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JOVAN KARAMATA
1902-1967

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Jovan Karamata comes from a Serbian merchant family from the city of Zemun, which is almost three centuries old. Business relations of his family, on the borders of the Austro-Hungarian and Turkish empires and Serbia, were very well known and widespread. Jovan Karamata was born in Zagreb on February 1, 1902, but he always considered Zemun as his native city. In 1914 he finished there the first three classes of high school. Because of the war and since Zemun was on the border at that time, Karamata's father has sent him, together with his brothers and his sister, to Switzerland. In Lausanne, in 1920, he finished high school oriented towards mathematics and sciences. In the same year he enrolled at the Faculty of Engineering of Belgrade University and, after finished two years of technical studies, he moved to the Faculty of Philosophy - Department of Mathematics, where he graduated in 1925. Let us mention that he submitted his doctoral thesis three months after his graduation. Obviously, during the last year of his studies he was involved in active research, and he studied the scientific literature. He spent the years 1927-28 in Paris, as a fellow of the Rockefeller Foundation, and in 1928 he became Assistant for Mathematics at the Faculty of Philosophy of Belgrade University. In 1930 he was Assistant Professor, in 1937 Associate Professor and, after the end of World War II, in

1950 he became Full Professor. In 1951 he was elected Full Professor at the University of Geneva.

In 1933 he was elected a Corresponding Member of the Yugoslav Academy of Sciences and Arts (Zagreb). and in 1936 a Corresponding Member of regia Societas Scientiarum Bohemica (Prague). In 1939 he became a Corresponding member, and in 1948 a Full Member of the Serbian Academy of Sciences and Arts.

He was one of the founders of the Mathematical Institute of the Serbian Academy of Sciences and Arts in 1946, and the first Editor-in-Chief of the journal Publications de l'Institut Mathématique (deuxième série). He had a great influence on many students of mathematics in Belgrade and Serbia during the years 1946-1953.

In 1954 he became member of the Editorial Board and Director of the journal L'Enseignement Mathématique from Geneva. His influence on the form and content of this journal was significant.

In 1931 he married Emilija Nikolajević, who gave birth to their two sons and a daughter. His wife died in 1959.

After a long illness, Jovan Karamata died on August 14, 1967 in Geneva. His ashes rest in his native town of Zemun.

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The development of mathematical sciences in Serbia starts with Mihailo Petrović (1864-1943). Petrović was a student of famous french mathematicians: Ch. Hermite, H. Poincaré, E. Picard, P. Painlevé. P. Appell. During his studies in Paris, at the École Normale Supérieure, his student fellows were: E. Borel. H. Lebesgue, E. Cartan. who marked this epoch in mathematics. Together with a few colleagues, Petrović not only brought to Serbia scientific ideas and theories from Europe, but also the spirit of science from the end of the 19th and the beginning of the 20th century. Young people, their students, accepted new ideas and theories, but their teachers attracted them also as scientists, not only as mathematicians. Jovan Karamata, who abandoned technical sciences, was attracted by the personality and work of Mihailo Petrović. But after World War I, in 1920, Petrović was tired and the world to which he belonged with all his soul, was vanishing. His creative years were 1900-1914, and his well-known results belonged to the past. Research in the area of differential equations, theory of functions and in particular of functions defined by entire series, was replaced by new fields of investigation and by new ideas. In spite of this, Petrović had influence on the research career of Jovan Karamata, especially at the beginning. Karamata accepted his advice that one should work in modern areas and seek solutions by new methods, and in particular that one should study the contemporary literature. Already as a student, he noticed the importance of the Stieltjes integral, and he used this integral to represent finite and infinite sums in a unique way. He mastered this tool and used it in his first works and also later. Still, Karamata is self-taught in science, and he sought for the proper inspiration in the literature. Perhaps he did not know the words of Abel, but he followed them: it is better to read masters than students. Karamata kept saying that his teacher of classical analysis was the famous book of PÓlya and Szegö [1]. From this book he learned what in classical analysis was called the mathematical style: using minimal means to acquire the greatest possibility of clear expression and natural order in proofs, without using phrases such as "it is easy to prove" and "obviously" ... Undoubtedly at that time, around 1925, there existed important results proved by ingenious and beautiful techniques, which could serve as a model and standard. A large number of such model proofs, which can serve as an example, can be found in the book of Pólya and Szegö mentioned above and in its references.

Today, classical analysis has no such importance and attraction as it had 70 years ago, in the time when Karamata devoted all his days to this branch of mathematics. Without certain basic notions of modern mathematics, such as the notion of structure, common properties of different mathematical topics, classical analysis looks nowadays for many mathematicians like a collection of isolated propositions. If some among them do not see the beauty of an isolated proposition, it may look to them as art for the art's sake. But the real results in classical mathematics and in every branch of it will not disappear in the future. These are the results which belong neither to some time, not to some special area - these are the results which open the way of progress in science. new results, new consequences and new theories originate from these results. They are the stimulus for new aspirations. D. Hilbert thought that the abundance and diversity of important areas of mathematics require precise and simple methods in resolving basic and essential problems of these areas, which then unite various ideas. But the discovered essence of these foundations, become unexpectedly new areas having their own independent development. On the other hand, new simple

[1] G. Pólya and G. Szegó. Aufgaben und Lehrsátze aus der Analysis, Springer (Berlin), 1925.

proofs which enlighten the whole problem, also give new results and bring important generalizations. Every discipline reaches its summit; only its substance remains.

The work of Jovan Karamata belongs to classical analysis, where it has reached its summit. Karama had the power and the properties of the great analysts of this epoch. He loved elegance in proof, which inspires enthusiasm for mathematics, and original proofs and unexpected ideas, which are more frequent among young mathematicians. He shared with G. H. Hardy the opinion that mathematics is a young people's game. The best three works of Karamata [15], [16], [17] (the numbers refer to the bibliography of Karamata) belongs to the time of his youth, and they became very famous.

As we said, Karamata has found inspiration for his thesis in the book of Pólya and Szegö (T. I, section 4), mentioned above. His thesis contains a generalization of Weyl's[2] notion of uniform distribution. The central problem studied in his thesis is the following one:

Let $\{a_{\nu n}\}$ *be a bounded, double sequence of real numbers,* $a < a_{\nu n} < b$, *and let* $f(x)$ *be an* R*-integrable function in* $[a,b]$; *find conditions on the sequence* $\{a_{\nu n}\}$ *and on the function* $f(x)$ *that guarantee the existence of the following limit*

$$\lim_{n\to\infty} A_n(f) = \lim_{n\to\infty}\left\{\frac{1}{n}\sum_{\nu=1}^{n} f(a_{\nu n})\right\} = A(f).$$

Karamata introduced first the notion of uniform distribution of a double sequence defined by

$$\nu(x) = \lim_{n\to\infty}\frac{r_n(x)}{n}$$

where $r_n(x)$ is the number of $a_{\nu n}$ less than x, for n fixed. For an ordinary sequence and if $\nu(x) = x$, this reduces to Weyl's problem of uniform distribution of numbers modulo one. He finds that $A(f)$ exists whenever the distribution function $\nu(x)$ exists, for all except countably many points of $[a,b]$, and excluding the discontinuities of $\nu(x)$ and $f(x)$. In this case

$$A(f) = \int_0^1 f[\mu(\xi)]d\xi,$$

where $\mu(x)$ is the inverse of $\nu(x)$. He gives several necessary and sufficient conditions for the existence of $\nu(x)$, generalizing the results of Weyl, Pólia

[2]H. Weyl, Úber die Gleichverteilung von Zahlen mod. Eins, Math. Ann., Bd. 77 (1916)

and Fejér. Using previous results, Karamata determines the domain of convergence of the series $\sum a_\nu P_\nu(x)$, where $P_\nu(x)$ are polynomials with real coefficients. This problem reduces to a problem of the distribution $\nu(x)$ of zeros of the polynomial $P_\nu(x)$. In the case when $P_\nu(x)$ are orthogonal polynomials, the problem was formulated are solved by Szegó[3].

Unfortunately, Karamata's thesis, published in Serbian, remained unknown to the majority of mathematicians working in the area of uniform distribution. let us mention that a number of results from his thesis was proved independently and almost at the same time in the beautiful thesis of Schoenberg[4].

Karamata published in Comptes Rendus [3] a theorem related to linear functionals $A_n(f)$, which he used in his thesis. It is the following theorem:

Let $A_n(f)$ and $A(f)$ be linear functionals of continuous functions on $[a, b]$, and let $\alpha_n(x)$ and $\alpha(x)$ be their generating functions, where $\alpha_n(a) = \alpha(a) = 0$ and $\alpha(x)$ is nondecreasing; then in order that the sequence $A_n(f)$ converge to $A(f)$ for every continuous function $f(x)$ it is necessary and sufficient that the sequence $\alpha_n(x)$ converges to $\alpha(x)$ in every continuity point of $\alpha(x)$.

Jacques Hadamard realized the importance of this simple theorem (see the same Comptes Rendus immediately after Karamata's note). This theorem can also be found in the well-known book of Riesz and Nagy[5] (page 121).

Karamata once said: " I Wanted to improve my knowledge of the foundations of the theory of functions. This is the reason why I first started to study the theory of series, but when I entered into this theory, I found so many old and new results that I remained there." In this area of analysis he obtained beautiful and important results, which made his name famous. In the first place these are his results about the summability, and especially the proof of Littlewood's[6] theorem on the converse of Abel's theorem. This

[3]G. Szegó, Úber die Entwickelung einer analytischen Funktion nach den Polynomen eines Orthogonalsystems, Math. Ann. 82 (1921) 188-212.

[4]I. Schoenberg, Über die asymptotische Verteilung reeller Zahlen mod. 1, Math. Zeit., Bd. 28 (1928) 171-199.

[5]Frédéric Riesz et Béla Sz.-Nagy, Leçons d'analyse fonctionnelle, Gautheir Villars, 1935.

[6]J. E. Littlewood, The converse of Abel's theorem on power series, proc. London Math. Soc. (2), vol. 9 (1910) 434-448; G. H. Hardy and J. E. Littlewood, Tauberian theorems concerning power series ..., Proc. London Math. Soc. (2) 13 (1914) 174-191.

is the classical theorem:

Suppose that $f(x) = \sum a_n x^n$ is convergent for $|x| < 1$ and that $f(x)$ tends to a finite limit s as $x \to 1-0$ (i.e. that the series $\sum a_n$ is A-summable); if $a_n = \mathcal{O}(1/n)$, then the series $\sum a_n$ converges and its sum is s.

Today there exists a large and rich literature on inverse theorems - called theorems of the Tauberian type.

The first theorem which carries Tauber's[7] name (1897) states that the convergence follows from A-summability (and from summability in general) under an additional condition, which is here $aa_n \to 0$. A part of real analysis bears his name. Littlewood[6] (1910) replaced the Tauberian condition $aa_n \to 0$ by the condition $na_n = \mathcal{O}(1)$, and this condition is so essential and important, that this branch of analysis starts with it. While the proof of the Tauberian theorem is very simple (though theorem is not at all trivial), the proof of Littlewood's theorem is not only complicated, but also involved and very long. Hardy and Littlewood[6] (1913) gave a new proof using auxiliary theorems connected with the relations between real functions and their derivatives. This proof, including the auxiliary results, has several pages and also some references to previous theorems. Still, their proof uses only simple methods of analysis. In 1925, Schmidt [8] gave the proof of this theorem using new notions and auxiliary theorems, but his proof was neither simple nor short. In his monograph, Landau [9] says that Littlewood's theorem is a very deep one, but the proof is complicated. then Karamata [16] published his spectacular proof of two pages.

The elementary Karamata method is based on the Weierstrass theorem on approximation by polynomials. If we put

$$g(x) = \begin{cases} 0, & 0 \le x \le \frac{1}{\mathrm{e}} \\ 1, & \frac{1}{\mathrm{e}} \le x \le 1 \end{cases} ,$$

[7]A. Tauber, Ein Satz aus der Theorie der unendlichen Reihen. Monatshefte für Math. 8 (1897), 273-277.

[8]R. Schmidt, Über das Borelsche Summierungsverfahren, Schriften der Königsberger gelehrten Gesellschaft, I (1925) 202-256; Über divergente Folgen und lineare Mittelbildungen, M. Zeit., 22 (1925) 89-152.

[9]E. Landau, Darsellung und Begründung einiger neurer Ergebnisse der Funktionentheorie, Springer, Berlin (1930); Bull. Acad. Belgique 1911, see Pólya and Szegö, vol. I, pp. 231, where the term "langsam wachsende Funktion" as well as the notation $L(x)$ and $L(cr)/L(r) \to 1$, $c > 1$, are introduced.

then a polynomial $p(x)$ such that

$$\int_0^1 |x^{-1}g(x) - p(x)|dx < \varepsilon$$

corresponds to every $\varepsilon > 0$. The proof is obvious for the power functions x^n, and for the polynomial the proof is carried by using approximation.

A survey of all methods used in the proofs of Littlewood's theorem can be found in the article of Ingham[10]. Among the elementary ones, Karamata's proof stands out, and it is cited as a model and a standard in all well-known monographs in this field, for example in those by E. C. Titchmarsh, G. H. Hardy. K. Knopp. A. Zygmund, D. V. Widder, G. Doetsh. The importance of Littlewood's theorem lies not only in the fact that it is one of the crucial inverse theorems, but that it is also an important result in the Theory of Functions. R. Schmidt, in his review in the Zentralblatt, says that Karama's proof is "an important discovery". K. Knopp[11] says that this is a simple, surprising proof (Ein úberaschend Beweis). In 1978, on the occasion of the 60th anniversary of the Mathematische Zeitschrift, this work of Karamata's was chosen among the first 50 most important papers selected between several hundreds of papers.

From Littlewood's theorem one can deduce the order of the remainder $|s_n - s| = o(1/\lg n)$, and this was done by Postnikov[12] and finally by Freud[13] and Korevaar[14]. The remainder is obtained essentially from the Weierstrass approximation theorem, i.e. from its generalization given by Jackson[15]. It is obtained by using classical approximation theorems (Weierstrass and Jackson) with a help of a method similar to Karamata's; and because of that it is sometimes called Karamata's elementary method. This order of the remainder term is used in number theory. This addition to the Littlewood theorem,

[10] A. E. Ingham, Proc. London Math. Soc. 14a (1965) 157-173.

[11] K. Knopp, Theorie und Anwendung der Unendlichen Reihen, Dritte Auflage, Springer, berlin, 1931.

[12] A. G. Postnikov, The remainder term in the Tauberian theorem of Hardy and Littlewood, Doklady Akad. Nauk. SSSR (N.S), 77 (1951), 193-196.

[13] G. Freud, Restglied eines Tauberschen Satzes I, Acta. Math. Acad. Sci. Hung. 2 (1951) 299-308.

[14] J. Korevaar, Best L_1 Approximation and the Remainder in Littlewood's Theorem, Konikl. Nederl. Akademie van Wetenschappen, Amsterdam, A. 56, 3; Indag. Math. 15 (3) (1953) 281-293.

[15] D. Jackson, Über die Genauigkeit der Annäherung stetiger Funktionen durch ganze rationale Funktionen gegebenen Grades und trigonometrische Summen gegebener Ordnung (Diss. Göttingen 1911); The theory of approximation, 1930.

which is numerically important, follows from elementary procedures, that is from Karamata's proof. Wielandt[16] showed that one can avoid the step in the proof of Littlewood's theorem using C-summability; that is, one can avoid Hardy's lemma and every auxiliary theorem in the proof, except for the Weierstrass approximation theorem. Let us note that the proof of Hardy's lemma is today quite simple.

The complete survey of Tauberian theorems with remainder was given by Ganelius [17]

The big and very important treatise of Wiener[18] appeared in 1932. Using Fourier transforms, Wiener gave the most general theory of Tauberian theorems. Wiener's theory not only includes the most important forms of general inverse theorems, but also contains new results outside this area, some of which are the deepest theorems in analysis, having a wide range of application, from integral equations to the distribution of primes (S. Ikehara). Every analyst can feel the depth of the ideas and of the techniques of many results that were scattered before Wiener. But the synthesis and the essence, and also the structure of the whole problem, can be understood only by studying Wiener's theory. This theory is based on Fourier transforms and convolutions. If we denote by $\bar{f}$ the Fourier transform of $f \in L_1$, i.e.

$$(2\pi)^{-1.2}\int_{-\infty}^{\infty} f(x)e^{iux}dx,$$

and by $\bar{K}_1$ the Fourier transform of K_1, then, if the condition $\bar{K}_1(u) \neq 0$ holds for every real u, we arrive to the special Wiener theorem of the Tauberian type which says:

If

$$\lim_{x\to\infty}\int K_1(x-y)a(y)dy = A\int K_1(y)dy,$$

where $\bar{K}_1(u) \neq 0$ for every real $u, K_1 \in L_1$, then

$$\lim_{y\to\infty} a(y) = A,$$

provided that $a(y)$ belongs to the class of functions satisfying the Tauberian conditions in a precisely defined way stated below. Today such functions are called *slowly oscillating functions.* The approximation by the functions of

[16] N. Wielandt, Zur Umkehrung des Abelschen Stetigkeitsatzes, Math. Zeit. 56, (1952) 206-207.

[17] T. Ganelius, Tauberian Remainder Theorems, Lecture Notes in Math., vol. 131, 1971.

[18] N. Wiener, Tauberian Theorems, Annals of mathematics, Vol. 33, No. 1 (1932) 1-100.

the subspace generated by the translations of K_1 plays a fundamental role in the whole of Wiener's theory: if $K_1 \in L_1$, $K \in L_1$, $\bar{K}_1(u) \neq 0$ for every real u, then to every $\varepsilon \geq 0$, there corresponds a sequence A_n and a sequence of reals ξ_n, such that

$$\int |K(x) - \sum A_n K_1(x + \xi_n)| dx \leq \varepsilon,$$

and $\bar{K}_1(u) \neq 0$ is the necessary and sufficient condition for this approximation. This theorem was very important in the creation of abstract harmonic analysis, and also in the development of some modern branches of contemporary mathematics.

In the scope of Wiener's theory one finds the greatest number of inverse theorems of the general form, even those which at the first glance do not appear to be of Tauberian nature. But numerical problems remain out of the reach of harmonic analysis. Special problems: the connections between the relationships of different summability procedures, estimation of the remainder, asymptotic expansion of the sum s_n, cannot be seen in the general theory. Here the breadth of general harmonic analysis can be seen in the general outlook on the whole problem including a number of scattered results, while classical analysis reveals the inner structure of a problem by using special means, by a breakthrough into the core of a single isolated problem.

Wiener's theory was presented by Pitt[19], 1938, in a most precise and concise way. In order to arrive to Karamata's result in this theory, through Fourier transforms, one should take the transform of $f(x) = \mathrm{e}^x \mathrm{e}^{\mathrm{e}^x}$, with the translation of the form $f(x + \lg\mu)$. Even with this, it is not easy to arrive to the result without long calculations. It is not easy, because the auxiliary propositions in the general theory require many explanations. This is why Karamata's proof remains classical, and this is why Wiener says: "the elegant method of Karamata (Wiener[18], p. 51) requires only one textbook theorem: the Weierstrass approximation theorem."

The abstract version of Wiener's theory was given by Beurling[20]. Among the others who have spread these abstract ideas one should mention the names of I. Gelfand, R. Godement and S. L. Segal.

[19] H. R. Pitt, General Tauberian Theorems, Proc. London Math. Soc., vol. 44 (1938) 243-288.

[20] A. Beurling, Un théorème sur les fonctions bornées uniformément continues sur l'axe réel, Acta Math., 77 (1945) 127-136.

The first one who noticed the role of approximation in Tauberian theorems was Schmidt[8], when he studied the moment problem. Karamata introduced the approximation using the Weierstrass Theorem as the main tool in the proof, and in this way he added an important theorem to the mathematical literature. Wiener's approximation of Fourier transforms was not only a breakthrough in harmonic analysis as a new method, but also a source of new discoveries. The crucial role there was played by approximations of various kinds in Tauberian theorems.

On the other hand, the weakening of conditions for convergence in Tauberian theorems is not only a formal matter, but raises the question of the scope of inverse theorems in general. Landau replaced Littlewood's condition $na_n = \mathcal{O}(1)$ by the one-sided condition: $na_n \geq -K$ and in this way he made an important step in Tauberian theorems. According to Wiener, the most important conditions for convergence are those which originate from Schmidt[8], Schmidt's conditions, contrary to those containing only one element, are such that one should consider a group of elements. These conditions can be expressed in the simplest way by using slowly varying sequences, where the sequence $s_n = a_0 + a_1 + \ldots + a_n$, such that $\lim\limits_{p\to\infty}(s_p - s_q) = 0$, where $p \to \infty$, $+q \to \infty$, $q \geq p$, and $w/p \to 1$, is *slowly varying.*

Sequence of this kind were introduced by Schmidt under the name "sehr langsam oscillierende Folgen". Today they are called *slowly varying sequences.* Already Landau[7] uses this name "sehr langsame Folgen". Obviously, functions satisfying the condition

$$\lim_{y-x\to 0,\, x\to\infty}[f(y) - f(x)] = 0$$

are also called *slowly varying.* Schmidt used some of the properties, which appear in calculations, usually in studying conditions for convergence of the general kind, which showed the importance of such functions in Tauberian theorems.

In the proof of Littlewood's theorem 1928-1929, Karamata noticed Schmidt's slowly varying functions, but he left aside convergence conditions in Tauberian theorems and started original investigations on the fundamental structure and meaning of these functions. Nobody suspected that under the surface of this notion, almost superficial after all, lies a notion of real analysis, not clearly defined though it is in fact almost elementary. The significance and the wide use of this notion, so many years after its appearance, and so many years after the death of Karamata, speaks about its role in real analysis, not only in Tauberian convergence conditions, but also in the final form of many other theorems. Karamata took from Schmidt (or already from Landau[7]) the term itself for such functions; he called them "slowly decreasing (increasing) functions (functions á croissance lente)" and denoted them by $L(x)$, Schmidt stated some properties of these functions (in the form of sequences) which were almost evident, but Karamata was the first to discover the structure of these functions $L(x)$ and their fundamental properties. Schmidt introduced these functions using differences, while Landau had already expressed them through the quotient $f(y)/f(x)$. Finally, Karamata [15], [37] gave a really precise definition: *a function $L(x)$ defined for $x \geq 0$ is slowly varying if it is positive, continuous such that*

$$\frac{L(\lambda x)}{L(x)} \to 1, \quad x \to \infty \tag{1}$$

for every fixed $\lambda \geq 0$.

It is Karamata's big achievement that he derived a whole theory from this simple definition [15], [37]. From (1) Karamata obtained the canonical representation for such functions:

$$L(x) = c(x)\exp \int_0^x \frac{\varepsilon(u)}{u} du, \tag{2}$$

where $c(x) \to c > 0$ and $\varepsilon(x) \to 0$ as $x \to \infty$, and also another fundamental property (about uniform convergence): the relation (1) holds uniformly with respect to $\lambda \in [a, b]$ for fixed $0 < a \leq b < \infty$. Some of the properties of slowly varying functions can be used as the initial definition.

The creation of the theory of slowly varying functions is a landmark. There are not many theories today which are so general, and which arise from such a simple definition as (1).

As stated before, a simple proof can sometimes give new results. Karamata replaces conditions of the form $x^k \lg_k^\alpha x$, where the iterated logarithm appears in Hardy-Littlewood theorem (1913), by $L(x)$ and in this way he gives - this is the opinion of many authors - an important theorem [19]:

Let $U(x)$ be a nondecreasing function in $[0, \infty)$ such that

$$w(x) = \int_0^\infty e^{-xs} d\{U(s)\}$$

is bounded for every $x > 0$ and if $\rho \geq 0$, $L(x)$ is slowly varying and $w(x) = x^{-\rho} L(1/x)$ as $x \to 0+$, then $U(x) \sim x^\rho L(x)/\Gamma(\rho + 1)$, $x \to \infty$. Analogously,

$L(1/x)$ can be replaced by $L(x)$ and $x \to 0$ by $x \to \infty$.

A function of the form $R(x) = x^\rho L(x)$ is then called *regularly varying with the regularity index ρ*. Its definition is given, analogously to (1), by

$$\lim_{\lambda\to\infty} \frac{R(\lambda x)}{R(\lambda)} = x^\rho. \tag{3}$$

for some ρ from the interval $-\infty < \rho < \infty$. If ρ lies between two finite bounds then such a function is called a *regularly bounded* $(R-0)$ function. The properties of such functions $(R-0)$, as well as their definition, was first given by Avakumović[21], a student of Karamata. Functions from the class $(R-0)$can also be defined through inequalities: if there exist number $\gamma > 0$ and $\Gamma > 0$ such that $x^\gamma g(x)$ is almost increasing and $x^{-\Gamma} g(x)$ almost decreasing, then $g(x)$ belongs to the class $(R-0)$.

Karamata's theorem, mentioned above, with a regularly varying function is known today as Karamata's Tauberian Theorem. The proof of Littlewiid's Theorem, the introduction of the slowly varying functions $L(x)$, i.e. the creation of the theory of slowly varying functions, and Karamata's Tauberian theorem are three results of lasting value in the mathematical literature, which cannot be left out. Karamata' s Tauberian theorem is the first, and also the nicest example of an application of slowly varying functions. Instead of something limited - the iterated logarithm - in this problem of Hardy and Littlewood we have the function $R(x) = x^\gamma L(x)$ which is as general as possible.

Among the three results mentioned above, there is no doubt that the most important place is occupied by the *theory of slowly varying functions.* Theories have their own destiny, just like books. Karamata published the first article on slowly varying functions in an almost unknown journal (Mathematica, Cluj). The revised and enlarged version [37] of this paper had no particular response either. For almost 25 years both articles remained obscure, and were almost forgotten. Neither Karamata nor his students, who started in the fifties to investigate these functions, used the term "theory of slowly varying functions". Foreign authors gave an appropriate name to this theory, and emphasized its importance.

In 1966 the large monograph of Feller[22] on probability theory ant its

[21] V. G. Avakumović, On the behavior of Laplace integrals on the boundary of convergence (Ph. D. Thesis, in Serbian), Beograd, 1940; Math. Zeit. 46 (1940), 62-66, 67-69; see Karamata [51]: Bemerkung úber die vorstehende Arbeit des Herrn Avakumović.

[22] W. Feller, An Introduction to Probability Theory and its Applications, I, II, Wiley and Sons, New York, 1971.

applications came out. Feller introduced Karamata's ideas and theorems about slowly varying functions in this field; he introduced the means and tools used to present more clearly many theorems and to give them their final form. This was the source of new results. Karamata was never interested in probability theory, not even in the basic facts of this branch. But many mathematicians who worked in probability theory have seen at once that the notion of regularly varying function and its properties are well adapted to many problems in this area. With this notion the final statements of the principal theorems then get a simple form. This is the case for example with the Central Limit Theorem of Khinchine, Feller, Lévy, 1935, where necessary and sufficient conditions are now expressed in the language of regular variation. This is also the case of the Weak Law of Large Numbers, where necessary and sufficient conditions for the convergence of arithmetic means $S_n/n \to \mu$ of random variables x_i are given, S_n being the sum of these random variables. For critical branching processes necessary and sufficient conditions for iterated solution of the functional equation $f(x) = x - a(x)R(x)$ are given, where $a(x) \to a$, and $R(x)$ is a regularly varying function of the form $x^k L(x)$. Then from $x_{n+1} = f(x_n)$ it follows that $x_n \sim a_k n^{-k_1} L(1/n)$, $n \to \infty$. Karamata [104] uses this functional equation as an example and considers it as a mathematical problem of lesser importance, while Seneta[23] uses this solution as a basic fact in some probabilistic problems. Such kind of a complete accordance of the problem with the solution can be found in many places in the literature where regularly varying functions are involved. This can be seen from many citations of Karamata, especially of the function denoted by $L(x)$.

Forty six years after the first work of Karamata, the monograph by Seneta[23] about regularly varying functions appeared. Seneta gave a concise and substantial survey of all results connected with Karamata functions known up to that time, as well as their historical development. But he also introduced numerous applications: from number theory through integrals with parameters, which reduce to the calculations with these functions, to Tauberian theorems and probability theory. There were numerous examples and also some generalizations of the notion of these functions. Already in 1949, Korevaar, van Aardenne-Ehrenfest and de Brujin[24], proved the

[23] E. Seneta, Regularly Varying functions, Lecture Notes in Mathematica, 580, Springer, Berlin, Heidelberg, New-York, 1976; Karamata's iteration theorems and normed regularly varying sequences in a historical light, Publ. Inst. Math. Beograd 48 (62) (1990) 45-51.

[24] van Aardenne-Ehrenfest T., de Brujin N. G., Korecaar J., A note on slowly oscillating functions, Nieuw. Arch. Wisk. 23 (1949) 77-86

main properties of R-functions under the assumption that $L(x)$ is Lebesque-integrable. Karamata and his students gave new generalizations, as well as a multidimensional variant for regularly varying functions. Twenty years after the death of Karamata the monograph of Bingham, Goldie and Teugels[25] was published, where this theory is presented in detail and with supplements. This book, published in the series Encyclopedia of mathematics and its Applications, contains all results up to the time of its publication. In includes 70 years of investigation in this area, and these investigations show that the time was not lost. The extensive literature about slowly varying functions speaks itself about its numerous applications in many areas of real analysis. Karamata is the most frequently cited Yugoslav mathematician.

At the beginning of the 20th century several mathematical theories had influence on the future development of mathematics. Let us mention two such theories: Lebesgue's integral (1904) and I. Fredholm's theory of integral equations (1903). Between Euclid and Lebesgue there is a time distance of 20 centuries. Lebesgue really explained the notions such as integral, length, area. On the 25 pages of his work, Fredholm gave a new start to some problems which were investigated and studied by generations of mathematicians working in mathematical physics or abstract spaces. These are the theories which link the past, the present, and the future of mathematics. many mathematical theories which came later contributed to the explosion of science. But there exist also theories which are their own goal, which do not unite old fields and do not give new results. Science develops through all sort of theories, which are the product of the human mind and of combination of ideas: abstract science has no bounds.

But some isolated proposition, some simple theorem can mark the turn in the development of mathematics. At the beginning of the 20th century, L. Fejér, an unknown young Hungarian mathematician, showed in 1903 by a beautiful and outstanding theorem that the arithmetic means of partial sums of Fourier series lead to the notion of convergence. the central point is the so called positive Fejér kernel, whose positivity is of greatest importance in many problems. In such a way the theory of Fourier series can be divided into two epochs: until Fejér's discovery and after it. Littlewood's theorem (1910) mentioned before is of that kind.

One can not measure and compare such big theories and big scientific discoveries with theories which frequently arise in all parts of mathematics,

[25] N. N. Bingham, C. M. Goldie and J. L. Teugels, Regular Variation, Cambridge University Press, Cambridge, 1987.

and which fade with the end of come fields. But when some work is applied to the solution of some problem from another area of mathematics, then this work has accomplished its goal. If some work remains for years a tool in different parts of mathematics, where it gives the definitive form to various results, if it is not only cited, but becomes a stimulus for new generalizations and new results, and if its originality is beyond doubt, then one can also speak about a theory. In further investigations the theory obtains its structure, its laws and its application. During 70 years regularly varying functions have shown new results, have united some dispersed results and have given them their final form. And when one speaks about asymptotic analysis, it is impossible to avoid regularly varying functions and their theory, so specific and so necessary. The name of Jovan Karamata, its chief creator, must be mentioned in connection with this theory.

Karamata had the most effective period of scientific and teaching work during 1929-1939, before World War II started. These ten years were marked not only by the appearance of his most important work, but also by attempts to introduce new directions in teaching. This has stimulated the awakening of mathematics in Yugoslavia which, until then, followed outdated fixed paths. Apart from greatest efforts in scientific work, Karamata also had other activities. He visited many universities in Europa and presented many lectures on his investigations. He participated in almost all of the largest scientific meetings and conferences between the two world wars, he met many scientists with whom he not only corresponded, but also had some sort of friendly relationship - among others several famous mathematicians of that time, such as: Landau, Knopp, I. Schur, Fejér, M. Riesz, P. Montel, W. Blachke, ... His reviews in the Zentralblatt für Mathematik and in the Jahrbuch úber die Fortschritte der Mathematik were noted for their criticism and conscientious comments.

After the important works from 1930-1932, in the period 1932-1939, Karamata had little interest not only for regularly varying functions themselves, but also for their applications.

Let us remember that the time when Karamata was most active was very unfavorable for scientific work. The economic crisis in the country, social events, big family, misunderstanding and envy from colleagues and the abundance of everyday small events which make the integral of negativity of life. He often spoke about his failed scientific life - he had hoped for something different - but the time went by and on the other hand science was not standing still, and this was exactly the time when a period of great research of classical analysis was ending, the time of Tauberian theorems and summability in its final form.

Wiener took active part in the English School of Analysis at the time of its heights, with Hardy and Littlewood and with the group of their most well-known students. Everyday exchange of thoughts and ideas in their circle brought new results and methods. On the other hand, Karamata's milieu was indifferent and not interested even in more important things, least of all in the science. In every milieu, where there is no interest, there is no progress - no competition, no results. Karamata was almost lonely in science, far away from the sources, without support. He could only follow the events and the development of science, far from the center. Karamata studied Wiener's new theory with the enthusiasm that was typical of him - he gave new scientific contributions, used and enlarged by the others, but he didn't reach the summit of scientific creativity of his youth (1928-1932). The basic ideas and discoveries belong to Norbert Wiener. Besides that, working on somebody else's ideas is not the same thing as one's own discovery. In the period 1933-39, he published a large number of treatises. That was the time of feverish work: to publish as soon as possible new results with modern tools, and to master old problems. During these few years he published more than 20 articles; on the other hand, as Karamata himself said, he was writing the paper on slowly varying functions for a years, obsessed with only one idea and with one thought. At this period he was a young and unknown mathematician and had no feeling of time. It was a period when passion for the unknown was hunting him, and when he gave the most of himself. This time didn't last long. When he wrote, he not only had something to say, but he mastered perfectly and analytic tools, he had style and precision and he tried to give a simple proof in the natural order, without tedious checking; and then every analyst wanted to read this. He cared a lot about the content too, and he polished many steps in proofs for several days, through obvious, but false difficulties. And just as H. R. Pitt, in 1938, reduced Wiener's theory to the simplest form, Karamata applied his new method to various theorems of Tauberian type. The range of this theory and its importance were visible. New elements appeared in this theory as Karamata applied the notion of normal families (Montel) [55]. This method has shown the relationship between different, apparently independent theorems, but their core became visible [39]. The synthesis of these works can be found in his small monograph published by Hermann [60]. By applying Schmidt's theorem on power series to Stieltjes integrals and to the Laplace transform, he weakened the conditions for convergence in Tauberian 0-theorem. New results followed from this. This shows the role of the Stieltjes integral as an analytic tool. As this time Karamata was interested in Borel summability (B-summability), especially in the inverse of Hardy-Littlewood's theorem concerned with B-summability. The lack of a simple proof of this theorem and the attempts of many authors T. Vijayaraghavan, O. Szasz, and in particular of Hardy and Littlewood, can be seen in Karamata's work in the so called shifting of indices. Among numerous problem in the articles of that time one can find a theorem of that type, but with new ideas and beyond real analysis, in the area of complex functions. Karamata [66] has proved the following theorem:

Let

$$f(z) = \sum a_k z^k \quad \textit{be convergent for } \ |z| \leq 1 \tag{4}$$

and let $f(z)$ *be bounded in* $K : |z - 1/2| = 1/2$. *In order that the series (4) be B-summable to* s *at* $z = 1$, *it is necessary and sufficient that the Fourier series of* f *on* K *converges to* s *at the same point.*

Karamata conjectured the general problem by replacing K with a parabola having the imaginary axis as a tangent of order $k-1$ $(k > 1)$ and in such a way he found the relationship between the Landau-Wiener case $(k = \infty)$ and the Littlewood-Schmidt case $(k = 1)$. The general case was solved by Karamata's student Avakumović[21] in his thesis.

Among the articles outside the scope of Tauberian theorems, one should select those which are cited and which represent new contributions in proofs and ideas. By estimating series with real, positive and monotone members, Karamata arrived to the general theorem of Jensen on convex functions. This work was cited by Hardy, Littlewood and Pólya[26]. Although Karamata thought that these authors were the first ones to publish this generalization of Jensen's theorem, they thought that the proof and the formulation belong to Karamata [30]. From Cauchy-Jensen's theorem he derives, as an almost evident consequence, several of Mercer's theorems of elementary nature [25], [26]. In [63], with the precision and the knowledge of analytic tools so characteristic of him, he derives several theorems on singular trigonometric integrals. This work was cited in the theory of singular integrals of Calderón and Zygmund, Acta Math. 88 (1952) 85-139.

Before the World War II, in 1939 approximately, Karamata felt that he was tired of such kind of work and of such mathematics. He was not only a mathematician of value and of good reputation, but he also had a

[26] G. H. Hardy, J. E. Littlewood, G. Pólya, Inequalities, Cambridge University Press, 1952.

natural, rare talent. He had a feeling for real mathematics, and not only for its superstructure and its formalism. After many efforts and work, an ordinary mathematician masters some field, its methods and technique, gets acquainted with the literature, learns how to write papers. Then from every article he can make a new one. Just by changing the assumptions. He can write new articles unlimitedly - all variants of the same work, of only one and the same idea. Probably the best one is his first work, connected with the thesis. The public and society especially respect the number of papers, nobody is interested in the content, and a wide audience does not understand this. Only the greatest ones, like Poincaré, can work simultaneously both on the New methods in celestial mechanics and on the foundations of algebraic topology. Hilbert changed his fields every 5-6 years, from foundations of geometry on integral equations. And they always made new scientific discoveries. But an ordinary mathematician forgets many fields which he once studied. To him, real science is his field, his property. All the rest is a fashion and an empty theory. Karamata felt that new areas should be carefully studied, that one should understand the foundations in order to comprehend the whole. About 1940, he discussed with M. Riesz and made notes about the fields on the border between the theory of functions and potential theory.

On the eve of World War II financial circumstances became a problem for his big family. The freedom that he had as a young scientist was gone. He didn't suspect that his country was entering the war, and something unknown...

Everything has vanished as if it never existed. Mathematics has disappeared, as well as his connections abroad, his students, his teaching. In the years 1941-1945 he didn't even look at the works he had started, the time went by. He awoke from the nightmare only in 1945. The University destroyed, without literature, without former friends, new ideas about teaching, about science; the earlier routine with old habits, and the new order and modern development of life and science. This is the way teaching started and the old fashioned scientific knowledge refreshed. For the teaching of the Theory of Complex Functions, Karamata prepared a small, but original, introduction. This is the book "Complex Number" [89] with many examples and geometric constructions avoiding classical notions - real and imaginary part. In less than weeks he wrote: "Theory and Practice of the Stieltjes Integral" [88], a model for an introduction to real analysis. Many examples in this textbook showed his knowledge and the wealth of all parts of real analysis. It is a pity that he didn't continue work on such kind of textbooks.

Finally, he returned to scientific work, after a quarter of a century, to the theory of regularly varying functions, but with new students. The preceding period of his works was ended. After 1946, Karamata and his students have finished the last chapters of this theory, in particular its applications. We give credit to Karamata for creating a circle of mathematicians which gathered around his ideas. His merit is even bigger because of the bad conditions for scientific work; he has also merit for the scientific work which has only started to develop and finally merit for the appearance of Yugoslav mathematics on the international stage of that time. Karamata himself and his students marked this development. Aljančić and Karamata in [112] derive that from the limit $\lim_{x\to\infty} x^{-1}\int_0^x f(t)dt = \rho$ it follows that the function $p(x) = \left\{\int_1^\infty (f(t)/t)dt\right\}$ is regularly varying with index ρ.

In [126], [127], Bojanić and Karamata found new classes of functions $f(x)$ related to a regularly varying $\rho(x)$ by its increments at infinity, through the relation

$$f(\lambda x) - f(x) = x(\lambda)o(\rho(x)), \quad x \to \infty.$$

The relation above holds uniformly in $\lambda \in [a, b]$, $0 < a < b < \infty$ for fixed a and b. The representation of the form

$$f(x) = \int_0^x \varepsilon(t)\frac{L(t)}{t}dt + o(L(x)), \quad x \to \infty,$$

under precise conditions on the function $\varepsilon(t)$, follows from this.

The first variant of regularly varying sequences was introduced by Schmidt (1925) in the article mentioned before, and in such a way he has started Karamata's theory, for which the final form, specially for the sequences, was given by Karamata's student M. Vuilleumier. There, given a sequence $r_n = n^\sigma L_n$, $n = 1, 2, \dots$, where $\{L_n\}$ is slowly varying sequence and σ is any real number, necessary and sufficient conditions are given for the matrix $[a_{nk}]$ to transform every slowly varying sequence into an asymptotically equivalent sequence.

In [130], Bajšanski and Karamata gave a new precise analysis of the notion of general order of growth in the transfinite problem. This is an old and a well known problem which originates from Hardy, but with a new understanding. In [128] the same authors study and find the principle of equicontinuity in the foundations of regularly varying functions.

D. Adamović and D. Arandjelović, students of S. Aljančić, gave significant contributions to the theory of regularly varying functions, as well as others who completed this theory in various ways. Many applications of these functions are natural and they show the width of this theory in different fields.

Karamata knew to perfection a part of classical analysis, during the time of its rise and also during the time of its decline. He knew not only basic ideas, but also the whole technique of work, methods and procedures as well as the literature of that time. His students partially continued this work, and they also spread the range of investigations. This was called then Karamata's circle. In the short period, after 1929, for about less than 10 years, and shortly after 1946, this circle was made, enlarged and then slowly lost power. Karamata left Belgrade in 1953, and most of his students scattered all over the world. Besides that, they have never been in the centers of investigations, such as the Department of Mathematics of the University of Belgrade, and they were not even in the University. At that time the Mathematical Institute had no permanent positions, but it was more like a club with meetings once a week. Most of Karamata's students were looking for positions abroad. They looked for jobs abroad and in their country they looked for the approval which didn't come. Here the affirmation of their knowledge and the acceptance of the milieu, there the recognition of their competence. Even in that period they have done something, first of all Karamata. From Karamata's circle something was recorded in international monographs and texts. The light of the faded star does not vanish, but travels still, and so the ideas like the regularly varying function and its applications do not fade. International scientific connections and the stream of information are today (written in 2000) almost instantaneous. At times immediately after the World War II, about 1946, it wasn't like that. In Belgrade then, there was no scientific literature at all. People were there, but no books, no journals. Still, those people tried to do something, what others have already been doing. regularly varying functions are an example of that kind. It took a lot of time to collect results scattered in journals and books.

Although the theory of regularly varying functions does not have a deep foundation, it has diverse applications. Later came its enlargement and additional results. Its long existence and its applications confirm that it deserves the name: theory. One should mention one more advantage of this theory: it is authentic and original. Its origin is in Belgrade. It was created there, it was developed mostly there and it still lines there. it is the work of this milieu, of the people who worked in it - Karamata's circle, and in particular it is the work of Jovan Karamata.

Karamata was a simple man, even slightly naive. He was maybe wrong in his judgments about many everyday problems, ordinary matters and about people. But he had the high morality of the real scientist. there is a difference between the moral norms of a scientist and of a simple moral man. Karamata distinguished those who knew their field, who judged objectively scientific truth, from those who didn't have these qualities. This is why Karamata couldn't claim the trivial and wrong to be deep and correct. He couldn't claim the unimportant to be beautiful and important. He left results of lasting value, he fought with enormous difficulties of creation, not with formal calculations and copies. And he loved mathematics more than ordinary achievements and, at least in his youth, he could sacrifice a lot to this end.

He could spend many days in vain, obsesses with some problem. For him, it was not only a profession, but the necessity of life in which he burned. On the other hand, an ordinary mathematician works as a routine, without particular enthusiasm. And, as a rule, his work is already forgotten as soon as it is written. Such works are a burden for our libraries. A quarter of a century after the death of Karamata and 70 years after the appearance of his first work, his works remain not only like a memorial, but they are the base for further developments of mathematics. When his works appeared, perhaps they were not of big importance, but their value, their real value, grows with time. They survive and they don't get embedded in other results generalizing them. Other people start with them, and they don't end with them. he created he didn't fabricate results, he invented, and he didn't only write. And Karamata succeeded in what every scientist at the end of his career and his life would like to achieve: the work outlived its creator.

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