The Cyclic Subgroup Separability of Certain HNN Extensions

P.C. Wong and K.B. Wong
Institute of Mathematical Sciences, University of Malaya, 50603 Kuala Lumpur, Malaysia
1wongpc@um.edu.my, 2kbwong@um.edu.my

Abstract. In this note we give characterisations for certain HNN extensions with central associated subgroups to be cyclic subgroup separable. We then apply our results to HNN extensions of polycyclic-by-finite groups and Fuchsian groups.

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1. Introduction

A group $G$ is called cyclic subgroup separable (or $\pi_c$ for short) if for each cyclic subgroup $H$ and $x \in G \setminus H$, there exists a normal subgroup $N$ of finite index in $G$ such that $x \notin HN$. Clearly a cyclic subgroup separable group is residually finite. The concept of cyclic subgroup separability was introduced by Stebe [14] in 1968 and he used it to prove the residual finiteness of a class of knot groups.

Many classes of groups, including the free groups and the polycyclic-by-finite groups are cyclic subgroup separable ([7], [10]). Also the finite extension of a cyclic subgroup separable group is again cyclic subgroup separable (Stebe [14]). On the other hand, the cyclic subgroup separability of HNN extensions are not much known. For example, the HNN extension $\langle h, t; t^{-1}ht = h^2 \rangle$ is residually finite but is not cyclic subgroup separable (see [1]) while another HNN extension, the Baumslag-Solitar group, $\langle h, t; t^{-1}h^2t = h^3 \rangle$ is not even residually finite (see [5]).

Kim [8] and Kim and Tang [9] gave characterisations for HNN extensions of cyclic subgroup separable groups with cyclic associated subgroups to be again cyclic subgroup separable. They then apply their results to give characterisations for the HNN extensions of a finitely generated abelian group with cyclic associated subgroups to be cyclic subgroup separable and show that certain HNN extensions of finitely generated torsion-free nilpotent groups with cyclic associated subgroups to be again cyclic subgroup separable.

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Andreadakis, Raptis and Varsos in a series of papers [2], [3], [4] and [11], gave characterisations for HNN extensions of a finitely generated abelian group to be residually finite. Some of these results were extended by Wong and Tang [15] to characterisations for HNN extensions of finitely generated abelian groups to be cyclic subgroup separable.

In this note we investigate the cyclic subgroup separability of HNN extensions with central associated subgroups. More precisely, let $G = \langle t, A; t^{-1}Ht = K, \varphi \rangle$ denote an HNN extension where $A$ is the base group, $H, K$ are the associated subgroups and $\varphi : H \rightarrow K$ is the associated isomorphism. We shall show that if $G = \langle t, A; t^{-1}Ht = K, \varphi \rangle$ is an HNN extension where $H$ and $K$ are subgroups in the center of $A$, $H \neq A \neq K$ and $A$ is subgroup separable, then $G$ is cyclic subgroup separable if and only if its subgroup (HNN extension) $G_1 = \langle t, HK; t^{-1}Ht = K, \varphi \rangle$ is cyclic subgroup separable. Thus we are able to use the results of [11] and [15] to give characterisations for HNN extensions of polycyclic-by-finite groups and Fuchsian groups with central associated subgroups to be cyclic subgroup separable.

More importantly, our result shows that the study of the cyclic subgroup separability of HNN extensions with central associated subgroups can be reduced to that of the cyclic subgroup separability of HNN extensions of abelian groups. Thus the characterisations given in the papers [11] and [15] can be applied to these HNN extensions.

The notation used here is standard. In addition, the following will be used for any group $G$: $N \trianglelefteq_f G$ means $N$ is a normal subgroup of finite index in $G$.

2. Preliminaries

Definition 2.1. A group $G$ is called $H$-separable for the subgroup $H$ if for each $x \in G \setminus H$, there exists $N \trianglelefteq_f G$ such that $x \notin HN$. The group $G$ is termed subgroup separable if $G$ is $H$-separable for every finitely generated subgroup $H$. The group $G$ is termed cyclic subgroup separable (or $\pi_c$ for short) if $G$ is $H$-separable for every cyclic subgroup $H$.

It is well known that free groups, polycyclic groups and surface groups are subgroup separable (M. Hall [7], Mal’cev [10], Scott [12]). Since a finite extension of a subgroup separable group is again subgroup separable, polycyclic-by-finite groups and Fuchsian groups (finite extension of surface groups) are subgroup separable.

3. The main results

In this section we will prove our main results, i.e., Theorem 3.2 and Theorem 3.3. To simplify our exposition we will use the term $\pi_c$ instead of cyclic subgroup separable for the rest of the paper.

The following lemma is an application of a result of Blass and Newman in [6].

Lemma 3.1. Let $G = \langle t, A; t^{-1}Ht = K, \varphi \rangle$ be an HNN extension. If $K = A$ and $H \neq A$, then $G$ is not $H$-separable.

Remark 3.1. In this note we consider the HNN extension $G = \langle t, A; t^{-1}Ht = K, \varphi \rangle$ where $H$ and $K$ are subgroups in the center of $A$. Note that if $H = A = K$, then $A$ is abelian. Furthermore $A$ is normal in $G$ and $G/A \cong \langle t \rangle$. Hence $G$ is polycyclic.
and by [10], \( G \) is subgroup separable and hence \( \pi_c \). Now by Lemma 3.1, if \( K = A, H \neq A \) and \( H \) is cyclic, then \( G \) is not \( \pi_c \). So we shall only consider the HNN extensions where \( H \neq A \neq K \).

**Theorem 3.1.** Let \( G = \langle t, A; t^{-1}Ht = K, \varphi \rangle \) be an HNN extension. Suppose \( H \) and \( K \) are subgroups in the center of \( A \) and \( H \neq A \neq K \). Let \( \Delta = \{ N \triangleleft_f A \mid \varphi(N \cap H) = N \cap K \} \). Then \( G \) is \( \pi_c \) if and only if \( \bigcap_{N \in \Delta} NH = H \), \( \bigcap_{N \in \Delta} NK = K \) and \( \bigcap_{N \in \Delta} N\langle a \rangle = \langle a \rangle, \forall a \in A \).

**Proof.** Suppose \( G \) is \( \pi_c \). Then \( G \) is residually finite. Suppose \( A = HK \). Since \( H \) and \( K \) are in the center of \( A \), the subgroup \( HK \) satisfies the nontrivial identity \( W(x, y) = x^{-1}y^{-1}xy \). Furthermore \( H \nsubseteq K \) and \( K \nsubseteq H \) because \( H \neq A = HK \neq K \). Therefore by [13, Theorem 3], \( \bigcap_{N \in \Delta} NH = H \) and \( \bigcap_{N \in \Delta} NK = K \).

Suppose \( A \neq HK \). Let \( c \in A - HK \). Then the subgroup generated by \( c \) and \( HK \), i.e., \( \langle c, HK \rangle \), is abelian and satisfies the nontrivial identity \( W(x, y) = x^{-1}y^{-1}xy \). Furthermore \( HK \) is properly contained in \( \langle c, HK \rangle \). Therefore by [13, Theorem 3], \( \bigcap_{N \in \Delta} NH = H \) and \( \bigcap_{N \in \Delta} NK = K \).

So in both cases we have \( \bigcap_{N \in \Delta} NH = H \) and \( \bigcap_{N \in \Delta} NK = K \).

Next we show that \( \bigcap_{N \in \Delta} N\langle a \rangle = \langle a \rangle \). Let \( a \in A \) and \( x \notin \langle a \rangle \). Since \( G \) is \( \pi_c \), there exists \( M \triangleleft_f G \) such that \( x \notin M\langle a \rangle \). Note that \( (M \cap A) \in \Delta \) and \( x \notin (M \cap A)\langle a \rangle \). This implies that \( \bigcap_{N \in \Delta} N\langle a \rangle = \langle a \rangle \).

The converse follows from [8, Theorem 2.2]. \( \square \)

By using Theorem 3.1, we shall prove Theorem 3.2 and Theorem 3.3. Before that, we state Lemma 3.2, whose proof is immediate from Theorem 3.1 and the result of [6].

**Lemma 3.2.** Let \( G = \langle t, A; t^{-1}Ht = K, \varphi \rangle \) be an HNN extension where \( H \) and \( K \) are subgroups in the center of \( A \) and \( H \neq A \neq K \). Suppose \( G \) is \( \pi_c \). If \( H \subseteq K \) or \( K \subseteq H \) then \( H = K \).

**Theorem 3.2.** Let \( G = \langle t, A; t^{-1}Ht = H, \varphi \rangle \) be an HNN extension where \( H \) is a finitely generated subgroup in the center of \( A \) and \( H \neq A \). If \( A \) is \( H^n \langle a \rangle \)-separable for every \( a \in A \) and every positive integer \( n \), then \( G \) is \( \pi_c \).

**Proof.** Let \( N \in \Delta \) where \( \Delta = \{ N \triangleleft_f A \mid \varphi(N \cap H) = N \cap H \} \). By Theorem 3.1, it is sufficient to show that \( \bigcap_{N \in \Delta} NH = H \) and \( \bigcap_{N \in \Delta} N\langle a \rangle = \langle a \rangle, \forall a \in A \).

First we show that \( \bigcap_{N \in \Delta} NH = H \). Let \( a \in A - H \). Since \( A \) is \( H \)-separable, there exists \( M_a \triangleleft_f A \) such that \( a \notin M_aH \). Note that \( M_aH \triangleleft_f A \) and \( M_aH \in \Delta \). This implies that \( \bigcap_{a \in A - H} M_aH = H \) and hence \( \bigcap_{N \in \Delta} NH = H \).

Next we show that \( \bigcap_{N \in \Delta} N\langle a \rangle = \langle a \rangle, \forall a \in A \). But first we construct a subgroup \( N_n \in \Delta \) for each \( n \geq 2 \). Let \( h_0, h_1, \ldots, h_n \) be coset representatives of \( H^n \) in \( H \) where \( h_0 = 1 \) and \( n \geq 2 \). Since \( A \) is \( H^n \)-separable, there exists \( M_n \triangleleft_f A \) such that
Suppose $j \in A$ then we are done. So we may assume that $m \neq b$, and therefore $b \notin b/a$ where $m \in M$ and $h \in H^n$. But then $h \in M_nH^n$, a contradiction. Hence $N_n \cap H \subseteq H^n$. Therefore $N_n \in \Delta$ for each $n \geq 2$.

Let $a \in A$ and $b \in A - \langle a \rangle$. We claim that $b \notin H^n(a)$ for some $n \geq 1$. If $b \notin H(a)$, then we are done. So we may assume that $b \in H(a)$. Let $b = ha^j$ for some $h \in H$ and integer $j$. Clearly $h \notin H \cap \langle a \rangle$. Since $H$ is finitely generated abelian, there exists an integer $n \geq 1$, such that $h \notin H^n(H \cap \langle a \rangle)$. This implies that $h \notin H^n(a)$, and therefore $b \notin H^n(a)$.

Since $A$ is $H^n\langle a \rangle$-separable, there exists $M_1 \trianglelefteq A$ such that $b \notin M_1H^n\langle a \rangle$. Note that for sufficiently large $n$, $H^n \subseteq M_1$. As above, we can construct $N_n \in \Delta$ such that $N_n \cap H = H^n$. Let $M = N_n \cap M_1$. Then $M \in \Delta$ since $M \cap H = H^n$. Furthermore $b \notin MH^n\langle a \rangle$. This implies that $b \notin M\langle a \rangle$ and so $\bigcap_{N \in \Delta} N\langle a \rangle = \langle a \rangle$.

The proof is now completed. \hfill \Box

**Theorem 3.3.** Let $G = \langle t, A; t^{-1}Ht = K, \varphi \rangle$ be an HNN extension where $H$ and $K$ are subgroups in the center of $A$ and $H \neq A \neq K$. Suppose $H \notin K, K \notin H$ and $A$ is $M\langle a \rangle$-separable for every subgroup $M \trianglelefteq HK$ and $a \in A$. Then $G$ is $\pi_c$ if and only if $G_1 = \langle t, HK; t^{-1}Ht = K, \varphi \rangle$ is $\pi_c$.

**Proof.** Suppose $G$ is $\pi_c$. Since $G_1$ is a subgroup of $G$, $G_1$ is $\pi_c$.

Suppose $G_1$ is $\pi_c$. Since $H \neq HK \neq K$, then by Theorem 3.1, $\bigcap_{M \in \Delta_1} MH = H, MK = K$ and $\bigcap_{M \in \Delta_1} M\langle x \rangle = \langle x \rangle, \forall x \in HK$, where $\Delta_1 = \{ M \trianglelefteq HK ; \varphi(M \cap H) = M \cap K \}$.

Let $\Delta = \{ N \trianglelefteq A ; \varphi(N \cap H) = N \cap K \}$. By Theorem 3.1, it is sufficient to show that $\bigcap_{N \in \Delta} NH = H, \bigcap_{N \in \Delta} NK = K$ and $\bigcap_{N \in \Delta} N\langle a \rangle = \langle a \rangle, \forall a \in A$.

We begin by constructing a subgroup $N_{(b,a,M)}(b,a,M) \in \Delta$ for each $M \in \Delta_1$, $a \in A$ and $b \in A - M\langle a \rangle$. Let $M \in \Delta_1, a \in A$ and $b \in A - M\langle a \rangle$. Let $h_0, h_1, \ldots, h_m$ be coset representatives of $M$ in $HK$ where $h_0 = 1$. Since $A$ is $M$-separable, there exists $P_M \trianglelefteq A$ such that $h_i \notin P_M$ for $1 \leq i \leq m$. Since $A$ is $M\langle a \rangle$-separable, there exists $P_{(b,a)} \trianglelefteq A$ such that $b \notin P_{(b,a)} M\langle a \rangle$. Let $P_{(b,a,M)} = P_{(b,a)} \cap P_M$ and $N_{(b,a,M)} = P_{(b,a,M)} M$. Then $N_{(b,a,M)} \trianglelefteq A$ and $b \notin N_{(b,a,M)}\langle a \rangle$. Next we claim that $N_{(b,a,M)} \cap HK = M$. Clearly we need only to show that $N_{(b,a,M)} \cap HK \subseteq M$.

Suppose $y \in N_{(b,a,M)} \cap HK - M$. Since $y \notin M$, then $y = h_i m_1$ where $h_i \neq 1$ is a coset representative of $M$ in $HK$ and $m_1 \in M$. On the other hand, since $y \in N_{(b,a,M)} = P_{(b,a,M)} M$, we have $y = pm_2$ where $p \in P_{(b,a,M)}$ and $m_2 \in M$. But this implies that $h_i \in P_{(b,a,M)} M \subseteq P_M M$, a contradiction. Thus $N_{(b,a,M)} \cap HK = M$. This implies that $N_{(b,a,M)} \cap H = M \cap H$ and $N_{(b,a,M)} \cap K = M \cap K$. Hence $N_{(b,a,M)} \in \Delta$.

Next we show that $\bigcap_{N \in \Delta} NH = H$. Let $a \in A - H$. Suppose $a \in A - HK).

Since $A$ is $HK$-separable, there exists $M_a \trianglelefteq A$ such that $a \notin M_a HK$. Note that $M_a HK \trianglelefteq A$ and $M_a HK \in \Delta$. Suppose $a \in HK$. Since $a \in HK - H$, there exists
\( M \in \Delta_1 \) such that \( a \notin MH \). As above, we can construct a subgroup \( N_{(a,1,M)} \) for \( M \in \Delta_1, 1 \in A \) and \( a \in A - M(1) \). We claim that \( a \notin N_{(a,1,M)}H \). Suppose \( a \in N_{(a,1,M)}H \). Then \( a = nh \) for some \( n \in N_{(a,1,M)} \) and \( h \in H \). This implies that \( n \in N_{(a,1,M)} \cap HK \). But \( N_{(a,1,M)} \cap HK = M \) by its construction above. Hence \( a \in MH \), a contradiction. Therefore \( a \notin N_{(a,1,M)}H \) and thus \( \bigcap_{N \in \Delta} NH = H \).

Similarly \( \bigcap_{N \in \Delta} NK = K \).

Finally we show that \( \bigcap_{N \in \Delta} N \langle a \rangle = \langle a \rangle, \forall a \in A \). Let \( a \in A \) and \( b \in A - \langle a \rangle \).

Suppose \( b \notin HK \langle a \rangle \). Since \( A \) is \( HK \langle a \rangle \)-separable, there exists \( M \triangleleft_f A \) such that \( b \notin MHK \langle a \rangle \). Note that \( MHK \triangleleft_f A \) and \( MHK \in \Delta \). Suppose \( b \in HK \langle a \rangle \). Then \( b = xa^i \) for some \( x \in HK \) and integer \( i \). Clearly \( x \notin HK \cap \langle a \rangle \). Therefore there exists \( M_1 \in \Delta_1 \) such that \( x \notin M_1(HK \cap \langle a \rangle) \). This implies that \( b \notin M_1 \langle a \rangle \). As above, we can construct a subgroup \( N_{(b,a,M_1)} \) for \( M_1 \in \Delta_1, a \in A \) and \( b \in A - M_1 \langle a \rangle \). From this construction, we have \( N_{(b,a,M)} \in \Delta \) and \( b \notin N_{(b,a,M)} \langle a \rangle \). Hence \( \bigcap_{N \in \Delta} N \langle a \rangle = \langle a \rangle \). The proof is now completed. \( \square \)

4. Applications

In this section we will apply the results in section 3 to HNN extensions of polycyclic-by-finite groups and Fuchsian groups. But first we have the following lemma.

**Lemma 4.1.** Let \( A \) be a group and \( H \) and \( K \) be isomorphic finitely generated subgroups in the center of \( A \) such that \( \varphi : H \longrightarrow K \) is an isomorphism from \( H \) onto \( K \). Let \( \Delta_1 = \{ M \triangleleft_f H ; \varphi(M \cap H) = M \cap K \} \). Suppose \( \bigcap_{M \in \Delta_1} MH = H \)

and \( \bigcap_{M \in \Delta_1} MK = K \). Then there exists \( N \in \Delta_1 \) such that \( N^n \in \Delta_1 \) for all \( n \geq 1 \).

**Proof.** Let \( i_{HK}(H) = \{ b \in HK; b^n \in H \) for some positive integer \( n \}\}. Then \( i_{HK}(H) \) is a group and \( H \) is of finite index in \( i_{HK}(H) \). Similarly let \( i_{HK}(K) = \{ b \in HK; b^n \in K \) for some positive integer \( n \}\}. Then \( i_{HK}(K) \) is a group and \( K \) has finite index in \( i_{HK}(K) \). Since \( \bigcap_{M \in \Delta_1} MH = H \), \( \bigcap_{M \in \Delta_1} MK = K \) and \( H \) and \( K \) are of finite index in \( i_{HK}(H) \) and \( i_{HK}(K) \) respectively, there exists \( N \in \Delta_1 \) such that \( N \cap \bigcap_{i \in i_{HK}(H)} = N \cap H \) and \( N \cap \bigcap_{i \in i_{HK}(K)} = N \cap K \). Furthermore \( N^n \cap H = (N \cap H)^n \) and \( N^n \cap K = (N \cap K)^n \). This implies that \( N^n \in \Delta_1 \) for all \( n \geq 1 \). \( \square \)

**Theorem 4.1.** Let \( G = \langle t, A; t^{-1}Ht = K, \varphi \rangle \) be an HNN extension where \( H \) and \( K \) are finitely generated subgroups in the center of \( A \). Suppose \( A \) is subgroup separable and \( H \neq A \neq K \). Then \( G \) is \( \pi_c \) if and only if one of the following holds:

(a) \( H = K \)

(b) \( H \not\unlhd K, K \not\unlhd H \) and there exists a torsion free subgroup \( N \triangleleft_f HK \) such that \( \varphi(N \cap H) = N \cap K \) and \( N \cap K, N \cap H \) are isolated in \( N \).

**Proof.** Suppose \( G \) is \( \pi_c \) and suppose \( H \neq K \). If \( H \subset K \), then by Lemma 3.2, \( H = K \), a contradiction. Therefore \( H \neq HK \) and similarly \( K \neq HK \). So by Theorem 3.3, \( G_1 = \langle t, HK; t^{-1}Ht = K, \varphi \rangle \) is \( \pi_c \) and therefore residually finite. Then by [11, Theorem], there exists a torsion free subgroup \( N \triangleleft_f HK \) such that \( \varphi(N \cap H) = N \cap K \) and \( N \cap K, N \cap H \) are isolated in \( N \).
Conversely suppose $H = K$. Then $G$ is $\pi_c$ by Theorem 3.2. Next suppose $H \neq K$. Then $H \neq HK \neq K$. If there exists a torsion free group $N \triangleleft HK$ such that $\varphi(N \cap H) = N \cap K$ and $N \cap K, N \cap H$ are isolated in $N$, then $G_1 = \langle t, HK; t^{-1}Ht = K, \varphi \rangle$ is residually finite by [11, Theorem]. We will show that $G_1$ is $\pi_c$. By Theorem 3.3, it is sufficient to show that $\bigcap_{M \in \Delta_1} MH = H$, $\bigcap_{M \in \Delta_1} MK = K$ and $\bigcap_{M \in \Delta_1} M(x) = \langle x \rangle$, $\forall x \in HK$, where $\Delta_1 = \{ M \triangleleft HK : \varphi(M \cap H) = M \cap K \}$.

First we show that $\bigcap_{N \in \Delta_1} NH = H$ and $\bigcap_{N \in \Delta_1} NK = K$. Since $H$ and $K$ are in the center of $G$, the subgroup $HK$ satisfies the nontrivial identity $W(x, y) = x^{-1}y^{-1}xy$. Therefore by [13, Theorem 3'], $\bigcap_{N \in \Delta_1} NH = H$ and $\bigcap_{N \in \Delta_1} NK = K$.

Next we will show that $\bigcap_{M \in \Delta_1} M(x) = \langle x \rangle$, $\forall x \in HK$. Let $y \in HK - \langle x \rangle$. Since $HK$ is abelian and so subgroup separable, there exists $M \triangleleft HK$ such that $y \notin M(x)$. By Lemma 4.1, there exists $N \in \Delta_1$ such that $N^n \in \Delta_1$ for all $n \geq 1$. Since $M$ is of finite index in $HK$, there exists a positive integer $n$ such that $N^n \subseteq M$. Thus $y \notin N^n \langle x \rangle$ and so $\bigcap_{M \in \Delta_1} M(x) = \langle x \rangle$. Hence we have shown that $G_1$ is $\pi_c$. Now by Theorem 3.3, $G$ is $\pi_c$.

**Corollary 4.1.** Let $G = \langle t, A; t^{-1}Ht = K, \varphi \rangle$ be an HNN extension where $A$ is a polycyclic-by-finite group or a Fuchsian group. Suppose $H$ and $K$ are finitely generated subgroups in the center of $A$ and $H \neq A \neq K$. Then $G$ is $\pi_c$ if and only if one of the following holds:

(a) $H = K$
(b) $H \not\subseteq K, K \not\subseteq H$ and there exists a torsion free subgroup $N \triangleleft HK$ such that $\varphi(N \cap H) = N \cap K$ and $N \cap K, N \cap H$ are isolated in $N$.

**Proof.** Since polycyclic-by-finite groups and Fuchsian groups are subgroup separable, the corollary follows from Theorem 4.1.

Another application from Section 3 is the following result.

**Corollary 4.2.** Let $G = \langle t, A; t^{-1}Ht = K, \varphi \rangle$ be an HNN extension where $A$ is a polycyclic-by-finite group or a Fuchsian group. Suppose $H$ and $K$ are finitely generated subgroups in the center of $A$ such that $H \cap K$ is finite. Then $G$ is $\pi_c$.

**Proof.** By Theorem 3.3, $G$ is $\pi_c$ if and only if $G_1 = \langle t, HK; t^{-1}Ht = K, \varphi \rangle$ is $\pi_c$. But by [15, Theorem 2], $G_1$ is $\pi_c$. Hence $G$ is $\pi_c$.

**References**


