infty} \frac{1}{(k-3)!} \left(\frac{\beta}{1+\alpha}\right)^{k-3} \\
&\qquad + 2\left(\frac{\beta}{1+\alpha}\right) \sum_{k=2}^{\infty} \frac{1}{(k-2)!} \left(\frac{\beta}{1+\alpha}\right)^{k-2} \\
&= \frac{\beta(2+2\alpha+\beta)}{(1+\alpha)^2} e^{\frac{\beta}{1+\alpha}}.
\end{aligned}$$

This completes the proof of (1.15). □

2. Univalence of Integral Operators Involving Rabotnov Functions

Our first main result provides an application of Lemma 1.1 which gives sufficient univalence conditions for the class of integral operators of the type (1.5) when the functions f_j $(j = 1, ..., n)$ are normalized forms of Rabotnov functions (1.2) involving various parameters.

Theorem 2.1. *Let the parameters* $\alpha_j \geq 0,\ \beta_j > 0$ $(j = 1, ..., n)$ *and*

$$\frac{\beta}{1+\alpha} = \max\left\{ \frac{\beta_j}{1+\alpha_j} \middle| (j = 1, ..., n) \right\} \tag{2.1}$$

be so constrained that

$$\frac{\beta}{1+\alpha} < \ln(2) \ \text{ with } \ \alpha \geq 0,\ \beta > 0.$$

Consider the functions $\mathbb{R}_{\alpha_j,\beta_j} : \mathbb{U} \to \mathbb{C}$ *defined by*

$$\mathbb{R}_{\alpha_j,\beta_j}(z) = z + \sum_{k=2}^{\infty} \frac{\beta_j^{k-1}\Gamma(1+\alpha_j)}{\Gamma((1+\alpha_j)k)} z^k. \tag{2.2}$$

Suppose also that

$$\Re(\mu) > 0,\ c \in \mathbb{C}\backslash\{-1\} \text{ with } |c| \leq 1 \quad \text{and} \quad \lambda_j \in \mathbb{C}\backslash\{0\} \ \ (j = 1, ..., n)$$

satisfy the following inequality:

$$|c| + \frac{\beta e^{\frac{\beta}{1+\alpha}}}{(1+\alpha)\left(2 - e^{\frac{\beta}{1+\alpha}}\right)} \sum_{j=1}^{n} \frac{1}{|\mu\lambda_j|} \leq 1.$$

Then the function $\mathcal{F}_{\alpha_1,...,\alpha_n,\beta_1,...,\beta_n,\lambda_1,...,\lambda_n,\mu}(z) : \mathbb{U} \to \mathbb{C}$*, defined by*

$$\mathcal{F}_{\alpha_1,...,\alpha_n,\beta_1,...,\beta_n,\lambda_1,...,\lambda_n,\mu}(z) = \left[\mu \int_0^z t^{\mu-1} \prod_{j=1}^{n} \left(\frac{\mathbb{R}_{\alpha_j,\beta_j}(t)}{t} \right)^{1/\lambda_j} dt \right]^{1/\mu} \tag{2.3}$$

is in the class $\mathcal{S}$ *of normalized univalent functions in* $\mathbb{U}$*.*

Proof. Let us set $\mu = 1$ in (2.3), then we have

$$\mathcal{F}_{\alpha_1,...,\alpha_n,\beta_1,...,\beta_n,\lambda_1,...,\lambda_n,1}(z) = \int_0^z \prod_{j=1}^{n} \left(\frac{\mathbb{R}_{\alpha_j,\beta_j}(t)}{t} \right)^{1/\lambda_j} dt. \tag{2.4}$$

We observe that since $\mathbb{R}_{\alpha_j,\beta_j} \in \mathcal{A}$ $(j = 1, ..., n)$ and also $\mathbb{R}_{\alpha_j,\beta_j}(0) = \mathbb{R}'_{\alpha_j,\beta_j}(0) - 1 = 0$, therefore $\mathcal{F}_{\alpha_1,...,\alpha_n,\beta_1,...,\beta_n,\lambda_1,...,\lambda_n,1} \in \mathcal{A}$, that is

$$\mathcal{F}_{\alpha_1,....\alpha_n,\beta_1,...,\beta_n,\lambda_1,...,\lambda_n,1}(0) = \mathcal{F}'_{\alpha_1,...,\alpha_n,\beta_1,...,\beta_n,\lambda_1,...,\lambda_n,1}(0) - 1 = 0.$$

Differentiating (2.4) with respect to z, we have

$$\mathcal{F}'_{\alpha_1,...,\alpha_n,\beta_1,...,\beta_n,\lambda_1,...,\lambda_n,1}(z) = \prod_{j=1}^{n} \left(\frac{\mathbb{R}_{\alpha_j,\beta_j}(z)}{z} \right)^{1/\lambda_j}. \tag{2.5}$$

Again differentiating (2.5) with respect to z, we obtain

$$\mathcal{F}''_{\alpha_1,...,\alpha_n,\beta_1,...,\beta_n,\lambda_1,...,\lambda_n,1}(z) = \sum_{j=1}^{n} \frac{1}{\lambda_j} \left(\frac{\mathbb{R}_{\alpha_j,\beta_j}(z)}{z} \right)^{(1-\lambda_j)/\lambda_j}$$

$$\cdot \left(\frac{z\mathbb{R}'_{\alpha_j,\beta_j}(z) - \mathbb{R}_{\alpha_j,\beta_j}(z)}{z^2} \right) \prod_{\substack{k=1 \\ (k \neq j)}}^{n} \left(\frac{\mathbb{R}_{\alpha_k,\beta_k}(z)}{z} \right)^{1/\lambda_k}. \tag{2.6}$$

Using (2.5) and (2.6), we get

$$\frac{z\mathcal{F}''_{\alpha_1,\ldots\alpha_n,\beta_1,\ldots,\beta_n,\lambda_1,\ldots,\lambda_n,1}(z)}{\mathcal{F}'_{\alpha_1,\ldots,\alpha_n,\beta_1,\ldots,\beta_n,\lambda_1,\ldots,\lambda_n,1}(z)} = \sum_{j=1}^{n} \frac{1}{\lambda_j}\left(\frac{z\mathbb{R}'_{\alpha_j,\beta_j}(z)}{\mathbb{R}_{\alpha_j,\beta_j}(z)} - 1\right).$$

Now applying the inequality (1.12) in Lemma 1.4 for each α_j, β_j ($j = 1, ..., n$), we get

$$\left|\frac{z\mathcal{F}''_{\alpha_1,\ldots,\alpha_n,\beta_1,\ldots,\beta_n,\lambda_1,\ldots,\lambda_n,1}(z)}{\mathcal{F}'_{\alpha_1,\ldots,\alpha_n,\beta_1,\ldots,\beta_n,\lambda_1,\ldots,\lambda_n,1}(z)}\right| \leq \sum_{j=1}^{n} \frac{1}{|\lambda_j|}\left|\frac{z\mathbb{R}'_{\alpha_j,\beta_j}(z)}{\mathbb{R}_{\alpha_j,\beta_j}(z)} - 1\right|$$
$$\leq \sum_{j=1}^{n} \frac{1}{|\lambda_j|} \frac{\frac{\beta_j}{(1+\alpha_j)} e^{\frac{\beta_j}{1+\alpha_j}}}{2 - e^{\frac{\beta_j}{1+\alpha_j}}}. \tag{2.7}$$

Suppose

$$\phi(x) = \frac{xe^x}{2 - e^x}, \tag{2.8}$$

then

$$\phi'(x) = \frac{e^x}{(2 - e^x)^2} \cdot (2x + 2 - e^x).$$

Clearly $\phi'(x)$ is positive for $x \in (0, \ln 2)$, and hence $\phi(x)$ is an increasing function of x. Hence, in view of (2.1), we have

$$\phi\left(\frac{\beta_j}{1+\alpha_j}\right) \leq \phi\left(\frac{\beta}{1+\alpha}\right).$$

Therefore from (2.7) and (2.8), we get

$$\left|\frac{z\mathcal{F}''_{\alpha_1,\ldots,\alpha_n,\beta_1,\ldots,\beta_n,\lambda_1,\ldots,\lambda_n,1}(z)}{\mathcal{F}'_{\alpha_1,\ldots,\alpha_n,\beta_1,\ldots,\beta_n,\lambda_1,\ldots,\lambda_n,1}(z)}\right| \leq \sum_{j=1}^{n} \frac{1}{|\lambda_j|} \frac{\frac{\beta}{(1+\alpha)} e^{\frac{\beta}{1+\alpha}}}{2 - e^{\frac{\beta}{1+\alpha}}}.$$

Finally, by using the triangle inequality and the assertion of Theorem 2.1, we obtain

$$\left|c|z|^{2\mu} + \left(1 - |z|^{2\mu}\right)\frac{z\mathcal{F}''_{\alpha_1,\ldots,\alpha_n,\beta_1,\ldots,\beta_n,\lambda_1,\ldots,\lambda_n,1}(z)}{\mu\mathcal{F}'_{\alpha_1,\ldots,\alpha_n,\beta_1,\ldots,\beta_n,\lambda_1,\ldots,\lambda_n,1}(z)}\right|$$
$$\leq |c| + \frac{1}{|\mu|}\left|\frac{z\mathcal{F}''_{\alpha_1,\ldots,\alpha_n,\beta_1,\ldots,\beta_n,\lambda_1,\ldots,\lambda_n,1}(z)}{\mathcal{F}'_{\alpha_1,\ldots\alpha_n,\beta_1,\ldots,\beta_n,\lambda_1,\ldots,\lambda_n,1}(z)}\right|$$
$$\leq |c| + \frac{1}{|\mu|} \frac{\frac{\beta}{(1+\alpha)} e^{\frac{\beta}{1+\alpha}}}{2 - e^{\frac{\beta}{1+\alpha}}} \sum_{j=1}^{n} \frac{1}{|\lambda_j|}$$
$$= |c| + \frac{\beta e^{\frac{\beta}{1+\alpha}}}{(1+\alpha)\left(2 - e^{\frac{\beta}{1+\alpha}}\right)} \sum_{j=1}^{n} \frac{1}{|\mu\lambda_j|} \leq 1,$$

which in view of Lemma 1.1 implies that $\mathcal{F}_{\alpha_1,\ldots,\alpha_n,\beta_1,\ldots,\beta_n,\lambda_1,\ldots,\lambda_n,\mu}(z) \in \mathcal{S}$. This completes the proof of Theorem 2.1. □

If we put $n = 1$, $\lambda_1 = \lambda, \mu = 1, \alpha_1 = \alpha$ and $\beta_1 = \beta$ in Theorem 2.1, then we get the following corollary:

Corollary 2.1. *Let the parameters $\alpha \geq 0,\ \beta > 0$ be so constrained that*

$$\frac{\beta}{1+\alpha} < \ln(2).$$

Consider the functions $\mathbb{R}_{\alpha,\beta} : \mathbb{U} \to \mathbb{C}$ defined by (1.2). Suppose also that $c \in \mathbb{C}\backslash\{-1\}$ with $|c| \leq 1$ and $\lambda \in \mathbb{C}\backslash\{0\}$ satisfy the following inequality:

$$|c| + \frac{\beta e^{\frac{\beta}{1+\alpha}}}{|\lambda|(1+\alpha)\left(2 - e^{\frac{\beta}{1+\alpha}}\right)} \leq 1.$$

Then the function $\mathcal{F}_{\alpha,\beta,\lambda}(z) : \mathbb{U} \to \mathbb{C}$, defined by

$$\mathcal{F}_{\alpha,\beta,\lambda}(z) = \int_0^z \left(\frac{\mathbb{R}_{\alpha,\beta}(t)}{t}\right)^{1/\lambda} dt$$

is in the class $\mathcal{S}$ of normalized univalent functions in $\mathbb{U}$.

Remark 2.1. If we put $c = 0, \lambda = 1$ in Corollary 2.1, we obtain then a known result Theorem 3.3 of [35].

We consider below few examples to illustrate the applications of Corollary 2.1.

Example 2.1. Next, if we choose $\alpha = 0,\ \beta = 1/4,\ c = 0$ and $\lambda = 1/2$ in Corollary 2.1 and use (1.3), then we have

$$\mathcal{F}_{0,1/4,1/2}(z) = \int_0^z \left(e^{t/4}\right)^2 dt = 2\left[e^{z/2} - 1\right] \in \mathcal{S}.$$

Example 2.2. Putting $\alpha = 1,\ \beta = 1/4,\ c = 0$ and $\lambda = 1$ in Corollary 2.1 and using (1.4), we have

$$\mathcal{F}_{1,1/4,1}(z) = 2\int_0^z \frac{\sinh\left(\frac{\sqrt{t}}{2}\right)}{\sqrt{t}} dt = 8\left[\cosh\left(\frac{\sqrt{z}}{2}\right) - 1\right] \in \mathcal{S}.$$

Our second main result contains sufficient univalence conditions for integral operators of the type (1.6) when the functions g_j $(j = 1, ..., n)$ are normalized forms of Rabotnov functions involving various parameters. The key tools in the proof are Lemma 1.2 and the inequality (1.12) of Lemma 1.4.

Theorem 2.2. *Let the parameters $\alpha_j \geq 0,\ \beta_j > 0\ \ (j = 1, ..., n)$ and*

$$\frac{\beta}{1+\alpha} = \max\left\{\frac{\beta_j}{1+\alpha_j}\middle|(j = 1, ..., n)\right\}$$

be so constrained that

$$\frac{\beta}{1+\alpha} < \ln(2) \ \ \text{with} \ \ \alpha \geq 0,\ \beta > 0.$$

Consider the functions $\mathbb{R}_{\alpha_j,\beta_j} : \mathbb{U} \to \mathbb{C}$ defined by (2.2), and let $\Re(\gamma) > 0$. Moreover, suppose that the following inequality holds true:

$$|\gamma| \leq \frac{(1+\alpha)\left(2 - e^{\frac{\beta}{1+\alpha}}\right)}{n\beta e^{\frac{\beta}{1+\alpha}}}\Re(\gamma).$$

Then the function $\mathcal{G}_{\alpha_1,\dots,\alpha_n,\beta_1,\dots,\beta_n,n,\gamma}(z) : \mathbb{U} \to \mathbb{C}$ *defined by*

$$\mathcal{G}_{\alpha_1,\dots,\alpha_n,\beta_1,\dots,\beta_n,n,\gamma}(z) = \left[(n\gamma+1)\int_0^z \prod_{j=1}^{n} \left(\mathbb{R}_{\alpha_j,\beta_j}(t)\right)^{\gamma} dt\right]^{1/(n\gamma+1)}$$

is in the class $\mathcal{S}$ *of normalized univalent functions in* $\mathbb{U}$.

Proof. Let us consider the function $\widetilde{\mathcal{G}}_{\alpha_1,\dots,\alpha_n,\beta_1,\dots,\beta_n,n,\gamma}(z) : \mathbb{U} \to \mathbb{C}$ defined by

$$\widetilde{\mathcal{G}}_{\alpha_1,\dots,\alpha_n,\beta_1,\dots,\beta_n,n,\gamma}(z) = \int_0^z \prod_{j=1}^{n} \left(\frac{\mathbb{R}_{\alpha_j,\beta_j}(t)}{t}\right)^{\gamma} dt.$$

Observe that $\widetilde{\mathcal{G}}_{\alpha_1,\dots,\alpha_n,\beta_1,\dots,\beta_n,n,\gamma} \in \mathcal{A}$, that is

$$\widetilde{\mathcal{G}}_{\alpha_1,\dots,\alpha_n,\beta_1,\dots,\beta_n,n,\gamma}(0) = \widetilde{\mathcal{G}'}_{\alpha_1,\dots,\alpha_n,\beta_1,\dots,\beta_n,n,\gamma}(0) - 1 = 0.$$

Using (1.12) of Lemma 1.4 for each α_j, β_j $(j = 1, ..., n)$ and hypothesis of Theorem 2.2, we have

$$\begin{aligned}
\frac{1-|z|^{2\Re(\gamma)}}{\Re(\gamma)} \left|\frac{z\widetilde{\mathcal{G}''}_{\alpha_1,\dots,\alpha_n,\beta_1,\dots,\beta_n,n,\gamma}(z)}{\widetilde{\mathcal{G}'}_{\alpha_1,\dots,\alpha_n,\beta_1,\dots,\beta_n,n,\gamma}(z)}\right| &\le \frac{|\gamma|}{\Re(\gamma)} \sum_{j=1}^{n} \left|\frac{z\mathbb{R}'_{\alpha_j,\beta_j}(z)}{\mathbb{R}_{\alpha_j,\beta_j}(z)} - 1\right| \\
&\le \frac{|\gamma|}{\Re(\gamma)} \sum_{j=1}^{n} \frac{\frac{\beta_j}{(1+\alpha_j)} e^{\frac{\beta_j}{1+\alpha_j}}}{2 - e^{\frac{\beta_j}{1+\alpha_j}}} \\
&\le \frac{|\gamma|}{\Re(\gamma)} \sum_{j=1}^{n} \frac{\frac{\beta}{(1+\alpha)} e^{\frac{\beta}{1+\alpha}}}{2 - e^{\frac{\beta}{1+\alpha}}} \\
&= \frac{|\gamma|}{\Re(\gamma)} \frac{n\beta e^{\frac{\beta}{1+\alpha}}}{(1+\alpha)\left(2 - e^{\frac{\beta}{1+\alpha}}\right)} \le 1.
\end{aligned}$$

Applying now Lemma 1.2 and the fact that $\Re(n\gamma+1) > \Re(\gamma)$, the function $\mathcal{G}_{\alpha_1,\dots,\alpha_n,\beta_1,\dots,\beta_n,n,\gamma}(z)$ defined by

$$\mathcal{G}_{\alpha_1,\dots,\alpha_n,\beta_1,\dots,\beta_n,n,\gamma}(z) = \left[(n\gamma+1)\int_0^z t^{n\gamma} \prod_{j=1}^{n} \left(\frac{\mathbb{R}_{\alpha_j,\beta_j}(t)}{t}\right)^{\gamma} dt\right]^{1/(n\gamma+1)},$$

then belongs to the class $\mathcal{S}$. The proof of Theorem 2.2 is complete. □

By choosing $n = 1$, and also $\alpha_1 = \alpha, \beta_1 = \beta$ in Theorem 2.2, we have the following result.

Corollary 2.2. *Let the parameters* $\alpha \ge 0$, $\beta > 0$ *be so constrained that*

$$\frac{\beta}{1+\alpha} < \ln(2).$$

Consider the functions $\mathbb{R}_{\alpha,\beta} : \mathbb{U} \to \mathbb{C}$ *defined by* (1.2), *and let* $\Re(\gamma) > 0$ *be such that the following inequality holds true:*

$$|\gamma| \le \frac{(1+\alpha)\left(2 - e^{\frac{\beta}{1+\alpha}}\right)}{\beta e^{\frac{\beta}{1+\alpha}}} \Re(\gamma).$$

Then the function $\mathcal{G}_{\alpha,\beta,\gamma}(z):\mathbb{U}\to\mathbb{C}$ *defined by*

$$\mathcal{G}_{\alpha,\beta,\gamma}(z)=\left[(\gamma+1)\int_0^z\left(\mathbb{R}_{\alpha,\beta}(t)\right)^{\gamma}dt\right]^{1/(\gamma+1)},$$

is in the class $\mathcal{S}$ *of normalized univalent functions in* $\mathbb{U}$.

By setting $n=1,\alpha_1=\alpha,\beta_1=\beta$ and $\gamma=1$ in Theorem 2.2, we get the following result.

Corollary 2.3. *Let the parameters* $\alpha\geq 0$, $\beta>0$ *be so constrained that*

$$\frac{\beta}{1+\alpha}\leq W(2e)-1,$$

where W(2e) is given by (1.9). *Consider the functions* $\mathbb{R}_{\alpha,\beta}:\mathbb{U}\to\mathbb{C}$ *defined by* (1.2)*, then the function* $\mathcal{G}_{\alpha,\beta}(z):\mathbb{U}\to\mathbb{C}$ *defined by*

$$\mathcal{G}_{\alpha,\beta}(z)=\left[2\int_0^z\mathbb{R}_{\alpha,\beta}(t)dt\right]^{1/2}$$

is in the class $\mathcal{S}$ *of normalized univalent functions in* $\mathbb{U}$.

As before, we give below few examples of Corollary 2.3.

Example 2.3. Putting $\alpha_1=0$, $\beta_1=1/4$ in Corollary 2.3 and using (1.3), we have

$$\mathcal{G}_{0,1/4}(z)=\left[2\int_0^z te^{t/4}dt\right]^{1/2}=2\left[2\left(z-4\right)\mathrm{e}^{\frac{z}{4}}+8\right]^{1/2}\in\mathcal{S}.$$

Example 2.4. Next, putting $\alpha=1$ and $\beta=1/4$ in Corollary 2.3 and using (1.4), we have

$$\begin{aligned}\mathcal{G}_{1,1/4}(z)&=2\left[\int_0^z\sqrt{t}\sinh\left(\frac{\sqrt{t}}{2}\right)dt\right]^{1/2}\\&=4\left[\left((z+8)\cosh\left(\frac{\sqrt{z}}{2}\right)-4\sqrt{z}\sinh\left(\frac{\sqrt{z}}{2}\right)\right)-8\right]^{1/2}\in\mathcal{S}.\end{aligned}$$

The following result contains another set of sufficient univalence conditions for integral operators of the type (1.7) when the functions h_j $(j=1,...,n)$ are normalized forms of Rabotnov functions involving various parameters. We shall apply Lemma 1.1 and the inequality (1.13) of Lemma 1.4 to prove the following theorem.

Theorem 2.3. *Let the parameters* $\alpha_j\geq 0$, $\beta_j>0$ *and*

$$\frac{\beta}{1+\alpha}=\max\left\{\frac{\beta_j}{1+\alpha_j}\middle|(j=1,...,n)\right\}\tag{2.9}$$

be so constrained that

$$\frac{\beta}{1+\alpha}<W(2e)-1\ \text{ with }\ \alpha\geq 0,\ \beta>0,$$

where $W(2e)$ *is given by* (1.9). *Consider the functions* $\mathbb{R}_{\alpha_j,\beta_j}:\mathbb{U}\to\mathbb{C}$ *defined by* (2.2)*, and also let*

$$\Re(\xi)>0,c\in\mathbb{C}\backslash\{-1\}\text{ with }|c|\leq 1\quad\text{ and }\quad\nu_j\in\mathbb{C}\backslash\{0\}\ \ (j=1,...,n)$$

satisfy the following inequality:

$$|c| + \frac{1}{|\xi|} \frac{\beta(2+2\alpha+\beta)e^{\frac{\beta}{1+\alpha}}}{(1+\alpha)\left(2(1+\alpha)-(1+\alpha+\beta)e^{\frac{\beta}{1+\alpha}}\right)} \sum_{j=1}^{n} |\nu_j| \leq 1.$$

Then the function $\mathcal{H}_{\alpha_1,\ldots,\alpha_n,\beta_1,\ldots,\beta_n,\nu_1,\ldots,\nu_n,\xi}(z) : \mathbb{U} \to \mathbb{C}$ *defined by*

$$\mathcal{H}_{\alpha_1,\ldots,\alpha_n,\beta_1,\ldots,\beta_n,\nu_1,\ldots,\nu_n,\xi}(z) = \left[\xi \int_0^z t^{\xi-1} \prod_{j=1}^{n} \left(\mathbb{R}'_{\alpha_j,\beta_j}(t)\right)^{\nu_j} dt\right]^{1/\xi}$$

is in the class $\mathcal{S}$ *of normalized univalent functions in* $\mathbb{U}$.

Proof. Our demonstration of proof of Theorem 2.3 is similar to that of Theorem 2.1. Indeed, by considering the function $\mathcal{H}_{\alpha_1,\ldots,\alpha_n,\beta_1,\ldots,\beta_n,\nu_1,\ldots,\nu_n}(z) : \mathbb{U} \to \mathbb{C}$ defined by

$$\mathcal{H}_{\alpha_1,\ldots,\alpha_n,\beta_1,\ldots,\beta_n,\nu_1,\ldots,\nu_n}(z) = \int_0^z \prod_{j=1}^{n} \left(\mathbb{R}'_{\alpha_j,\beta_j}(t)\right)^{\nu_j} dt,$$

we observe that $\mathcal{H}_{\alpha_1,\ldots,\alpha_n,\beta_1,\ldots,\beta_n,\nu_1,\ldots,\nu_n} \in \mathcal{A}$, that is

$$\mathcal{H}_{\alpha_1,\ldots,\alpha_n,\beta_1,\ldots,\beta_n,\nu_1,\ldots,\nu_n}(0) = \mathcal{H}'_{\alpha_1,\ldots,\alpha_n,\beta_1,\ldots,\beta_n,\nu_1,\ldots,\nu_n}(0) - 1 = 0.$$

On the other hand, it is easy to see that

$$\mathcal{H}'_{\alpha_1,\ldots\alpha_n,\beta_1,\ldots,\beta_n,\nu_1,\ldots,\nu_n}(z) = \prod_{j=1}^{n} \left(\mathbb{R}'_{\alpha_j,\beta_j}(z)\right)^{\nu_j} \tag{2.10}$$

and

$$\mathcal{H}''_{\alpha_1,\ldots,\alpha_n,\beta_1,\ldots,\beta_n,\nu_1,\ldots,\nu_n}(z) = \sum_{j=1}^{n} \nu_j \left(\mathbb{R}'_{\alpha_j,\beta_j}(z)\right)^{\nu_j-1} \cdot \mathbb{R}''_{\alpha_j,\beta_j}(z) \prod_{\substack{k=1 \\ (k\neq j)}}^{n} \left(\mathbb{R}'_{\alpha_k,\beta_k}(z)\right)^{\nu_k}. \tag{2.11}$$

Applying (2.10) and (2.11), we have

$$\frac{z\mathcal{H}''_{\alpha_1,\ldots,\alpha_n,\beta_1,\ldots,\beta_n,\nu_1,\ldots,\nu_n}(z)}{\mathcal{H}'_{\alpha_1,\ldots,\alpha_n,\beta_1,\ldots,\beta_n,\nu_1,\ldots,\nu_n}(z)} = \sum_{j=1}^{n} \nu_j \frac{z\mathbb{R}''_{\alpha_j,\beta_j}(z)}{\mathbb{R}'_{\alpha_j,\beta_j}(z)}.$$

Now, by making use of the inequality (1.13) of Lemma 1.4 for each α_j, β_j ($j = 1, \ldots, n$), we obtain

$$\begin{aligned}\left|\frac{z\mathcal{H}''_{\alpha_1,\ldots,\alpha_n,\beta_1,\ldots,\beta_n,\nu_1,\ldots,\nu_n}(z)}{\mathcal{H}'_{\alpha_1,\ldots,\alpha_n,\beta_1,\ldots,\beta_n,\nu_1,\ldots,\nu_n}(z)}\right| &\leq \sum_{j=1}^{n} |\nu_j| \left|\frac{z\mathbb{R}''_{\alpha_j,\beta_j}(z)}{\mathbb{R}'_{\alpha_j,\beta_j}(z)}\right| \\ &\leq \sum_{j=1}^{n} |\nu_j| \frac{\beta(2+2\alpha_j+\beta_j)e^{\frac{\beta_j}{1+\alpha_j}}}{(1+\alpha_j)\left(2(1+\alpha_j)-(1+\alpha_j+\beta_j)e^{\frac{\beta_j}{1+\alpha_j}}\right)}.\end{aligned} \tag{2.12}$$

Let

$$\psi(x) = \frac{x(2+x)e^x}{2-(1+x)e^x} \implies \psi'(x) = \frac{e^x \cdot \left[2x^2+8x+4-\left(x^2+2x+2\right)e^x\right]}{\left[2-(x+1)\,e^x\right]^2}.$$

Clearly, $\psi'(x)$ is positive for $x \in (0, W(2e)-1)$ and hence $\psi(x)$ is an increasing function of x for $x \in (0, W(2e)-1)$. Thus, in view of (2.9), we have

$$\psi\left(\frac{\beta_j}{1+\alpha_j}\right) \leq \psi\left(\frac{\beta}{1+\alpha}\right).$$

Using now (2.12), we obtain

$$\left|\frac{z\mathcal{H}''_{\alpha_1,\ldots,\alpha_n,\beta_1,\ldots,\beta_n,\nu_1,\ldots,\nu_n}(z)}{\mathcal{H}'_{\alpha_1,\ldots,\alpha_n,\beta_1,\ldots,\beta_n,\nu_1,\ldots,\nu_n}(z)}\right| \leq \frac{\beta(2+2\alpha+\beta)e^{\frac{\beta}{1+\alpha}}}{(1+\alpha)\left(2(1+\alpha)-(1+\alpha+\beta)e^{\frac{\beta}{1+\alpha}}\right)}\sum_{j=1}^{n}|\nu_j|.$$

Finally, from the triangle inequality and the assertion of Theorem 2.3, we get

$$\begin{aligned}\left|c|z|^{2\xi}+\left(1-|z|^{2\xi}\right)\frac{z\mathcal{H}''_{\alpha_1,\ldots,\alpha_n,\beta_1,\ldots,\beta_n,\nu_1,\ldots,\nu_n}(z)}{\xi\mathcal{H}'_{\alpha_1,\ldots,\alpha_n,\beta_1,\ldots,\beta_n,\nu_1,\ldots,\nu_n}(z)}\right| &\leq |c|+\frac{1}{|\xi|}\left|\frac{z\mathcal{H}''_{\alpha_1,\ldots,\alpha_n,\beta_1,\ldots,\beta_n,\nu_1,\ldots,\nu_n}(z)}{\mathcal{H}'_{\alpha_1,\ldots,\alpha_n,\beta_1,\ldots,\beta_n,\nu_1,\ldots,\nu_n}(z)}\right| \\ &\leq |c|+\frac{\beta(2+2\alpha+\beta)e^{\frac{\beta}{1+\alpha}}}{(1+\alpha)\left(2(1+\alpha)-(1+\alpha+\beta)e^{\frac{\beta}{1+\alpha}}\right)}\sum_{j=1}^{n}\frac{|\nu_j|}{|\xi|} \leq 1,\end{aligned}$$

which in view of Lemma 1.1 implies that $\mathcal{H}_{\alpha_1,\ldots,\alpha_n,\beta_1,\ldots,\beta_n,\nu_1,\ldots,\nu_n,\xi}(z) \in \mathcal{S}$. This evidently completes the proof of Theorem 2.3. $\square$

By setting $n=1$, and also $\alpha_1=\alpha, \beta_1=\beta, \nu_1=\nu$ and $\xi=1$ in Theorem 2.3, we immediately obtain the following result.

Corollary 2.4. *Let the parameters $\alpha \geq 0$ and $\beta > 0$ be so constrained that*

$$\frac{\beta}{1+\alpha} < W(2e)-1.$$

Consider the functions $\mathbb{R}_{\alpha,\beta}: \mathbb{U} \to \mathbb{C}$ defined by (1.2), and also let

$$c \in \mathbb{C}\backslash\{-1\} \text{ with } |c| \leq 1 \quad \text{and} \quad \nu \in \mathbb{C}\backslash\{0\}$$

satisfy the following inequality:

$$|c|+\frac{\beta(2+2\alpha+\beta)e^{\frac{\beta}{1+\alpha}}}{(1+\alpha)\left(2(1+\alpha)-(1+\alpha+\beta)e^{\frac{\beta}{1+\alpha}}\right)}|\nu| \leq 1.$$

Then the function $\mathcal{H}_{\alpha,\beta,\nu}(z): \mathbb{U} \to \mathbb{C}$ defined by

$$\mathcal{H}_{\alpha,\beta,\nu}(z) = \int_0^z \left(\mathbb{R}'_{\alpha,\beta}(t)\right)^{\nu} dt$$

is in the class $\mathcal{S}$ of normalized univalent functions in $\mathbb{U}$.

Example 2.5. If we put $\alpha=0$, $\beta=1/8, c=0$ and $\nu=2$ in Corollary 2.4 and use (1.3), we have then

$$\begin{aligned}\mathcal{H}_{0,1/8,2}(z) &= \int_0^z \left(\frac{t}{8}+1\right)^2 e^{t/4}dt \\ &= \frac{1}{16}\left[\left(z^2+8z+32\right)e^{z/4}-32\right] \in \mathcal{S}.\end{aligned}$$

Example 2.6. Next, by setting $\alpha=1$, $\beta=1/4, c=0$ and $\nu=1$ in Corollary 2.4 and using (1.4), we have

$$\mathcal{H}_{1,1/4,1}(z) = \mathbb{R}_{1,1/4}(z) = 2\sqrt{z}\sinh\left(\frac{\sqrt{z}}{2}\right) \in \mathcal{S}.$$

By applying Lemma 1.3 and the inequality (1.11) of Lemma 1.4, we can establish the following result which follows straightforwardly from the details mentioned in the proof of Theorem 2.3.

Theorem 2.4. *Let the parameters $\alpha \geq 0$ and $\beta > 0$ be so constrained that*

$$\frac{\beta}{1+\alpha} < W(2e) - 1,$$

where $W(2e)$ is given by (1.9). Consider the functions $\mathbb{R}_{\alpha,\beta} : \mathbb{U} \to \mathbb{C}$ defined by (1.2), and also let $\Re(\delta) \geq 1$ and

$$|\delta| \leq \frac{3\sqrt{3}}{2\left(1 + \frac{\beta}{1+\alpha}\right) e^{\frac{\beta}{1+\alpha}}}.$$

Then the function $\mathcal{Q}_{\alpha,\beta,\delta}(z) : \mathbb{U} \to \mathbb{C}$ defined by

$$\mathcal{Q}_{\alpha,\beta,\delta}(z) = \left[\delta \int_0^z t^{\delta-1} \left(e^{\mathbb{R}_{\alpha,\beta}(t)}\right)^{\delta} dt\right]^{1/\delta}$$

is in the class $\mathcal{S}$ of normalized univalent functions in $\mathbb{U}$.

For $\delta = 1$ in Theorem 2.4, we immediately have the following result.

Corollary 2.5. *Let the parameters $\alpha \geq 0$ and $\beta > 0$ be so constrained that*

$$\frac{\beta}{1+\alpha} < W(2e) - 1.$$

Consider the functions $\mathbb{R}_{\alpha,\beta} : \mathbb{U} \to \mathbb{C}$ defined by (1.2). Then the function $\mathcal{Q}_{\alpha,\beta}(z) : \mathbb{U} \to \mathbb{C}$ defined by

$$\mathcal{Q}_{\alpha,\beta}(z) = \int_0^z e^{\mathbb{R}_{\alpha,\beta}(t)} dt$$

is in the class $\mathcal{S}$ of normalized univalent functions in $\mathbb{U}$.

3. Convexity Of Integral Operators Involving The Rabotnov Functions

In this section, we first prove the following theorem which gives the sufficient conditions of convexity for integral operators of the type (1.5) when the functions f_j $(j = 1, , n)$ are normalized forms of Rabotnov functions. The proof is based on the application of Lemma 1.1.

Theorem 3.1. *Let the parameters $\alpha_j \geq 0$, $\beta_j > 0$ and*

$$\frac{\beta}{1+\alpha} = \max\left\{\frac{\beta_j}{1+\alpha_j}\middle|(j = 1, ..., n)\right\}$$

be so constrained that

$$\frac{\beta}{1+\alpha} < \ln(2) \quad \textit{with} \quad \alpha \geq 0,\ \beta > 0.$$

Consider the functions $\mathbb{R}_{\alpha_j,\beta_j} : \mathbb{U} \to \mathbb{C}$ defined by (2.2), and suppose also that

$$\lambda_j \in \mathbb{C}\backslash\{0\} \quad (j = 1, ..., n)$$

satisfy the following inequality:

$$1 - \frac{\frac{\beta}{(1+\alpha)} e^{\frac{\beta}{1+\alpha}}}{2 - e^{\frac{\beta}{1+\alpha}}} \sum_{j=1}^{n} \frac{1}{|\lambda_j|} \geq 0. \tag{3.1}$$

Then the function $\mathcal{F}_{\alpha_1,\ldots,\alpha_n,\beta_1,\ldots,\beta_n,\lambda_1,\ldots,\lambda_n}(z) : \mathbb{U} \to \mathbb{C}$ *defined by*

$$\mathcal{F}_{\alpha_1,\ldots,\alpha_n,\beta_1,\ldots,\beta_n,\lambda_1,\ldots,\lambda_n}(z) = \int_0^z \prod_{j=1}^{n} \left(\frac{\mathbb{R}_{\alpha_j,\beta_j}(t)}{t} \right)^{1/\lambda_j} dt$$

is in the class $\mathcal{K}$ *of normalized convex functions in* $\mathbb{U}$.

Proof. Proceeding similarly as in Theorem 2.1, we have

$$\frac{z\mathcal{F}''_{\alpha_1,\ldots.\alpha_n,\beta_1,\ldots,\beta_n,\lambda_1,\ldots.,\lambda_n}(z)}{\mathcal{F}'_{\alpha_1,\ldots.\alpha_n,\beta_1,\ldots,\beta_n,\lambda_1,\ldots.,\lambda_n}(z)} = \sum_{j=1}^{n} \frac{1}{\lambda_j} \left(\frac{z\mathbb{R}'_{\alpha_j,\beta_j}(z)}{\mathbb{R}_{\alpha_j,\beta_j}(z)} - 1 \right).$$

Hence,

$$\begin{aligned}
\Re\left[1 + \frac{z\mathcal{F}''_{\alpha_1,\ldots,\alpha_n,\beta_1,\ldots,\beta_n,\lambda_1,\ldots,\lambda_n}(z)}{\mathcal{F}'_{\alpha_1,\ldots,\alpha_n,\beta_1,\ldots,\beta_n,\lambda_1,\ldots,\lambda_n}(z)}\right] &= \Re\left[1 + \sum_{j=1}^{n} \frac{1}{\lambda_j} \left(\frac{z\mathbb{R}'_{\alpha_j,\beta_j}(z)}{\mathbb{R}_{\alpha_j,\beta_j}(z)} - 1 \right)\right] \\
&= 1 + \Re\left[\sum_{j=1}^{n} \frac{1}{\lambda_j} \left(\frac{z\mathbb{R}'_{\alpha_j,\beta_j}(z)}{\mathbb{R}_{\alpha_j,\beta_j}(z)} - 1 \right)\right] \\
&\geq 1 - \left|\sum_{j=1}^{n} \frac{1}{\lambda_j} \left(\frac{z\mathbb{R}'_{\alpha_j,\beta_j}(z)}{\mathbb{R}_{\alpha_j,\beta_j}(z)} - 1 \right)\right| \\
&\geq 1 - \sum_{j=1}^{n} \frac{1}{|\lambda_j|} \left| \frac{z\mathbb{R}'_{\alpha_j,\beta_j}(z)}{\mathbb{R}_{\alpha_j,\beta_j}(z)} - 1 \right| \\
&\geq 1 - \sum_{j=1}^{n} \frac{1}{|\lambda_j|} \frac{\frac{\beta_j}{(1+\alpha_j)} e^{\frac{\beta_j}{1+\alpha_j}}}{2 - e^{\frac{\beta_j}{1+\alpha_j}}} \qquad \text{(using (1.12))} \\
&\geq 1 - \frac{\frac{\beta}{(1+\alpha)} e^{\frac{\beta}{1+\alpha}}}{2 - e^{\frac{\beta}{1+\alpha}}} \sum_{j=1}^{n} \frac{1}{|\lambda_j|} \text{ (in view of (2.8))} \\
&\geq 0 \text{ (in view of (3.1))}.
\end{aligned}$$

This implies that $\mathcal{F}_{\alpha_1,\ldots,\alpha_n,\beta_1,\ldots,\beta_n,\lambda_1,\ldots,\lambda_n}(z) \in \mathcal{K}$ and the proof of Theorem 3.1 is complete. □

Setting $n = 1, \lambda_1 = \lambda$, and also $\alpha_1 = \alpha, \beta_1 = \beta$ in Theorem 3.1, we immediately obtain the following result.

Corollary 3.1. *Let the parameters* $\alpha \geq 0$, $\beta > 0$ *and* $\lambda \in \mathbb{C}\backslash\{0\}$ *be so constrained that*

$$\frac{\beta}{1+\alpha} \leq W(2|\lambda|e^{|\lambda|}) - |\lambda|.$$

Consider the functions $\mathbb{R}_{\alpha,\beta} : \mathbb{U} \to \mathbb{C}$ *defined by* (1.2), *then the function* $\mathcal{F}_{\alpha,\beta,\lambda}(z) : \mathbb{U} \to \mathbb{C}$ *defined by*

$$\mathcal{F}_{\alpha,\beta,\lambda}(z) = \int_0^z \left(\frac{\mathbb{R}_{\alpha,\beta}(t)}{t} \right)^{1/\lambda} dt$$

is in the class $\mathcal{K}$ *of normalized convex functions in* $\mathbb{U}$.

Remark 3.1. If we put $\lambda = 1$ in Corollary 3.1, we then get the same result as obtained earlier in Theorem 3.4 of [35].

Example 3.1. Putting $\alpha = 0$, $\beta = 1/4$ and $\lambda = 1/2$ in Corollary 3.1 and using (1.3), we have

$$\mathcal{F}_{0,1/4,1/2}(z) = \int_0^z \left(e^{t/4}\right)^2 dt = 2\left[e^{z/2} - 1\right] \in \mathcal{K} \text{ (See below Figure(a)).}$$

Example 3.2. Next, putting $\alpha = 1$ and $\beta = 1/4$ and $\lambda = 1$ in Corollary 3.1 and using (1.4), we get

$$\mathcal{F}_{1,1/4,1}(z) = 2\int_0^z \frac{\sinh\left(\frac{\sqrt{t}}{2}\right)}{\sqrt{t}} dt = 8\left[\cosh\left(\frac{\sqrt{z}}{2}\right) - 1\right] \in \mathcal{K}.$$

The following result contains another set of sufficient convexity conditions for integral operators of the type (1.7) when the functions h_j $(j = 1, ..., n)$ are normalized forms of Rabotnov functions involving various parameters. The key tools in the proof are Lemma 1.1 and the inequality (1.13) of Lemma 1.4.

Theorem 3.2. *Let the parameters* $\alpha_j \geq 0$, $\beta_j > 0$ *and*

$$\frac{\beta}{1+\alpha} = \max\left\{\frac{\beta_j}{1+\alpha_j}\middle| (j = 1, ..., n)\right\}$$

be so constrained that

$$\frac{\beta}{1+\alpha} < W(2e) - 1 \quad \textit{with} \quad \alpha \geq 0,\ \beta > 0.$$

Consider the functions $\mathbb{R}_{\alpha_j,\beta_j} : \mathbb{U} \to \mathbb{C}$ *defined by* (2.2), *also let*

$$\nu_j \in \mathbb{C}\backslash\{0\} \quad (j = 1, ..., n)$$

satisfy the following inequality:

$$1 - \frac{\beta(2+2\alpha+\beta)e^{\frac{\beta}{1+\alpha}}}{(1+\alpha)\left(2(1+\alpha) - (1+\alpha+\beta)e^{\frac{\beta}{1+\alpha}}\right)} \sum_{j=1}^n |\nu_j| \geq 0.$$

Then the function $\mathcal{H}_{\alpha_1,...,\alpha_n,\beta_1,...,\beta_n,\nu_1,...,\nu_n}(z) : \mathbb{U} \to \mathbb{C}$, *defined by*

$$\mathcal{H}_{\alpha_1,...,\alpha_n,\beta_1,...,\beta_n,\nu_1,...,\nu_n}(z) = \int_0^z \prod_{j=1}^n \left(\mathbb{R}'_{\alpha_j,\beta_j}(t)\right)^{\nu_j} dt$$

is in the class $\mathcal{K}$ *of normalized convex functions in* $\mathbb{U}$.

Proof. Proceeding similarly as in Theorem 2.3, we have

$$\begin{aligned}\Re\left[1 + \frac{z\mathcal{H}''_{\alpha_1,...,\alpha_n,\beta_1,...,\beta_n,\nu_1,...,\nu_n}(z)}{\mathcal{H}'_{\alpha_1,...,\alpha_n,\beta_1,...,\beta_n,\nu_1,...,\nu_n}(z)}\right] &\geq 1 - \left|\frac{z\mathcal{H}''_{\alpha_1,...,\alpha_n,\beta_1,...,\beta_n,\nu_1,...,\nu_n}(z)}{\mathcal{H}'_{\alpha_1,...,\alpha_n,\beta_1,...,\beta_n,\nu_1,...,\nu_n}(z)}\right| \\ &\geq 1 - \frac{\beta(2+2\alpha+\beta)e^{\frac{\beta}{1+\alpha}}}{(1+\alpha)\left(2(1+\alpha) - (1+\alpha+\beta)e^{\frac{\beta}{1+\alpha}}\right)} \sum_{j=1}^n |\nu_j| \geq 0.\end{aligned}$$

This implies that $\mathcal{H}_{\alpha_1,...,\alpha_n,\beta_1,...,\beta_n,\nu_1,....,\nu_n}(z) \in \mathcal{K}$ and the proof of Theorem 3.2 is complete. □

If we set $n = 1$, and also $\alpha_1 = \alpha, \beta_1 = \beta, \nu_1 = \nu$ in Theorem 3.2, we immediately obtain the following result.

Corollary 3.2. *Let the parameters $\alpha \geq 0$ and $\beta > 0$ be so constrained that*

$$\frac{\beta}{1+\alpha} < W(2e) - 1.$$

Consider the functions $\mathbb{R}_{\alpha,\beta} : \mathbb{U} \to \mathbb{C}$ defined by (1.2), and also $\nu \in \mathbb{C}\backslash\{0\}$ satisfy the following inequality:

$$1 - \frac{\beta(2+2\alpha+\beta)e^{\frac{\beta}{1+\alpha}}}{(1+\alpha)\left(2(1+\alpha)-(1+\alpha+\beta)e^{\frac{\beta}{1+\alpha}}\right)}|\nu| \geq 0.$$

Then the function $\mathcal{H}_{\alpha,\beta,\nu}(z) : \mathbb{U} \to \mathbb{C}$ defined by

$$\mathcal{H}_{\alpha,\beta,\nu}(z) = \int_0^z \left(\mathbb{R}'_{\alpha,\beta}(t)\right)^{\nu} dt$$

is in the class $\mathcal{K}$ of normalized convex functions in $\mathbb{U}$.

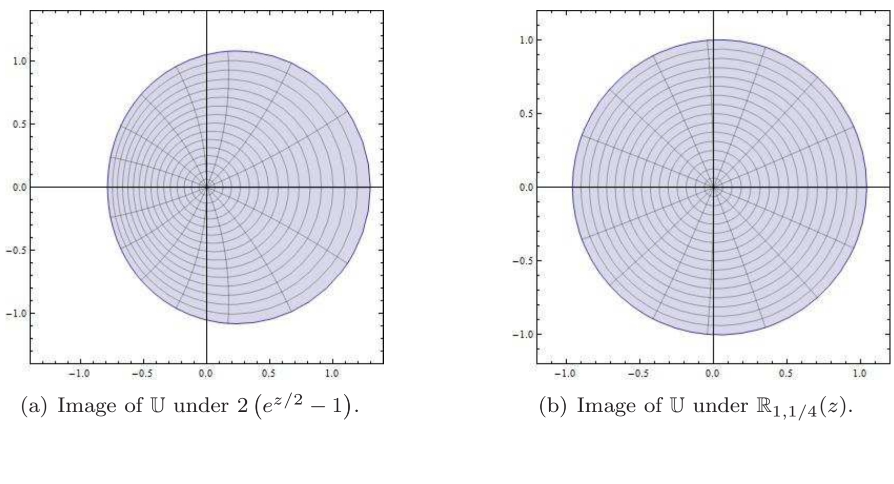
Remark 3.2. Setting $n = 1$, and also $\alpha_1 = \alpha, \beta_1 = \beta, \nu_1 = 1$ in Corollary 3.2, we obtain the same result as obtained in Theorem 3.2 of [35].

Example 3.3. Next, putting $\alpha = 0$, $\beta = 1/8$ and $\nu = 2$ in above Corollary 3.2 and using (1.3), we have

$$\begin{aligned}\mathcal{H}_{0,1/8,2}(z) &= \int_0^z \left(\frac{t}{8}+1\right)^2 e^{t/4} dt \\ &= \frac{1}{16}\left[\left(z^2+8z+32\right)e^{z/4} - 32\right] \in \mathcal{K}.\end{aligned}$$

Example 3.4. Further, putting $\alpha = 1$, $\beta = 1/4$ and $\nu = 1$ in Corollary 3.2 and using (1.4), we have

$$\mathcal{H}_{1,1/4,1}(z) = \mathbb{R}_{1,1/4}(z) = 2\sqrt{z}\sinh\left(\frac{\sqrt{z}}{2}\right) \in \mathcal{K} \text{ (See Figure(b)).}$$



(a) Image of $\mathbb{U}$ under $2\left(e^{z/2}-1\right)$.

(b) Image of $\mathbb{U}$ under $\mathbb{R}_{1,1/4}(z)$.

Data availability statement: Data is available within the article.

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