Bulletin T.CXXII de l'Académie Serbe des Sciences et des Arts - 2001
Classe des Sciences mathématiques et naturelles
Sciences mathématiques, № 26

STRUCTURAL THEOREMS FOR FAMILIES OF FOURIER HYPERFUNCTIONS

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(Presented at the 7th Meeting, held on October 27, 2000)

A b s t r a c t. A structural characterization of convergent and bounded families of Fourier hyperfunctions is given.

AMS Mathematics Subject Classification (2000): 46F15
Key Words: Fourier hyperfunctions, families of hyperfunctions

1. *Introduction*

Let $\{f_h;\ h \in H\}$ be a family of Fourier hyperfunctions which is convergent or bounded. It is of interest for the theory or applications to know whether this family can be given by a unique differential operator $J(D)$ and a family of continuous or smooth functions $\{p_h;\ h \in H\}$ such that $f_h = J(D)p_h,\ h \in H$, where $\{p_h;\ h \in H\}$ is convergent or bounded but in some space of functions.

This kind of results for distributions one can find already by Schwartz [9] and in [1], [4], [5], [8] for ultradistributions. In [2] some results have been proved which relate to convergent sequences of hyperfunctions with supports belonging to a compact set K. In [6], [7] convergent sequences of Fourier hyperfunctions have been treated and in [11], Fourier hyperfunctions having the S-asymptotics. In this paper we prove a theorem for any convergent or bounded net without new conditions, which generalizes the results in [6], [7] and [11].

2. *Notation and definitions*

Let $\mathbf{O}$ be the sheaf of analytic functions defined on $\mathbf{C}^n$.

We denote by $\mathbf{D}^n$ the radial compactification of $\mathbf{R}^n$, and supply it with the usual topology. The sheaf $\tilde{\mathbf{O}}^{-\delta}$, $\delta \geq 0$, on $\mathbf{D}^n + i\mathbf{R}^n$ is defined as follows: For any open set $U \subset \mathbf{D}^n + i\mathbf{R}^n$, and $\delta \geq 0$, $\tilde{\mathbf{O}}^{-\delta}(U)$ consists of those elements F of $\mathbf{O}(U \cap \mathbf{C}^n)$ which satisfy $|F(z)| \leq C_{V,\varepsilon} \exp(-(\delta - \varepsilon)|Rez|)$ uniformly for any open set $V \subset \mathbf{C}^n$, $\bar{V} \subset U$, and for every $\varepsilon > 0$. By $\tilde{\mathbf{O}}$ we denote the sheaf on $\mathbf{D}^n + i\mathbf{R}^n$, $\tilde{\mathbf{O}}(U) = \tilde{\mathbf{O}}^0(U)$. The derived sheaf $\mathcal{H}^n_{\mathbf{D}^n}(\tilde{\mathbf{O}})$, denoted by $\mathcal{Q}$, is called the sheaf of Fourier hyperfunctions. It is a flabby sheaf on $\mathbf{D}^n$.

Let I be a convex neighbourhood of $0 \in \mathbf{R}^n$ and $U_j = \{(\mathbf{D}^n + iI) \cap \{Imz_j \neq 0\}\}$, $j = 1, ..., n$. The family $\{\mathbf{D}^n + iI,\ U_j;\ j = 1, ..., n\}$ gives a relative Leray covering for the pair $\{\mathbf{D}^n + iI,\ (\mathbf{D}^n + iI) \setminus \mathbf{D}^n\}$ relative to the sheaf $\tilde{\mathbf{O}}$. Thus

$$\mathcal{Q}(\mathbf{D}^n) = \tilde{\mathbf{O}}((\mathbf{D}^n + iI)\#\mathbf{D}^n) \Big/ \sum_{j=1}^{n} \tilde{\mathbf{O}}((\mathbf{D}^n + iI)\#_j\mathbf{D}^n), \tag{1}$$

where $(\mathbf{D}^n + iI)\#\mathbf{D}^n = U_1 \cap ... \cap U_n$ and $(\mathbf{D}^n + iI)\#_j\mathbf{D}^n = U_1 \cap ... \cap U_{j-1} \cap U_{j+1} \cap ... \cap U_n$. Similarly, $\mathcal{Q}^{-\delta}$, $\delta > 0$ is defined using $\tilde{\mathbf{O}}^{-\delta}$ instead of $\tilde{\mathbf{O}}$ (cf. Definition 8.2.5. in [3]).

We shall use the notation Λ for the set of $n-$vectors with entry $\{-1, 1\}$; the corresponding open orthants in $\mathbf{R}^n$ will be denoted by Γ_σ, $\sigma \in \Lambda$. A global section $f = [F] \in \mathcal{Q}(\mathbf{D}^n)$ is defined by $F \in \tilde{\mathbf{O}}((\mathbf{D}^n + iI)\#\mathbf{D}^n); F = (F_\sigma;\ \sigma \in \Lambda)$, where $F_\sigma \in \tilde{\mathbf{O}}(\mathbf{D}^n + iI_\sigma)$, $I_\sigma = I \cap \Gamma_\sigma$, $\sigma \in \Lambda$. F is the *defining function* for f.

Recall the topological structure of $\mathcal{Q}(\mathbf{D}^n)$. Let $f = [F]$, and K be a compact set in $\mathbf{R}^n$ then by $P_{K,\varepsilon}(F) = \sup_{z \in \mathbf{R}^n + iK} |F(z) \exp(-\varepsilon|Rez|)|, \varepsilon > 0$, $K \subset\subset I \setminus \{0\}$, is defined as the family of semi-norms in $\tilde{\mathbf{O}}((\mathbf{D}^n + iI)\#\mathbf{D}^n)$; $\tilde{\mathbf{O}}((\mathbf{D}^n + iI)\#\mathbf{D}^n)$ is a Fréchet and Montel space, as well as the quotient space $\mathcal{Q}(\mathbf{D}^n)$ with the family of seminorms $p_{K,\varepsilon}([F]) = \inf_G P_{K,\varepsilon}(F + G)$, where G belongs to the denominator in (1). In $\mathcal{Q}(\mathbf{D}^n)$ a weak bounded set is bounded. We associate to $f = [F^*]$

$$f(x) \cong \sum_{\sigma\in\Lambda} F_\sigma(x + i\Gamma_\sigma 0),\ F_\sigma \in \tilde{\mathbf{O}}(\mathbf{D}^n + iI_\sigma),\ F_\sigma = sgn\sigma F^*_\sigma. \qquad (2)$$

Let $\mathbf{P}_* = \operatorname{ind}\lim_{I\ni 0}\operatorname{ind}\lim_{\delta\downarrow 0}\tilde{\mathbf{O}}^{-\delta}(\mathbf{D}^n + iI)$. $\mathbf{P}_*$ and $\mathcal{Q}(\mathbf{D}^n)$ are topologically dual to each other ([3, Theorem 8.6.2]).

The Fourier transform on $\mathcal{Q}(\mathbf{D}^n)$ is defined by the use of functions $\chi_\sigma = \chi_{\sigma_1}...\chi_{\sigma_n}$, where $\sigma_k = \pm 1,\ k = 1,...,n,\ \sigma = (\sigma_1,...,\sigma_n)$ and $\chi_1(t) = e^t/(1+e^t),\ \chi_{-1}(t) = 1/(1+e^t),\ t\in\mathbf{R}$. Let f be given by (2). The Fourier transform of f is defined by

$$\mathcal{F}(f) \cong \sum_{\sigma\in\Lambda}\sum_{\tilde\sigma\in\Lambda}\mathcal{F}(\chi_{\tilde\sigma}F_\sigma)(\xi - i\Gamma_{\tilde\sigma}0), \qquad (3)$$

where $\mathcal{F}(\chi_{\tilde\sigma}F_\sigma) \in \tilde{\mathbf{O}}(\mathbf{D}^n - iI_{\tilde\sigma})$ and $\mathcal{F}(\chi_{\tilde\sigma}F_\sigma)(z) = O(e^{-w|x|})$ for a suitable $w > 0$ along the real axis outside the closed $\sigma-$orthant (cf. Proposition 8.3.2 in [3]).

A function v defined on $\mathbf{R}^n$ (on $\mathbf{C}^n$) is of *infra-exponential type* if for every $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that $|v(z)| \le C_\varepsilon e^{\varepsilon|x|},\ z\in\mathbf{R}^n$ $(z\in\mathbf{C}^n)$. A local operator $J(D) = \sum\limits_{|\alpha|\ge 0} b_\alpha D^\alpha$ with $\lim\limits_{|\alpha|\to\infty}\sqrt[|\alpha|]{|b_\alpha|\alpha!} = 0$ acts on $\mathcal{Q}(\mathbf{D}^n)$ as a sheaf homomorphism and continuously on $\mathcal{Q}(\mathbf{D}^n)$.

3. *Main results*

Theorem 1. *Let $f_h = [F^*_h] \in \mathcal{Q}(\mathbf{D}^n),\ F^*_h \in \tilde{\mathbf{O}}((\mathbf{D}^n + iI)\#\mathbf{D}^n),\ h\in H$. If:*

a) The net $\{f_h\}_{h\in H}$ converges in $\mathcal{Q}(\mathbf{D}^n)$ or

b) $\{f_h; h\in H\}$ is a bounded set in $\mathcal{Q}(\mathbf{D}^n)$.

Then there exist an elliptic local operator $J(D)$ and nets of functions $\{q_{h,s}\}_{h\in H},\ s\in\Lambda$, such that:

1. $q_{h,s}(x), h\in H,\ s\in\Lambda$, are smooth functions and of exponential type on $\mathbf{R}^n$.

2. $q_{h,s}(z) \in \tilde{\mathbf{O}}(\mathbf{D}^n + iI_s),\ s\in\Lambda,\ h\in H$, where $I_s,\ s\in\Lambda$, does not depend on $h\in H$.

3. $f_h = J(D)\sum\limits_{s\in\Lambda} q_{h,s},\ (x + i\varepsilon s),\ h\in H,\ 0 < \varepsilon \le \varepsilon_0$.

4. There exists $\epsilon_0 > 0$ such that for any compact sets $K_1 \subset\subset \mathbf{R}^n$ and $K_2 \subset\subset (0,\epsilon_0)$:

In case a) nets $\{q_{h,s}(x+i\epsilon s)\}_{h\in H}, s\in\Lambda$ *converge uniformly in* $x\in K_1$ *and* $\epsilon\in K_2$;

In case b) sets $\{q_{h,s}(x+i\epsilon s)\}_{h\in H}, s\in\Lambda$, *are uniformly bounded for* $x\in K_1$ *and* $\epsilon\in K_2$.

P r o o f. The idea of the proof is the same as in [11]. Let $f_h=[F_h^*]$ be given by (2) and their Fourier transform by (3). Let φ be a monotone increasing continuous, positive valued function $\varphi(r),\ r\geq 0$, which satisfies $\varphi(0)=1,\ \varphi(r)\to\infty,\ r\to\infty$.
By Lemma 1.2 in [2] there exists an elliptic local operator $J(D)$ whose Fourier transform $J(\zeta)$ satisfies the estimate:

$$|J(\zeta)|\geq C\exp(|\zeta|/\varphi(|\zeta|),\quad |Im\zeta|\leq 1. \tag{4}$$

By (4), $J^{-2}(\zeta)\in\tilde{\mathbf{O}}(\mathbf{D}^n+i\{|\mu|<1\})$. Denote by $g=\mathcal{F}^{-1}(1/J^2)$. By Theorem 8.2.6 in [3], $g\in\mathcal{Q}^{-1}(\mathbf{D}^n)$. Consequently $\delta=J_0(D)g, J_0=J^2$, and

$$f_h=J_0(D)(g*f_h),\quad h\in H. \tag{5}$$

By the properties of the Fourier transform, cited properties of $\chi_{\tilde\sigma},\tilde\sigma\in\Lambda$, and supposition on $F_h^*,\ h\in H$, we have for every $h\in H$:

(a) $\mathcal{F}(F_{h,\sigma}\chi_{\tilde\sigma})J^{-2})\in\tilde{\mathbf{O}}(x-iI_{\tilde\sigma})$ and decreases exponentially outside any cone containing $\overline{\Gamma}_\sigma$ as a proper subcone.

(b) $\mathcal{F}(F_{h,\sigma}\chi_{\tilde\sigma})J^{-2}\chi_s\in\tilde{\mathbf{O}}(x-iI_{\tilde\sigma})$ and decreases exponentially outside any cone containing $\overline{\Gamma}_\sigma$ and $\overline{\Gamma}_s$ as proper subcones.

(c) $\mathcal{F}^{-1}(\mathcal{F}(F_{h,\sigma}\chi_{\tilde\sigma})J^{-2}\chi_s)\in\tilde{\mathbf{O}}(x+i(I_\sigma\cup I_s))$ and decreases exponentially outside any cone containing $\overline{\Gamma}_{\tilde\sigma}$ as a proper subcone. We shall use these properties considering Fourier hyperfunctions $f_h*g,\ h\in H$, given in (5). The analysis of f_h*g is very similar to the analysis of $f*g$ in [11]. However we give it because of the integrity of the proof.

$$\begin{aligned} f_h*g &= \mathcal{F}^{-1}(\mathcal{F}(f_h)\mathcal{F}(g)) \\ &\cong \frac{1}{(2\pi)^n}\sum_{\sigma\in\Lambda}\sum_{\tilde\sigma\in\Lambda}\int_{\mathbf{R}^n} e^{iz_\sigma\zeta_{\tilde\sigma}}\mathcal{F}(\chi_{\tilde\sigma}F_{h,\sigma})(\zeta_{\tilde\sigma})/J^2(\zeta_{\tilde\sigma})d\xi, h\in H,\end{aligned}$$

where $\zeta_{\tilde\sigma}=\xi+i\eta_{\tilde\sigma},\ \eta_{\tilde\sigma}\in -I_{\tilde\sigma}$ and $z_\sigma\in\mathbf{R}^n+iI_\sigma$.

For fixed σ, for all $\tilde\sigma\in\Lambda$ and $z_\sigma\in\mathbf{R}^n+iI_\sigma$

$$S_{h,\sigma,\tilde\sigma}(z_\sigma)=\frac{1}{(2\pi)^n}\int_{\mathbf{R}^n} e^{iz_\sigma\zeta_{\tilde\sigma}}\mathcal{F}(\chi_{\tilde\sigma}F_{h,\sigma})(\zeta_{\tilde\sigma})/J^2(\zeta_{\tilde\sigma})d\xi;$$

$$|S_{h,\sigma,\tilde{\sigma}}(z_\sigma)| \leq \frac{1}{(2\pi)^n} \int\limits_{\mathbf{R}^n} e^{-x\eta_{\tilde{\sigma}} - y_\sigma \xi} |\mathcal{F}(\chi_{\tilde{\sigma}} F_{h,\sigma})(\zeta_{\tilde{\sigma}})/J^2(\zeta_{\tilde{\sigma}})| d\xi, h \in H.$$

One can see that $S_{h,\sigma,\tilde{\sigma}}(z_\sigma), h \in H$, are continuable to the real axis. The obtained functions $S_{h,\sigma,\tilde{\sigma}}(x)$ are continuous and of infra exponential type on $\mathbf{R}^n$. By Lemma 8.4.7 in [3], $S_{h,\sigma,\tilde{\sigma}}(x) \cong S_{h,\sigma,\tilde{\sigma}}(x + i\Gamma_\sigma 0),\ \tilde{\sigma} \in \Lambda,\ h \in H$ and

$$(f_h * g)(x) = \sum_{\sigma \in \Lambda} \sum_{\tilde{\sigma} \in \Lambda} S_{h,\sigma,\tilde{\sigma}}(x),\ h \in H. \tag{6}$$

The functions $S_{h,\sigma,\tilde{\sigma}}(z_\sigma)$ can be written in the following form

$$S_{h,\sigma,\tilde{\sigma}}(z_\sigma) = \frac{1}{(2\pi)^n} \sum_{s \in \Lambda} \int\limits_{\mathbf{R}^n} e^{iz_\sigma \zeta_{\tilde{\sigma}}} \mathcal{F}(\chi_{\tilde{\sigma}} F_{h,\sigma})(\zeta_{\tilde{\sigma}}) \chi_s(\zeta_{\tilde{\sigma}})/J^2(\zeta_{\tilde{\sigma}}) d\xi,\ h \in H.$$

Denote by

$$S_{h,\sigma,\tilde{\sigma},s}(z_\sigma) = \frac{1}{(2\pi)^n} \int\limits_{\mathbf{R}^n} e^{iz_\sigma \zeta_{\tilde{\sigma}}} \mathcal{F}(\chi_{\tilde{\sigma}} F_{h,\sigma})(\zeta_{\tilde{\sigma}}) \chi_s(\zeta_{\tilde{\sigma}})/J^2(\zeta_{\tilde{\sigma}}) d\xi,\ \ h \in H.$$

Functions $S_{h,\sigma,\tilde{\sigma},s}(z_\sigma),\ \sigma, \tilde{\sigma}, s \in \Lambda,\ h \in H$, are also continuable to the real axis and the obtained functions $S_{h,\sigma,\tilde{\sigma},s}(x)$ are continuous and of infra exponential type on $\mathbf{R}^n$. Moreover, for every $h \in H$

$$S_{h,\sigma,\tilde{\sigma},s}(x) \cong S_{h,\sigma,\tilde{\sigma},s}(x + i\Gamma_\sigma 0) \quad \text{and} \quad S_{h,\sigma,\tilde{\sigma}}(x) = \sum_{s \in \Lambda} S_{h,\sigma,\tilde{\sigma},s}(x). \tag{7}$$

Let us analyse the functions

$$I_{s,\epsilon}(\zeta) = J^{-2}(\zeta) e^{-\epsilon s \zeta} \chi_s(\zeta),\ \ \zeta \in \mathbf{R}^n + i\{|\eta| < 1\},$$

where $0 < \epsilon < 1$. These functions are elements of $\mathbf{P}_*$ because of

$$\begin{aligned} |I_{s,\epsilon}(\zeta)| \ &= |J^{-2}(\zeta)| \exp(-\epsilon \sum_{i=1}^n s_i \xi_i) \prod_{i=1}^n |\chi_{s_i}(\zeta_i)| \\ &\leq |J^{-2}(\zeta)| \prod_{i=1}^n |\chi_{s_i}(\zeta_i)| \exp(-\epsilon s_i \xi_i) \\ &\leq C \exp(-\epsilon \sum_{i=1}^n |\xi_i|),\ |\eta| < 1,\ \zeta = \xi + i\eta,\ s \in \Lambda. \end{aligned}$$

Therefore, $I_{s,\epsilon} \in \tilde{\mathbf{O}}^{-\epsilon}(\mathbf{D}^n + i\{|\eta| < 1\}),\ s \in \Lambda$. Since the Fourier transform maps $\mathbf{P}_*$ onto $\mathbf{P}_*$, there exists $\psi_{s,\epsilon} \in \mathbf{P}_*$ such that $\mathcal{F}(\psi_{s,\epsilon}) = I_{s,\epsilon},\ s \in \Lambda$. By Proposition 8.2.2 in [3],

$$\psi_{s,\epsilon} \in \tilde{\mathbf{O}}^{-1}(\mathbf{D}^n + i\{|y| < \epsilon\}),\quad s \in \Lambda. \tag{8}$$

Denote by

$$\begin{aligned} q_{h,s}(x) &= \sum_{\sigma\in\Lambda}\sum_{\tilde{\sigma}\in\Lambda} S_{h,\sigma,\tilde{\sigma},s}(x) \\ &\cong \sum_{\sigma\in\Lambda}\sum_{\tilde{\sigma}\in\Lambda} \mathcal{F}^{-1}(\mathcal{F}(F_{h,\sigma}\chi_{\tilde{\sigma}})J^{-2}\chi_s)(x + i(\Gamma_\sigma \cup \Gamma_s)0), s \in \Lambda, h \in H. \end{aligned} \tag{9}$$

Let us prove that the functions $q_{h,s},\ s \in \Lambda,\ h \in H$ have properties 1. - 4. cited in Theorem.

Property 1 follows from (9) and (c). Property 2 is satisfied because of (6) and (7). Property 3 follows by (5), (6) and (9). It remains only the property 4. Let us prove it.

If $f_h \in \mathcal{Q}(\mathbf{D}^n),\ h \in H$, and $\varphi \in \mathbf{P}_*$, then, because of the supposition on $F_h^*, h \in H$, $f_h * \varphi \in \tilde{\mathbf{O}}(\mathbf{D}^n + iI')$ (cf. [10]), where I' is an interval containing zero. We shall use this fact and the properties of the functions $I_{s,\epsilon}$, we analysed.

For a fixed $s \in \Lambda$ and $h \in H$ there exists $\epsilon_0 > 0$, such that ϵs belongs to all infinitesimal wedges of the form $\mathbf{R}^n + i(\Gamma_\sigma \cup \Gamma_s)0$ which apear in (9). For $\epsilon,\ 0 < \epsilon \leq \epsilon_0$ we have

$$\begin{aligned} & q_{h,s}(x + i\epsilon s) = \\ &= \sum_{\sigma\in\Lambda}\sum_{\tilde{\sigma}\in\Lambda} \frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} e^{i(x+i\epsilon s)\zeta_{\tilde{\sigma}}} \mathcal{F}(F_{h,\sigma}\chi_{\tilde{\sigma}})(\zeta_{\tilde{\sigma}}) J^{-2}(\zeta_{\tilde{\sigma}})\chi_s(\zeta_{\tilde{\sigma}}) d\xi \\ &= \sum_{\sigma\in\Lambda}\sum_{\tilde{\sigma}\in\Lambda} \frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} e^{ix\zeta_{\tilde{\sigma}}} \mathcal{F}(F_{h,\sigma}\chi_{\tilde{\sigma}})(\zeta_{\tilde{\sigma}}) \mathcal{F}(\psi_{s,\epsilon})(\zeta_{\tilde{\sigma}}) d\xi \\ &= \sum_{\sigma\in\Lambda}\sum_{\tilde{\sigma}\in\Lambda} ((F_{h,\sigma}\chi_{\tilde{\sigma}}) * \psi_{s,\epsilon})(x) = ((\sum_{\sigma\in\Lambda} F_{h,\sigma}) * \psi_{s,\epsilon})(x) \\ &= (f_h * \psi_{s,\epsilon})(x) = \langle f_h(t),\ \psi_{s,\epsilon}(x - t)\rangle,\ s \in \Lambda, h \in H \end{aligned} \tag{10}$$

Now, 4. a) and 4. b) follows from (10).

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