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# ON CERTAIN SUMS OVER ORDINATES OF ZETA-ZEROS

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*A b s t r a c t.* *Let $\gamma$ denote imaginary parts of complex zeros of $\zeta(s)$. Certain sums over the $\gamma$'s are evaluated, by using the function $G(s) = \sum_{\gamma>0}\gamma^{-s}$ and other techniques. Some integrals involving the function $S(T)$ are also considered.*

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## 1. *The function $G(s)$*

Define, for $\sigma = \Re\mathrm{e}\, s > 1$,

$$G(s) = \sum_{\gamma>0} \gamma^{-s}, \tag{1.1}$$

where $\gamma$ denotes ordinates of complex zeros of the Riemann zeta-function $\zeta(s)$. The aim of this note is to provide the (unconditional) study of $G(s)$ and some applications to the evaluation of sums over the $\gamma$'s and some related integrals. The function $G(s)$ is mentioned, in a perfunctory way, in the work of Chakravarty [2] and in more detail by Delsarte [5]. A related zeta-function, namely

$$\sum_{\gamma>0} \gamma^{-s}\sin(\alpha\gamma) \qquad (\alpha>0),$$

was studied by Fujii [6], but its properties are different from the properties of $G(s)$, and we shall not consider it here. Both Chakravarty and Delsarte (as well as Fujii) assume the Riemann Hypothesis (that all complex zeros of $\zeta(s)$ satisfy $\Re\mathrm{e}\, s = \frac12$, RH for short) in dealing with $G(s)$. Delsarte [5] obtains its analytic continuation to $\mathbb{C}$ under the RH. This will be obtained later in Section 3 by an argument which is different from Delsarte's, who employed a sort of a modular relation to deal with $G(s)$.

To begin the study of $G(s)$ we need some notation. As usual, let the function

$$N(T) \;=\; \sum_{0<\gamma\le T} 1$$

count the number of positive imaginary parts of all complex zeros which do not exceed $T$. We have (see [4, Chapter 15] or [13, Section 9.3])

$$\begin{aligned} N(T) &= \sum_{0<\gamma\le T} 1 = \frac{1}{\pi}\vartheta(T) + 1 + S(T),\\ \vartheta(T) &= \Im\mathrm{m}\,\Bigl\{\log\Gamma(\tfrac14+\tfrac12 iT)\Bigr\} - \tfrac12 T\log\pi, \end{aligned}$$

where $\vartheta(T)$ is continuously differentiable, and if $T$ is not an ordinate of a zero

$$S(T) = \frac{1}{\pi}\arg\zeta(\tfrac12+iT) = \frac{1}{\pi}\Im\mathrm{m}\,\Bigl\{\log\zeta(\tfrac12+iT)\Bigr\} \ll \log T. \tag{1.2}$$

Here the argument of $\zeta(\frac12+iT)$ is obtained by continuous variation along the straight lines joining the points $2$, $2+iT$, $\frac12+iT$, starting with the value $0$. If $T$ is an ordinate of a zero, then $S(T) = S(T+0)$.

It is clear then that the series in (1.1) converges absolutely for $\sigma>1$, and to obtain its analytic continuation to the region $\sigma\le 1$ we use Stirling's formula for the gamma-function (see [8]) and write the formula for $N(T)$ as

$$N(T) = \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + \frac78 + S(T) + f(T),\ f(T)\ll\frac{1}{T},\ f'(T)\ll\frac{1}{T^2}. \tag{1.3}$$

Since the smallest positive ordinate of a zeta-zero is 14.13..., we have

$$\begin{aligned} G(s) &= \int_1^\infty x^{-s}\,\mathrm{d}N(x) = \int_1^\infty x^{-s}\left\{\frac{1}{2\pi}\log(\frac{x}{2\pi})\,\mathrm{d}x + \mathrm{d}\left(S(x)+f(x)\right)\right\} \\ &= \frac{1}{2\pi}\left(\frac{x^{1-s}}{1-s}\log(\frac{x}{2\pi})\Big|_1^\infty - \int_1^\infty \frac{x^{1-s}}{1-s}\cdot\frac{\mathrm{d}x}{x}\right) \\ &+ x^{-s}\left(S(x)+f(x)\right)\Big|_1^\infty + s\int_1^\infty \left(S(x)+f(x)\right)x^{-s-1}\,\mathrm{d}x. \end{aligned}$$

In view of the bounds in (1.2) and (1.3) the last integral is seen to converge absolutely. Thus by the principle of analytic continuation we have, for $\sigma > 0$,

$$G(s) = \frac{1}{2\pi(s-1)^2} - \frac{\log 2\pi}{2\pi(s-1)} + C_1 + s\int_1^\infty \left(S(x)+f(x)\right)x^{-s-1}\,\mathrm{d}x, \tag{1.4}$$

where $C_1$ is a suitable constant. A relation similar to (1.4) was established by Chakravarty [3, p. 490]. Further analytic continuation will follow by integrating by parts the last integral. This will give, for $\sigma > -1$,

$$\begin{aligned} G(s) &= \frac{1}{2\pi(s-1)^2} - \frac{\log 2\pi}{2\pi(s-1)} + C_1 \\ &+ s\int_1^\infty f(x)x^{-s-1}\,\mathrm{d}x + s(s+1)\int_1^\infty\int_1^x S(u)\,\mathrm{d}u\cdot x^{-s-2}\,\mathrm{d}x, \end{aligned} \tag{1.5}$$

since we have the bound (see [13])

$$\int_0^T S(t)\,\mathrm{d}t \;=\; O(\log T). \tag{1.6}$$

It follows that (1.5) gives

$$G(s) \ll t^2 \qquad (\sigma > -1,\ |t| \ge t_0). \tag{1.7}$$

Hence by convexity (the Phragmén-Lindelöf principle, see [8]) we have

$$G(s) \ll_\varepsilon |t|^\varepsilon(1+|t|^{1-\sigma}) \qquad (\sigma > -1,\ |t| \ge t_0), \tag{1.8}$$

since $G(s) \ll 1$ for $\sigma > 1$. A sharper bound than (1.8), at least for $0 \le \sigma \le 1$, can be obtained as follows. We have (initially for $\sigma > 1$, then by analytic continuation for $\sigma > 0$)

$$G(s) = \sum_{0<\gamma\le X}\gamma^{-s} + \sum_{\gamma > X}\gamma^{-s} = \sum\nolimits_1(s,X) + \sum\nolimits_2(s,X), \tag{1.9}$$

say. The function $\sum_1(s,X)$ is entire, and we have by partial summation (since $N(T) \ll T\log T$)

$$\sum\nolimits_1(s,X) \ll X^{1-\sigma}\log X + \log^2 X \qquad (0 \le \sigma \le 1).$$

Henceforth we suppose that $T \le t \le 2T$ and we shall choose $X = X(T)$ ($\ge 2$) appropriately a little later. Integration by parts gives

$$\begin{aligned}\textstyle\sum_2(s,X) &= \int_X^\infty x^{-s}\left(\frac{1}{2\pi}\log\left(\frac{x}{2\pi}\right)\,\mathrm{d}x + \,\mathrm{d}(S(x)+f(x))\right)\\ &= \frac{X^{1-\sigma}}{s-1}\cdot\frac{1}{2\pi}\log\left(\frac{X}{2\pi}\right) + \frac{1}{s-1}\int_X^\infty \frac{x^{-s}}{2\pi}\,\mathrm{d}x \\ &+O(X^{-\sigma}\log^2 X) + s\int_X^\infty (S(x)+f(x))x^{-s-1}\,\mathrm{d}x.\end{aligned}\tag{1.10}$$

This gives

$$\begin{aligned}&G(s) \ll\\ &\ll X^{1-\sigma}\log X + X^{-\sigma}\log^2 X + X^{1-\sigma}|t|^{-2}\log X + X^{-\sigma}|t|\log\log X + \log^2 X\\ &\ll |t|^{1-\sigma}\log|t| + \log^2|t| \quad (X = T,\ 0 < \sigma < 1).\end{aligned}$$

Therefore by continuity we obtain a sharpening of (1.8) for $0 \le \sigma \le 1$, namely

$$G(s) \ll |t|^{1-\sigma}\log|t| + \log^2|t| \qquad (0 \le \sigma \le 1,\ |t| \ge t_0 > 0).\tag{1.11}$$

In estimating the last integral in (1.10) we used the Cauchy-Schwarz inequality for integrals and the mean square bound for $S(t)$ (see (4.3)).

## 2. *Mean square estimates for $G(s)$*

We pass now to mean square estimates for $G(s)$, for which as usual we expect to smoothen the irregularites of the integrand. If $0 \le \sigma \le 1$, then we can write

$$\begin{aligned}\int_T^{2T}|G(\sigma+it)|^2\,\mathrm{d}t &\ll \int_T^{2T}|\textstyle\sum_1(\sigma+it,X)|^2\,\mathrm{d}t +\\ &+ \int_T^{2T}|\textstyle\sum_2(\sigma+it,X)|^2\,\mathrm{d}t = I_1(T)+I_2(T),\end{aligned}\tag{2.1}$$

say, where $\sum_1$ and $\sum_2$ are defined by (1.9). To bound $I_1(T)$ we use the mean value theorem for Dirichlet polynomials (see e.g., [8, Th. 5.2]) in the form

$$\int_0^T |\sum_{n\le N} a_n n^{-it}|^2 \,\mathrm{d}t = T \sum_{n\le N} |a_n|^2 + O(\sum_{n\le N} n|a_n|^2). \tag{2.2}$$

If $0 < \gamma_1 \le \gamma_2 \le \cdots$ denote positive ordinates of zeta zeros, then we can write

$$\sum\nolimits_1(\sigma+it, X) = \sum_{\gamma_n \le X} \gamma_n^{-\sigma}\gamma_n^{-it}, \qquad \gamma_n \asymp n\log n.$$

Hence with $X = T$ and $a_n = \gamma_n^{-\sigma}$ we obtain from (2.2)

$$I_1(T) \ll \begin{cases} T & (\sigma > \frac12),\\ T\log^2 T & (\sigma = \frac12),\\ T^{2-2\sigma}\log T & (\sigma < \frac12). \end{cases} \tag{2.3}$$

To bound $I_2(T)$, we recall Parseval's formula for Mellin transforms (see [12]) in the form

$$\int_0^\infty f(x)g(x)x^{2\sigma-1}\,\mathrm{d}x = \frac{1}{2\pi i}\int_{(\sigma)} F(s)\overline{G(s)}\,\mathrm{d}s, \tag{2.4}$$

provided that

$$H(s) = \int_0^\infty h(x)x^{s-1}\,\mathrm{d}x, \quad x^{\sigma-\frac12}h(x) \in L^2(0,\infty)$$

with $h(x) = f(x)$ or $h(x) = g(x)$. As usual $\int_{(c)}$ denotes $\lim_{T\to\infty}\int_{c-iT}^{c+iT}$. From (2.4) one obtains

$$\int_1^\infty f(x)g(x)x^{1-2\sigma}\,\mathrm{d}x = \frac{1}{2\pi i}\int_{(\sigma)} F^*(s)\overline{G^*(s)}\,\mathrm{d}s, \tag{2.5}$$

provided that

$$H^*(s) = \int_1^\infty h(x)x^{-s}\,\mathrm{d}x, \quad x^{\frac12-\sigma}h(x) \in L^2(0,\infty)$$

with $h(x) = f(x)$ or $h(x) = g(x)$. Setting in (2.5) $f(x) = g(x)$ if $a \le x \le b\,(1 \le a < b)$ and $f(x) = 0$ otherwise, it follows that

$$\int_T^{2T} \Big|\int_a^b g(x)x^{-\sigma-it}\,\mathrm{d}x\Big|^2 \mathrm{d}t \le 2\pi\int_a^b g^2(x)x^{1-2\sigma}\,\mathrm{d}x. \tag{2.6}$$

Applying (1.10), (2.6) and (4.3) we obtain ($X = T$, $0 < \sigma \le 1$)

$$
\begin{aligned}
I_2(T) &\ll T^{-1}X^{2-2\sigma}\log^2 X + TX^{-2\sigma}\log^4 X + \\
&+T^2\int_X^\infty (S^2(x) + x^{-2})x^{-1-2\sigma}\,{\rm d}x \\
&\ll T^{2-2\sigma}\log\log T.
\end{aligned}
\tag{2.7}
$$

Combining (2.3) and (2.7), replacing $T$ by $T2^{-j}$ and summing all the results we finally deduce

**Theorem 1.** *For $\sigma$ fixed we have*

$$
\int_1^T |G(\sigma+it)|^2\,{\rm d}t \ll \begin{cases} T & (\frac12 < \sigma \le 1), \\ T\log^2 T & (\sigma = \frac12), \\ T^{2-2\sigma}\log T & (0 < \sigma < \frac12). \end{cases}
\tag{2.8}
$$

The lower limit of integration in (2.8) is 1 and not 0 to avoid the pole of $G(s)$ at $s = 1$. It is not difficult to see that, by using (1.5), the validity of the last bound in (2.8) can be extended to the range $-1 < \sigma < \frac12$, and the first bound in (2.8) to $\sigma > 1$ as well. A natural problem is to try to show that for $\sigma = \frac12$ the integral in (2.8) is asymptotic to $CT\log^2 T$.

## 3. *A multiple sum over zeta-zeros*

For a fixed $n \in \mathbb{N}$, let $\gamma^{(1)}, \dots, \gamma^{(n)}$ denote ordinates of zeta-zeros. By absolute convergence and the classical integral

$$
e^{-z} = \frac{1}{2\pi i}\int_{(c)} w^{-z}\Gamma(w)\,{\rm d}w \qquad (\Re{\rm e}\, z > 0, c > 0),
$$

we have

$$
\begin{aligned}
\sum_{\gamma^{(1)}>0,\dots,\gamma^{(n)}>0} e^{-\gamma^{(1)}\dots\gamma^{(n)}/X} &= \frac{1}{2\pi i}\int_{(2)} \sum_{\gamma^{(1)}>0,\dots,\gamma^{(n)}>0} (\gamma^{(1)}\dots\gamma^{(n)}/X)^{-s}\Gamma(s)\,{\rm d}s \\
&= \frac{1}{2\pi i}\int_{(2)} \Gamma(s)G^n(s)X^s\,{\rm d}s.
\end{aligned}
\tag{3.1}
$$

Since $G(s)$ has a double pole at $s = 1$, the function $G^n(s)$ will have a pole of order $2n$ at $s = 1$, but otherwise it is regular for $\sigma > -1$ and $G^n(s) \ll$

$(1+|t|)^{4n}$ in this region. Hence by the residue theorem and Stirling's formula for the gamma-function we obtain

$$\frac{1}{2\pi i}\int_{(2)}\Gamma(s)G^n(s)X^s\,\mathrm{d}s = X(A_{2n-1,n}\log^{2n-1}X+\cdots+A_{1,n}\log X+A_{0,n})$$
$$+G^n(0)+\frac{1}{2\pi i}\int_{(\varepsilon-1)}\Gamma(s)G^n(s)X^s\,\mathrm{d}s$$
$$= X(A_{2n-1,n}\log^{2n-1}X+\cdots+A_{1,n}\log X+A_{0,n})+G^n(0)+O_\varepsilon(X^{\varepsilon-1}),$$

where $A_{2n-1,n}\neq 0,\ldots,A_{0,n}$ are effectively computable constants. Thus we have

**Theorem 2.** *For fixed $n\in\mathbb{N}$ there exist effectively computable constants $A_{2n-1,n}\neq 0,\ldots,A_{0,n}$ such that*

$$\sum_{\gamma^{(1)}>0,\ldots,\gamma^{(n)}>0} e^{-\gamma^{(1)}\cdots\gamma^{(n)}/X} = X(A_{2n-1,n}\log^{2n-1}X+\cdots+A_{1,n}\log X+A_{0,n})$$
$$+G^n(0)+O_\varepsilon(X^{\varepsilon-1}), \tag{3.2}$$

*where $\gamma^{(1)},\ldots,\gamma^{(n)}$ denote ordinates of complex zeros of $\zeta(s)$.*

If the Riemann Hypothesis holds, then the asymptotic formula (3.2) can be considerably sharpened. Namely we have (see [13, eq. (14.13.8)])

$$S_n(t) \;=\; O\left(\frac{\log t}{(\log\log t)^{n+1}}\right),$$

where

$$S_n(t) \;:=\; \int_0^t S_{n-1}(u)\,\mathrm{d}u \qquad (n\geq 1,\ S_0(t)\equiv S(t)).$$

On the other hand, the function $f(x)$ in (1.3) admits (unconditionally) an asymptotic expansion in terms of negative odd powers of $x$, in view of Stirling's formula for the gamma-function. Thus from (1.5) we obtain, by successive integrations by parts and the above bound for $S_n(T)$, that on the RH the function $G(s)$ admits analytic continuation to $\mathbb{C}$, and is of polynomial growth in $\Im\mathrm{m}\,|s|$, provided that $s$ stays away from its poles: the double pole at $s=1$ and simple poles at $s=-1,-3,\ldots$. As mentioned in Section 1, these facts have been established by a different method in Delsarte [5, p. 431]. The converse problem seems to be interesting, namely what can be deduced about the location of zeros of $\zeta(s)$ from the fact that $G(s)$ has analytic continuation to, say, $\sigma>-A\,(1<A<\infty)$?

It transpires that if in the above proof we shift the line of integration (assuming RH) to $\Re\mathrm{e}\, s = -A$, where $A = k + \frac{1}{2} \ge \frac{3}{2}$ is half of an odd natural number, then we shall obtain in (3.2) additional main terms coming from the poles at $s = -1, -2, \dots, -k$ of the integrand, plus an error term which will be $\ll X^{-A}$.

We can obtain an unconditional result analogous to (3.2), namely

$$\begin{aligned}\sum_{\rho^{(1)},\dots,\rho^{(n)}} e^{-|\rho^{(1)}\dots\rho^{(n)}|/X} &= X(\alpha_{2n-1,n}\log^{2n-1}X + \cdots + \alpha_{1,n}\log X + a_{0,n}) \\ &\quad + R^n(0) + O_\varepsilon(X^{\varepsilon-1}),\end{aligned}\tag{3.3}$$

where $\alpha_{2n-1,n} \ne 0, \dots, \alpha_{0,n}$ are effectively computable constants, $\rho^{(1)}, \dots, \rho^{(n)}$ denote complex zeros of $\zeta(s)$ and, for $\sigma > 1$,

$$R(s) \;=\; \sum_\rho |\rho|^{-s} = 2\sum_{\gamma>0} |\rho|^{-s},$$

and otherwise $R(s)$ is defined by analytic continuation. This can be obtained in the region $\sigma > -1$ by writing

$$\begin{aligned} R(s) &= 2G(s) + 2\sum_{\gamma>0}(|\rho|^{-s} - \gamma^{-s}) \\ &= 2G(s) - 2s\sum_{\gamma>0}\int_\gamma^{|\rho|} x^{-s-1}\,\mathrm{d}x. \end{aligned}\tag{3.4}$$

But with $\rho = \beta + i\gamma$, $\gamma > 0$ we have

$$\left|\int_\gamma^{|\rho|} x^{-s-1}\,\mathrm{d}x\right| \le (|\rho| - \gamma)\gamma^{-\sigma-1} = (\sqrt{\beta^2+\gamma^2} - \gamma)\gamma^{-\sigma-1} \le \tfrac{1}{2}\gamma^{-\sigma-2}$$

since $0 < \beta < 1$. Hence

$$H(s) := 2s\sum_{\gamma>0}(|\rho|^{-s} - \gamma^{-s})$$

is regular for $\sigma > -1$ and in that region it satisfies

$$H(s) \;\ll\; |s|.$$

Therefore (3.4) provides analytic continuation of $R(s)$ to $\sigma > -1$. By using the method of proof of Theorem 1 we obtain

$$R(s) \;\ll_\varepsilon\; |t|^{1-\sigma+\varepsilon} \qquad (-1 < \sigma \le 1,\ |t| \ge t_0 > 0) \tag{3.5}$$

and also

$$\int_1^T |R(\sigma+it)|^2\,\mathrm{d}t \ll_\varepsilon \begin{cases} T^{2-2\sigma+\varepsilon} & (-1<\sigma\le\frac12), \\ T^{1+\varepsilon} & (\sigma\ge\frac12). \end{cases} \tag{3.6}$$

Using then (3.5) (or (3.6)) one obtains (3.3) similarly to the way (3.2) was obtained.

## 4. *Some integrals involving $S(T)$*

Certain types of integrals involving the function $S(T)$ (see (1.2)) are closely related to sums over zeta-zeros, and thus to $G(s)$. In this section we shall investigate the evaluation of some such integrals, which do not appear to have been treated in the literature before. We start by proving

**Theorem 3.** *Let $f(t)\in C[1,T]$ satisfy*

$$\int_1^T f^2(t)\,\mathrm{d}t \ll T\log^C T \qquad (C\ge 0). \tag{4.1}$$

*Then for fixed $r\in\mathbb{N}$ we have*

$$\begin{aligned}&\int_1^T S^r(t)f(t)\,\mathrm{d}t \\ &\ll_\varepsilon \min\left(T\log^{\frac{C}{2}}T(\log\log T)^{\frac{r}{2}},\, T+(\log\log T)^{\frac32 r+\varepsilon}\int_1^T |f(t)|\,\mathrm{d}t\right).\end{aligned} \tag{4.2}$$

P r o o f. The first bound in (4.2) follows from (4.1), the Cauchy-Schwarz inequality and the bound of K.-M. Tsang [14]

$$\int_T^{2T} S^{2k}(t)\,\mathrm{d}t \ll T(ck)^{2k}(\log\log T)^k \quad (C>0), \tag{4.3}$$

which is uniform in $k\in\mathbb{N}$. To obtain the second bound in (4.2) let, for a given constant $\delta>0$,

$$H_\delta(T) := \left\{t\,:\,T\le t\le 2T,\,|S(t)|\ge(\log\log T)^{\frac12+\delta}\right\}.$$

Then (4.3) gives ($\mu(\cdot)$ denotes measure)

$$\mu(H_\delta(T))(\log\log T)^{k+2k\delta} \ll T(ck)^{2k}(\log\log T)^k,$$

and consequently

$$\mu(H_\delta(T)) \ll T\left(\frac{ck}{(\log\log T)^\delta}\right)^{2k}. \tag{4.4}$$

Choose

$$k = \left[\frac{1}{2c}(\log\log T)^\delta\right].$$

Then for $T$ large enough $k \in \mathbb{N}$, and (4.4) implies

$$\mu(H_\delta(T)) \ll T2^{-2k} \le Te^{-A(\log\log T)^\delta} \qquad \left(A = \frac{\log 4}{4c}\right). \tag{4.5}$$

Thus if $\delta > 1$, then for any fixed $C_1 > 0$ we have from (4.5)

$$\mu(H_\delta(T)) \ll T(\log T)^{-C_1}. \tag{4.6}$$

Now suppose that $\delta > 1$. Then using (1.2) and (4.6) we have

$$\begin{aligned}
&\int_T^{2T} S^r(t)f(t)\,\mathrm{d}t = \int_{H_\delta(T)} + \int_{[T,2T]\setminus H_\delta(T)}\\
&\ll \left(\int_{H_\delta(T)} S^{2r}(t)\,\mathrm{d}t\right)^{1/2}\left(\int_T^{2T} f^2(t)\,\mathrm{d}t\right)^{1/2}\\
&+(\log\log T)^{r(\frac12+\delta)}\int_T^{2T}|f(t)|\,\mathrm{d}t\\
&\ll (T(\log T)^{2r-C_1})^{1/2}(T\log^C T)^{1/2} + (\log\log T)^{r(\frac12+\delta)}\int_T^{2T}|f(t)|\,\mathrm{d}t\\
&\ll T + (\log\log T)^{\frac{3r}{2}+\varepsilon}\int_T^{2T}|f(t)|\,\mathrm{d}t
\end{aligned}$$

with $C_1 = 2r + C$, $\delta = 1 + \varepsilon/r$. Replacing $T$ by $T2^{-j}$ ($j \in \mathbb{N}$) and adding up the resulting estimates we complete the proof of (4.2).

The integrals which seem of interest are e.g.,

$$\int_1^T S(t)|\zeta(\tfrac12+it)|^2\,\mathrm{d}t, \quad \int_1^T S^2(t)|\zeta(\tfrac12+it)|^2\,\mathrm{d}t \tag{4.7}$$

and

$$\int_1^T |\zeta(\tfrac12+it)|^2\,\mathrm{d}S(t). \tag{4.8}$$

An integration by parts shows that the integral in (4.8) equals

$$|\zeta(\tfrac12+it)|^2 S(t)\Big|_1^T - 2\int_1^T S(t)Z(t)Z'(t)\,\mathrm{d}t,$$

where Hardy's function $Z(t)$ (see [8], [11]) is a real-valued function of $t$ satisfying $|Z(t)| = |\zeta(\frac12+it)|$, and given by

$$Z(t) := \zeta(\tfrac12+it)\chi^{-1/2}(\tfrac12+it), \quad \chi(s) = \frac{\zeta(s)}{\zeta(1-s)} = 2^s\pi^{s-1}\sin(\tfrac12\pi s)\Gamma(1-s).$$

Since

$$\int_0^T |Z(t)Z'(t)|\,\mathrm{d}t \le \left(\int_0^T |\zeta(\tfrac12+it)|^2\,\mathrm{d}t\right)^{1/2}\left(\int_0^T |Z'(t)|^2\,\mathrm{d}t\right)^{1/2} \ll T\log^2 T,$$

it follows on using Theorem 3 that

$$\int_1^T |\zeta(\tfrac12+it)|^2\,\mathrm{d}S(t) \ll_\varepsilon T\log^2 T(\log\log T)^{\frac32+\varepsilon}. \tag{4.9}$$

Similarly we have

$$\int_1^T |\zeta(\tfrac12+it)|^2 S(t)\,\mathrm{d}t \ll_\varepsilon T\log T(\log\log T)^{\frac32+\varepsilon}, \tag{4.10}$$

and

$$\int_1^T |\zeta(\tfrac12+it)|^2 S^2(t)\,\mathrm{d}t \ll_\varepsilon T\log T(\log\log T)^{3+\varepsilon}. \tag{4.11}$$

The bounds (4.9)–(4.11) appear to be, at present, the strongest unconditional bounds that can be obtained.

On the other hand, the above integrals can be related to sums over zeta-zeros. For example, the integral in (4.8) is

$$\begin{aligned}\textstyle\int_1^T |\zeta(\tfrac12+it)|^2\,\mathrm{d}N(t) \quad &- \int_1^T \frac{1}{2\pi}\log\frac{t}{2\pi}\cdot|\zeta(\tfrac12+it)|^2\,\mathrm{d}t + O(1)\\ &= \sum_{0<\gamma\le T}|\zeta(\tfrac12+i\gamma)|^2 - \frac{T}{2\pi}\log^2 T + O(T\log T).\end{aligned}$$

This gives, on using (4.9),

$$\begin{aligned}\sum_{0<\gamma\le T}|\zeta(\tfrac12+i\gamma)|^2 \quad &= \frac{T}{2\pi}\log^2 T + O(T\log T) + \int_1^T |\zeta(\tfrac12+it)|^2\,\mathrm{d}S(t)\\ &\ll_\varepsilon T\log^2 T(\log\log T)^{\frac32+\varepsilon}.\end{aligned} \tag{4.12}$$

We recall the standard notation (see [8] and [9])

$$\int_0^T |\zeta(\tfrac12+it)|^2\,\mathrm{d}t = T\log\left(\frac{T}{2\pi}\right) + (2C_0-1)T + E(T),$$

where $C_0$ denotes Euler's constant. Then by using integration by parts, (1.6) and the bound $E(T) \ll T^c$ with suitable $c < 1/3$ (see [8]) we have

$$\begin{aligned}
&\int_0^T S(t)|\zeta(\tfrac12+it)|^2\,\mathrm{d}t = \int_0^T S(t)\left(\log\frac{t}{2\pi} + 2C_0 + E'(t)\right)\,\mathrm{d}t\\
&= O(\log^2 T) + \int_0^T S(t)E'(t)\,\mathrm{d}t = O(T^{1/3}) - \int_0^T E(t)\,\mathrm{d}S(t)\\
&= O(T^{1/3}) - \int_0^T E(t)\left(\,\mathrm{d}N(t) - \frac{1}{2\pi}\log\frac{t}{2\pi}\,\mathrm{d}t + \,\mathrm{d}O(\frac{1}{t})\right)\\
&= -\sum_{0<\gamma\le T} E(\gamma) + O(T^{1/3}) + \frac{1}{2\pi}\int_0^T E(t)\log\frac{t}{2\pi}\,\mathrm{d}t.
\end{aligned}$$

The last integral equals

$$\begin{aligned}
\textstyle\int_0^T (E(t)-\pi+\pi)\log\frac{t}{2\pi}\,\mathrm{d}t &= O(T^{3/4}\log T) + \pi\int_0^T \log\frac{t}{2\pi}\,\mathrm{d}t\\
&= \pi T\log T + O(T),
\end{aligned}$$

since we have (see [9])

$$\int_0^T (E(t)-\pi)\,\mathrm{d}t \ll T^{3/4}.$$

Therefore by using (4.10) we obtain

$$\begin{aligned}
\sum_{0<\gamma\le T} E(\gamma) &= \tfrac12 T\log T + O(T) - \int_0^T S(t)|\zeta(\tfrac12+it)|^2\,\mathrm{d}t\\
&\ll_\varepsilon T\log T(\log\log T)^{\frac32+\varepsilon}.
\end{aligned}\tag{4.13}$$

A similar calculation will also give (see [9] and [10])

$$\sum_{0<\gamma\le T} E_2(\gamma) \ll T^{3/2}\log T,\qquad \sum_{0<\gamma\le T} E_2(\gamma) = \Omega_\pm(T^{3/2}\log T),\tag{4.14}$$

where $E_2(T)$ is the error term in the asymptotic formula for the fourth power of $|\zeta(\frac12+it)|$.

The importance of the sum

$$\sum_{0<\gamma\le T} |\zeta(\tfrac{1}{2}+i\gamma)|^2 \tag{4.15}$$

lies in the fact that it identically vanishes if the Riemann Hypothesis holds. The unconditional bound (4.12) seems to be very weak. However this reflects the enormous difficulty of settling the Riemann Hypothesis. It may be remarked that a more general sum than the one in (4.15) was treated by S.M. Gonek [7]. He proved, assuming the Riemann Hypothesis, that

$$\sum_{0<\gamma\le T}\left|\zeta\left(\frac{1}{2}+i\left(\gamma+\frac{\alpha}{L}\right)\right)\right|^2 = \left(1-\left(\frac{\sin\pi\alpha}{\pi\alpha}\right)^2\right)\frac{T}{2\pi}\log^2 T + O(T\log T) \tag{4.16}$$

holds uniformly for $|\alpha|\le\frac{1}{2}L$, where $L=\frac{1}{2\pi}\log(\frac{T}{2\pi})$. It would be interesting to recover this result unconditionally, but our method of proof does not seem capable of achieving this.

One can treat the integrals in (4.9)-(4.11) by using Lemma 2 of Bombieri-Hejhal [1], which (after taking the imaginary part) provides an explicit expression for $S(T)$. The best this could give (in view of $O(1)$ in the error term) for the integral in (4.9) is the bound $O(T\log^2 T)$, which is still quite weak. Assuming the RH Lemma 2 of [1] will yield

$$\int_0^T S(t)|\zeta(\tfrac{1}{2}+it)|^2\,\mathrm{d}t \;=\; O(T\log T). \tag{4.17}$$

It remains elusive whether the bound in (4.17) gives the correct order of magnitude for the integral on the left-hand side. Is the integral $\Omega_\pm(T\log T)$? This seems to be difficult to settle, even if the Riemann Hypothesis is assumed.

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