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A CLASS OF EXPONENTIALLY BOUNDED DISTRIBUTION SEMIGROUPS

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A b s t r a c t. A structural theorem for a vector valued exponentially bounded distribution is used for introducing and studyng of a class distribution semigroups. An infinitesimal generator of such a semigroup is not necessarily densely defined, but if it is the case, then it corresponds to a distribution semigroup introduced by Lions. This result is obtained by Wang and Kunstmann for a class of exponentially bounded quasi-distribution semigroups. In fact we show that our class of distribution semigroup is identical to Wang-Kunstmann's one. Our approach is completely different and gives new characterizations. Applications to equations $\frac{\partial u}{\partial t} = Au + f$, *where* A *is not necessarily densely defined and* f *is an exponential vector valued distribution supported by* $[0, \infty)$, *are given.*

This paper is written much before the publishing of Wang's and Kunstmann's paper but because of various reasons it is published with a very long delay. Here it is given in the primary version as an original approach although some parts are consequences of published results of Wang and Kunstmann.

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0. *Introduction*

One-time integrated and n-times integrated exponentially bounded semigroups (n-t.i.e.b.s., in short), $n \in \mathbb{N}$, of operators on a Banach space introduced by Arendt, were developed in [1-2], [13-16], [19], [22], [29-30] and applied to abstract Cauchy problems with operators which do not generate C_0-semigroups. More generally, α-times integrated semigroups $\alpha \in \mathbb{R}^+ \cup \{0\} = [0, \infty)$ are introduced and analyzed in [13-15] and in [21] in connection with certain classes of partial differential and pseudodifferential operators on L^p spaces. Distribution semigroups were introduced and analyzed by Lions (cf. [18] and after that in [5-11], [32] and many other papers. By results of Sova in [25], Arendt had proved in [2] that every exponentially bounded distribution semigroup is an n-th distributional derivative of an n-t.i.e.b.s. with densely defined infinitesimal generator, where n is sufficiently large. The corresponding result for a distribution semigroup which is not exponentially bounded and which infinitesimal generator is not densely defined is proved in [35] and [17]. Local n-times integrated semigroups are introduced and analyzed in [3], [20], [23] and [28], where the relations with distribution semigroups were given.

Wang and Kunstmann have introduced and analyzed in [35] and [17] a quasi-distribution semigroup, QDSG, in short; specially in the case when it is exponentially bounded, EQDSG. A $\mathcal{G} \in \mathcal{D}'_+(L(E))$ (the notation is given in the next section) is a QDSG if and only if $\mathcal{G}$ is a distributional derivative of order n of an n.t.i.e.b.s. We call it a 0-distribution semigroup; in the case when it is exponentially bounded, we denote it by 0-EDSG. C_0-semigroup is among them.

We will investigate 0-EDSG having not densely defined infinitesimal generators. Lions has studied in [18] distribution semigroups with densely defined generators using the structural properties and advantages of the space of tempered distributions. Wang has developed his own approach by constructing a space of generalized functions. The natural frame for such investigations is the space of exponentially bounded distributions $\mathcal{K}'_1$ ([12]). We analyze 0-EDSG using this space.

Let us present the results of the paper. An infinitesimal generator A of an n-t.i.e.b.s. is the generator of a 0-EDSG and conversely, where n is sufficiently large. If A is densely defined, then a 0-EDSG is an exponentially bounded distribution semigroup, EDSG in short (cf. [18], Definition 6.1).

The composition law for a 0-EDSG is given by

$$\langle S(t+s, x), \varphi(t, s)\rangle = \langle S(t, S(s, x)), \varphi(t, s)\rangle,$$

$$\varphi \in \mathcal{K}_1(\mathbb{R}^2),\ \operatorname{supp}\varphi \subset [0,\infty) \times [0,\infty).$$

If $S = G'$ where G is strongly continuous and supported by $[0,\infty)$, then the above condition is sufficient for S being an 0-EDSG. Relations between a 0-EDSG and its infinitesimal generator are determined. They are not the same as in the case of an n-t.i.e.b.s.

It is known that when an operator A generates an n-t.i.e.b.s. on a Banach space E, then there exists a Banach space $E_1 \subset D(A^n)$ continuously imbedded in E such that the part of A in E_1 is the generator of a strongly continuous semigroup (cf. [4]). In this paper a Banach space $E_0 \subset E$ is constructed such that a 0-EDSG in E has the restriction on E_0 forming an EDSG.

In example 1 the results are applied to equation $u' = Au + f$, where $E = C_b(\mathbb{R})$ or $E = L^\infty(\mathbb{R})$,

$$A = \sum_{j=0}^{k} a_j \Big(\frac{d}{dx}\Big)^j, \quad Re \sum_{j=0}^{k} a_j (ix)^j < \infty, \ \ f \in \mathcal{K}_1'(E_0)$$

and E_0 is a suitable subspace of E. Note, A is not densely defined. In example 2 is solved

$$\frac{\partial u}{\partial t} - iH_m(\partial)u = f,\ f \in \mathcal{K}_1'(L^p(\mathbb{R}^n)),\ \operatorname{supp} f \subset [a,\infty),$$

where $H_m(\partial)$ is a pseudodifferential operator $i(\Delta)^{\frac{m}{2}}$, $m > 0$. Another approach to this equation in the framework of L^p spaces is given in [5-6] for $m = 2$ and [15]. The assumption $f \in \mathcal{K}_1'(L^p(\mathbb{R}^n))$ justifies our approach.

1. *Preliminaries from the theory of distributions*

Denote by E a Banach space with a norm $\|\cdot\|$; $L(E) = L(E,E)$ is the space of bounded linear operators from E into E and $C(\mathbb{R}, L(E))$ is the space continuous mappings from $\mathbb{R}$ into $L(E)$. We refer to [26-27] and [31] for the definitions of spaces $\mathcal{D}(\mathbb{R}), \mathcal{E}(\mathbb{R}), \mathcal{S}(\mathbb{R})$, their strong duals and $\mathcal{S}'(E) = L(\mathcal{S}(\mathbb{R}), E)$. Moreover, we refer to [33] for the space $\mathcal{S}_+ = \{\varphi; |t^k \varphi^{(v)}(t)| < C_{k,v},\ t \in [0,\infty),\ k,v \in \mathbb{N}_0\}$ ($\mathbb{N}_0 = \mathbb{N} \cup \{0\}$) and its dual $\mathcal{S}_+'$, which consists of tempered distributions supported by $[0,\infty)$. Recall ([12]), the space of exponentially decreasing test functions on the real line $\mathbb{R}$ is defined by $\mathcal{K}_1(\mathbb{R}) = \{\varphi; |e^{k|t|} \varphi^{(v)}(t)| < C_{k,v},\ t \in \mathbb{R},\ k,v \in \mathbb{N}_0\}$. The space $\mathcal{K}_1(\mathbb{R}^2)$ is defined in an appropriate way. This space has the same topological properties as $\mathcal{S}(\mathbb{R})$. Note,

$$f \in \mathcal{K}_1'(\mathbb{R}) \text{ if and only if } e^{-r|x|}f \in \mathcal{S}'(\mathbb{R}) \text{ for some } r \in \mathbb{R}. \tag{1}$$

The strong dual of $\mathcal{K}_1(\mathbb{R})$, $\mathcal{K}_1'(\mathbb{R})$ is the space of exponential distributions. The space $\mathcal{K}_{1+}' \subset \mathcal{K}_1'(\mathbb{R})$ consists of distributions which are supported by $[0,\infty)$.

We will also use the space $\mathcal{K}_{1+} = \{\varphi; |e^{k|t|}\varphi^{(v)}(t)| < C_{k,v},\ t \in [0,\infty),$ $k, v \in \mathbb{N}_0\}$ which has the same topological properties as $\mathcal{S}_+$. Note, its dual space is $\mathcal{K}_{1+}'$.

Essential role will play spaces $\mathcal{K}_{1_0}$ and $\mathcal{D}_0$, subspaces of $\mathcal{K}_1$ and $\mathcal{D}$, respectively, which elements are supporte by $[0,\infty)$. In the sequal, we will use the family of distributions

$$f_n(t) = \begin{cases} \dfrac{H(t)t^{n-1}}{(n-1)!}, & n \in \mathbb{N} \\ \\ f_{n+n_1}^{(n_1)}(t), & -n \in \mathbb{N}_0, n_1 \in \mathbb{N}, n+n_1 > 0, t \in \mathbb{R}, \end{cases}$$

where H is Heaviside's function. Note $f_{-1} = \delta'$.

Let $\mathcal{K}_1'(E) = L(\mathcal{K}_1, E)$ denote the space of continuous linear functions $\mathcal{K}_1 \to E$ with respect to the topology of uniform convergence on bounded sets of $\mathcal{K}_1$. Denote $\mathcal{K}_{1+}'(E) = L(\mathcal{K}_{1+}, E)$. It is a subspace of $\mathcal{K}_1'(E)$ with elements supported by $[0,\infty)$. There holds $\mathcal{K}_1'(E) = \mathcal{K}_1'(\mathbb{R})\widehat{\otimes} E = L(\mathcal{K}_1, E)$ where $\widehat{\otimes}$ denotes the completion of tensor product with respect to the $\mathcal{E}$-topology which is equivalent with the π-topology, since $\mathcal{K}_1'(\mathbb{R})$ is nuclear (cf. [31]). Also we have $\mathcal{K}_{1+}'(E) = \mathcal{K}_{1+}'\widehat{\otimes} E$.

The convolution of $f \in \mathcal{K}_{1+}'(E)$ and $g \in \mathcal{K}_{1+}'$ is defined by $\langle f * g, \varphi\rangle = \langle f, \check{g} * \varphi\rangle$, $\varphi \in \mathcal{K}_1(\mathbb{R})$, $(\check{g}(t) = g(-t))$. One can prove easily that $f * g = g * f \in \mathcal{K}_{1+}'(E)$.

Let $T : [0,\infty) \to L(E)$ be strongly continuous. Then, it is exponentially bounded at infinity if there exist $M \geq 0$ and $\omega \geq 0$ such that

$$\|T(t)\| \leq Me^{\omega t}, \quad t \geq 0, \tag{2}$$

In this case $\varphi \to \int_0^\infty T(t)\varphi(t)dt$, $\varphi \in \mathcal{K}_1(\mathbb{R})$, defines an element of $\mathcal{K}_{1+}'(E)$.

We need the following representation for elements of $\mathcal{K}_{1+}'(L(E))$.

Theorem 1. *Let* $S \in \mathcal{K}_{1+}'(L(E))$.

a) There exists $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ there exist a strongly continuous function $F_n : \mathbb{R} \to L(E)$, $\mathrm{supp} F_n \subset [0, \infty)$ and positive constants m_n and C_n, such that

$$\|F_n(t)\| \leq C_n e^{m_n t},\ t \geq 0,\ S = F_n^{(n)} \quad (^{(n)} \text{ is the distributional } n\text{-th derivative}).$$

b) Let $\psi, \varphi \in \mathcal{K}_1(\mathbb{R})$. Then

$$\langle S(t, \langle S(s, x), \psi(s)\rangle), \varphi(t)\rangle = \int S_{n_0}(t, S_{n_0}(s, x))\psi^{(n_0)}(s)\varphi^{(n_0)}(t)ds\,dt. \quad (3)$$

c) Let $\varphi(t, s) \in \mathcal{K}_1(\mathbb{R}^2)$ and $\varphi_v(t)$, $\psi_v(s)$ be sequences in $\mathcal{D}(\mathbb{R})$ such that the product sequence $\varphi_v(t) \cdot \psi_v(s)$ converge to $\varphi(t, s)$ in $\mathcal{K}_1(\mathbb{R}^2)$ as $v \to \infty$. Then, the limit

$$\lim_{v \to \infty} \langle S(t, \langle S(s, x), \psi_v(s)\rangle), \varphi_v(t)\rangle$$

defined by the left hand side of (3) exists and defines an element of $\mathcal{K}_1'(\mathbb{R}^2)$ which we denote by $S(t, S(s, x))$ i.e.

$$\langle S(t, S(s, x)), \varphi(t, s)\rangle = \lim_{v \to \infty} \int S_{n_0}(t, S_{n_0}(s, x))\psi_v^{(n_0)}(s)\varphi_v^{(n_0)}(t)ds\,dt, \quad (4)$$

where $\varphi \in \mathcal{K}_1(\mathbb{R}^2)$.

d) Also, we have for $\varphi \in \mathcal{K}_1(\mathbb{R}^2)$ and $r, p \in \mathbb{N}$,

$$(i) \quad \Big\langle \frac{\partial^r}{\partial t^r} S(t, S(s, x)), \varphi(t, s)\Big\rangle = (-1)^r \langle S(t, S(s, x)),\ \frac{\partial^r}{\partial t^r}\varphi(t, s)\rangle;$$

$$(ii) \quad \Big\langle \frac{\partial^r}{\partial s^p} S(t, S(s, x)), \varphi(t, s)\Big\rangle = \Big\langle S(t, \frac{\partial^p}{\partial s^p} S(s, x)), \varphi(t, s)\Big\rangle$$

$$= (-1)^p \langle S(t, S(s, x)),\ \frac{\partial^p}{\partial s^p}\varphi(t, s)\rangle.$$

P r o o f. a) Let $k \in \mathbb{N}$ and $\mathcal{K}_{1,k}(\mathbb{R})$ be the spaces of functions $\varphi \in C^k(\mathbb{R})$, such that $\lim\limits_{|x| \to \infty} \sup\limits_{i \leq k}\{e^{k|x|}|\varphi^{(i)}(x)|\} = 0$. Then, $\mathcal{K}_{1,k}(\mathbb{R})$, $k \in \mathbb{N}$, are Banach spaces with the norms $\|\varphi\| = \sup\{e^{k|x|}|\varphi^{(i)}(x)|;\ x \in \mathbb{R},\ i \leq k\}$ and

$$\mathcal{K}_1(\mathbb{R}) = \mathrm{proj} \lim_{k \to \infty} \mathcal{K}_{1,k}(\mathbb{R}), \quad \mathcal{K}_1'(L(E)) = \mathrm{ind} \lim_{k \to \infty} \mathcal{K}_{1,k}'(L(E))$$

in the sense of strong topologies because the inclusion mappings $\mathcal{K}_{1,k+1}(\mathbb{R}) \to \mathcal{K}_{1,k}(\mathbb{R})$ are compact, $k \in \mathbb{N}$. Thus, there exists $n_1 \in \mathbb{N}$ such that $S \in$

$\mathcal{K}'_{1,n_1}(L(E))$. Also, there exists n_0 such that for every $t \in \mathbb{R}$, $f_{n_0}(t-\cdot)\theta(\cdot) \in \mathcal{K}_{1,n_1}(\mathbb{R})$, where $\theta \in C^\infty(\mathbb{R})$, $\theta(x) = 0$, for $x \le -1$ and $\theta(x) = 1$, for $x \ge -\frac{1}{2}$. Then

$$(S * f_{n_0})(t) = \Big\langle S(u), \frac{(t-u)^{n_0-1}}{\Gamma(n_0)} H(t-u)\theta(u) \Big\rangle, \ \ t \in \mathbb{R},$$

it is continuous, supported by $[0,\infty)$, of exponential growth and $(S*f_{n_0})^{(n_0)} = S$. This implies the proof of assertion for every $n \ge n_0$.

b) Assume that φ and ψ are supported by $[\alpha, \beta]$.

$$\langle S(s,x), \psi(x) \rangle = (-1)^{n_0} \int_\alpha^\beta S_{n_0}(s,x)\psi^{(n_0)}(s)ds$$

$$= (-1)^{n_0} \lim_{v\to\infty} \sum_{i=0}^{v} S_{n_0}(s_i,x)\psi^{(n_0)}(s_i)\Delta s_i.$$

The continuity of S_{n_0} implies

$$(-1)^{n_0} \int_\alpha^\beta S_{n_0}(t, \langle S_{n_0}(s,x), \psi^{(n_0)}(s)\rangle)\varphi^{(n_0)}(t)dt$$

$$= \lim_{v\to\infty} \sum_{i=0}^{v} \int_\alpha^\beta S_{n_0}(t, S_{n_0}(s_i,x))\psi^{(n_0)}(s_i)\varphi^{(n_0)}(t)\Delta s_i \, dt$$

$$= \int_\alpha^\beta S_{n_0}(t, S_{n_0}(s,x))\psi^{(n_0)}(s)\varphi^{(n_0)}(t)ds\, dt.$$

Note $\|S_{n_0}(t, S_{n_0}(s,x))\| \le e^{k|t|}e^{k|s|}\|x\|$, $x \in E$. Thus, both sides in (3) exist for $\varphi, \psi \in \mathcal{K}_1(\mathbb{R})$. By taking sequences φ_v and ψ_v of test functions in $\mathcal{D}(\mathbb{R})$ which converge in $\mathcal{K}_1(\mathbb{R})$ to φ and ψ, respectively, we obtain (3).

c) This assertion directly follows from a) by using the integral form given in (3).

d) (i) Assertion b) and (4) imply

$$\Big\langle \frac{\partial^r}{\partial t^r} S(t, S(s,x)), \psi(s)\varphi(t) \Big\rangle$$

$$= (-1)^r \int S_{n_0}(t, S_{n_0}(s,x))\psi^{(n_0)}(s)\varphi^{(n_0+r)}(t)ds\, dt, \quad \varphi, \psi \in \mathcal{K}_1(\mathbb{R})$$

and this gives the assertion.

(ii) The same arguments imply

$$\Big\langle \frac{\partial^p}{\partial s^p} S(t, S(s,x)), \psi(s)\varphi(t) \Big\rangle$$

$$= (-1)^p \int S_{n_0}(t, S_{n_0}(s,x))\psi^{(n_0+p)}(s)\varphi^{(n_0)}(t) ds\, dt$$

$$= \langle S_{n_0}(t, \langle S_{n_0}(s,x), \psi^{(n_0+p)}(s)\rangle), \varphi^{(n_0)}(t)\rangle, \quad \varphi, \psi \in \mathcal{K}_1(\mathbb{R}). \qquad \square$$

By using (1) one can prove easily

$$f \in \mathcal{K}'_{1+}(L(E)) \text{ if and only if } e^{-r|x|} f \in \mathcal{S}'_+(L(E)) \text{ for some } r \geq 0. \tag{5}$$

Let f satisfy (5). Then the Laplace transformation of f is defined by

$$\mathcal{L}(f)(\lambda) = \widehat{f}(\lambda) = \langle f(t), e^{-\lambda t}\eta(t)\rangle, \quad Re\lambda > r,$$

where $\eta \in \mathbb{C}^\infty(\mathbb{R})$, $\operatorname{supp}\eta = [-\epsilon, \infty)$, $\epsilon > 0$ and $\eta \equiv 1$ on $[0, \infty)$. As in the case of tempered distributions, one can easily show that this definition does not depend on η (cf. [27]). If $f \in L^1([0,\infty), E)$ (which means $\| \int f(t)dt\|_E < \infty$), then

$$\widehat{f}(\lambda) = \int_0^\infty e^{-\lambda t} f(t) dt = \langle f(t), e^{-\lambda t}\rangle, \quad Re\lambda > 0,$$

where integral is taken in Bochner's sense.

2. *0-exponentially bounded distribution semigroup*

Let $T : (0,\infty) \to L(E)$ be strongly continuous, integrable in a neighborhood of 0, i.e., integrable on $(0,\epsilon)$ for some $\epsilon > 0$ and exponentially bounded at infinity i.e. satisfies (2) for some $M \geq 0$ and $\omega \in \mathbb{R}$. The Laplace transformation of T is defined by $\mathcal{L}(T)(\lambda) = R(\lambda) = \int\limits_0^\infty e^{-\lambda t}T(t)dt$, $Re\lambda > \omega$, where the integral is understood in Bochner's sense.

Let $(T(t))_{t\geq 0}$ be a C_0-semigroup. It is well-known that A is the infinitesimal generator of this semigroup if and only if there exists $\omega \in \mathbb{R}$ such that $(\omega, \infty) \subset \rho(A)$ and $R : \{\lambda \in \mathbb{C}; Re\lambda > \omega\} \to L(E)$, defined by $R(\lambda) = (\lambda I - A)^{-1}$, $Re\lambda > \omega$, is the Laplace transformation of $(T(t))_{t\geq 0}$.

Arendt ([2]) defined an n-t.i.e.b.s. as follows.

Let $(S(t))_{t\geq 0}$ be a strongly continuous (on $[0,\infty)$) exponentially bounded family in $L(E)$ and $n \in \mathbb{N}$. Then, it is called n-t.i.e.b.s. if $S(0,x)=0$ and

$$S(t,S(s,x)) = \tfrac{1}{(n-1)!}\Big[\int_t^{t+s}(t+s-r)^{n-1}S(r,x)dr - \int_0^s (t+s-r)^{n-1}S(r,x)dr\Big],\ t,s\geq 0,\ x\in E. \tag{6}$$

Let $S:(0,\infty)\to L(E)$ be strongly continuous, exponentially bounded, integrable in a neighbourhood of 0, satisfy (2) for some $M>0$, $\omega\in\mathbb{R}$ and

$$R(\lambda) = \lambda^n \int_0^\infty e^{-\lambda t}S(t)dt, \quad Re\lambda > \omega,\ \ n\in\mathbb{N}. \tag{7}$$

Then, Arendt proved that $(R(\lambda))_{Re\lambda>\omega}$ is a pseudoresolvent if and only if (6) holds.

Theorem 2. *Let $S\in\mathcal{K}'_{1^+}(L(E))$ and $R(\lambda)=\mathcal{L}(S)(\lambda),\ Re\lambda>\omega$.*

a) Then, $(R(\lambda))_{Re\lambda\geq\omega}$ is a pseudoresolvent if and only if there exists $n_0\in\mathbb{N}$ such that $S_{n_0}(t)=(S*f_{n_0})(t),\ t\in\mathbb{R}$, is continuous, $S_{n_0}(0)=0$ and satisfies

$$\begin{aligned}\langle S(t,S(s,x)),\varphi(t)\psi(s)\rangle &= \langle (S_{n_0}(t,S_{n_0}(s,x)))^{(n_0,n_0)},\varphi(t)\psi(s)\rangle \\ &\Big\langle \tfrac{1}{(n_0-1)!}\Big(\int_t^{t+s}(t+s-r)^{n_0-1}S_{n_0}(r,x)dr \\ &-\int_0^s(t+s-r)^{n_0-1}S_{n_0}(r,x)dr\Big)^{(n_0,n_0)},\ \varphi(t)\psi(s)\Big\rangle,\end{aligned} \tag{8}$$

for every $\varphi,\psi\in\mathcal{K}_{1_0}(\mathbb{R})$ and $x\in E$.

b) If $(R(\lambda))_{Re\lambda>\omega}$ is a pseudoresolvent then

$$\langle S(t,S(s,x)),\varphi(t,s)\rangle = \langle S(t+s,x),\varphi(t,s)\rangle,\ \ \varphi\in\mathcal{K}_{1_0}(\mathbb{R}^2),\ x\in E. \tag{9}$$

If (9) holds and $S=G'$, where G is strongly continuous, $G:\mathbb{R}\to L(E)$ and supp$G\subset[0,\infty)$, *then $(R(\lambda))_{Re\lambda>\omega}$ is a pseudoresolvent.*

Remark. If (8) holds, then it holds for every $n\geq n_0$ with $S_n=S*f_n$ because $S_{n_0}=S^{(n-n_0)}_{n_0+(n-n_0)}$.

P r o o f.

a) Since the necessity simply follows, we will prove the sufficiency of (8). We have $S = S_{n_0}^{(n_0)}$. Let $x \in E$. Then, (8) implies

$$(S_{n_0}(t, S_{n_0}(s,x)))^{(n_0,n_0)} = (F(t,s,x))^{(n_0,n_0)} \quad \text{(in the sense of distributions)},$$

in open sets not intersecting lines $x = 0$ and $y = 0$, where

$$F(t,s,x) = \frac{1}{(n_0-1)!}\Big[\int_t^{t+s} (t+s-r)^{n_0-1} S_{n_0}(r,x)dr$$

$$-\int_0^s (t+s-r)^{n_0-1} S_{n_0}(r,x)dr\Big], \ t, s \geq 0,$$

$$F(t,s,x) = 0, \quad t \leq 0 \quad \text{or} \quad s \leq 0.$$

Since both sides are supported by $[0,\infty)$ and $S_{n_0}(t, S_{n_0}(s,x))$ and $F(t,s,x), t, s \in \mathbb{R}$ are continuous, it follows that

$$S_{n_0}(t, S_{n_0}(s,x)) = F(t,s,x), \ \ t, s \geq 0.$$

Thus, $R(\lambda) = \lambda^{n_0}\mathcal{L}(S_{n_0})(\lambda), \ Re\lambda > \omega$, is a pseudoresolvent.

b) Let $(R(\lambda))_{Re\lambda>\omega}$ be a pseudoresolvent. By assertion a) and (4)

$$\langle S(t, S(s,x)), \varphi(t,s)\rangle = \langle S_{n_0}(t, S_{n_0}(s,x)), \varphi^{(n_0,n_0)}(t,s)\rangle$$

$$= \Big\langle D_t^{n_0} D_s^{n_0}\Big(\frac{1}{(n_0-1)!}\Big[\int_0^{t+s}(t+s-r)^{n_0-1}S_{n_0}(r,x)dr - \int_0^t (t+s-r)^{n_0-1}S_{n_0}(r,x)dr$$

$$-\int_0^s (t+s-r)^{n_0-1}S_{n_0}(r,x)dr\Big]\Big), \varphi(t,s)\Big\rangle, \ \varphi \in \mathcal{K}_{1_0}(\mathbb{R}^2),$$

where $D_t^{n_0} = \dfrac{\partial^{n_0}}{\partial t^{n_0}}$. Then,

$$\langle S(t, S(s,x)), \varphi(t,s)\rangle$$

$$= \Big\langle D_t^{n_0} D_s^{n_0}\Big(\frac{1}{(n_0-1)!}\Big[\int_0^{t+s}(t+s-r)^{n_0-1}S_{n_0}(r,x)dr - \int_0^t (t+s-r)^{n_0-1}S_{n_0}(r,x)dr$$

$$- \int_0^s (t+s-r)^{n_0-1} S_{n_0}(r,x)dr\Big]\Big),\ \varphi(t,s)\Big\rangle$$

$$= \Big\langle D_t^{n_0}\Big(S_{n_0}(t+s-r) - S_{n_0}(s,x) - \sum_{j=1}^{n_0-2} \frac{t^j}{j!} S^{(j+1)}(s,x)\Big), \varphi(t,s)\Big\rangle$$

$$= \langle S(t+s,x),\ \varphi(t,s)\rangle.$$

This implies

$$\langle S(t,S(s,x)), \varphi(t,s)\rangle = \langle S(t+s,x), \varphi(t,s)\rangle,\quad \varphi \in \mathcal{K}_{1_0}(\mathbb{R}^2).$$

Now we prove the second part of b). By (9) we have

$$S(\varphi * \psi, x) = S(\varphi, S(\psi,x)),\quad \varphi,\psi \in \mathcal{K}_{1_0},\ x\in E. \tag{10}$$

Let $\varphi,\psi \in \mathcal{K}_{1_0}$ and $x \in E$. Then by using

$$\mathcal{G}(\varphi, \mathcal{G}(\psi,x)) = \langle S(t,S(s,x)), \varphi'(t)\psi'(s)\rangle,$$

$$\mathcal{G}(\varphi * \psi, x) = -\langle S(t+s,x)), \varphi'(t)\psi(s)\rangle,$$

we obtain

$$\int\int S(t,S(s,x))\varphi'(t)\psi'(s)dt\,ds = -\int\int S(t+s,x))\varphi'(t)\psi(s)dt\,ds$$

$$= \int\int \Big(\int_t^{t+s} S(t+s-r,x)dr - \int_0^s S(r,x)dr\Big)\varphi'(t)\psi'(s)dt\,ds.$$

Put $P(t,s,x)=0$ if $t\le 0$ or $s \le 0$ and

$$P(t,s,x) = \int_t^{t+s} S(t+s-r,x)dr - \int_0^s S(r,x)dr - S(t,S(s,x)),\quad t>0,\ s>0.$$

This is a continuous function on $\mathbb{R}^2$ and $P(t,0,x)=0,\ t\in\mathbb{R},\ P(0,s,x) = 0,\ s \in \mathbb{R}$. We have

$$\Big\langle P(t,s,x), \varphi'(t)\psi'(s)\Big\rangle = 0,\quad \varphi,\psi \in \mathcal{K}_{1_0}(\mathbb{R}). \tag{11}$$

Put, for fixed φ,

$$p(s,x) = \int P(t,s,x)\varphi'(t)dt,\quad s\in\mathbb{R}.$$

It is continuous on $\mathbb{R}$ and $p(0,x)=0$.

Now by (11) we have

$$\langle p(s,x),\psi'(s)\rangle = 0,\;\; \psi\in\mathcal{K}_{1_0}(\mathbb{R}).$$

The continuity argument implies $p(s,x)=c$.

Thus, we have $\langle P(t,s,x),\varphi'(t)\rangle = 0$, $\forall\varphi\in\mathcal{K}_{1_0}$. This implies $P(t,s,x)=C(s)$, $t>0$, $s>0$, where C is a continuous function. With the same procedure, but first applying $P(t,s,x)$ on $\psi'(s)$ we obtain

$$P(t,s,x)=C_1(t),\;\; t>0,\;\; s>0,$$

where C_1 is a continuous function.

This implies $C_1(t)=C(s)$, $t>0$, $s>0$. Thus $C_1(t)=C(s)$ and $P(t,s,x)=0$, $t>0$, $s>0$. □

Definition 1. *Let $S\in\mathcal{K}'_{1+}(L(E))$. Then, S is called a 0-exponentially bounded distribution semigroup, a 0-EDSG, in short, if there exists $n_0\in\mathbb{N}$, such that $S_{n_0}=S*f_{n_0}$ is continuous on $\mathbb{R}$, supported by $[0,\infty)$, exponentially bounded and satisfies (8). It is called non-degenerate if $\langle S(t,x),\varphi(t)\rangle = 0$ for all $\varphi\in\mathcal{K}_{1_0}$, implies $x=0$. We will also use the notation $(S(t))_{t\geq 0}$ for an 0-EDSG.*

Clearly, C_0-semigroup is a 0-EDSG.

Let $(S(t))_{t\geq0}$ be a 0-EDSG, and $R(\lambda)=\mathcal{L}(S)(\lambda)$, where $Re\lambda>\omega$. Then, by the resolvent equation, $ker R(\lambda)$ is independent of $Re\lambda>\omega$. Hence, by the uniqueness theorem $R(\lambda)$ is injective if and only if $(S(t))_{t\geq0}$ is non-degenerate. In that case there exists a unique operator $A: D(A)\to E$, $(D(A)\subset E)$ satisfying $(\omega,\infty)\subset\rho(A)$ such that $R(\lambda)=(\lambda I-A)^{-1}$, $Re\lambda>\omega$. This operator is called the generator of $(S(t))_{t\geq0}$. We put this in the following definition.

Definition 2. *A closed operator A is the generator of a 0-EDSG $(S(t))_{t\geq0}$ if $(a,\infty)\subset\rho(A)$ for some $a\in\mathbb{R}$ and the function $\lambda\mapsto(\lambda I-A)^{-1}=\mathcal{L}(S)(\lambda)$, $Re\lambda>a$, and it is injective, where the Laplace transformation is understood in the sense of distribution theory.*

Theorem 2 and the above definition directly imply the next Proposition.

Proposition 1. *Let $(S_n(t))_{t\geq0}$, $n\in\mathbb{N}$ be an n-t.i.e.b.s. Then S_n*f_{-n} is a 0-EDSG. If $(S(t))_{t\geq0}$ is a 0-EDSG, then there is $n_0\in\mathbb{N}$ such that $S*f_n$ is an n-t.i.e.b.s. for every $n\geq n_0$.*

Also, A is the generator of an n-t.i.e.b.s. $(S(_n(t))_{t\geq 0}$ if and only if A is the generator of a 0-EDSG, $S_n * f_{-n}$.

The relations of a 0-EDSG $(S(t))_{t\geq 0}$ and its infinitesimal generator A are slightly different then in the case of an n-t.i.e.b.s. However, in the proofs we have to use results for n-t.i.e.b.s. and after that to apply the n-th distributional derivative.

Theorem 3. *Let A be a generator of a 0-EDSG $(S(t))_{t\geq 0}$. Then, for all $\varphi \in \mathcal{K}_1$, we have*

a) $A\langle S(t,x), \varphi(t)\rangle = \langle S(t, Ax),\ \varphi(t)\rangle,\ \ x \in D(A)$

b) For every $x \in E$, $\langle S(t,x), \varphi(t)\rangle \in D(A)$.

c) $\langle S(t,x), \varphi(t)\rangle = \langle f_1(t,x), \varphi(t)\rangle + \langle (f_1 * S)(t, Ax), \varphi(t)\rangle,\ x \in D(A)$ and

$$A\langle (f_1 * S)(t,x), \varphi(t)\rangle = \langle S(t,x), \varphi(t)\rangle - \langle f_1(t,x), \varphi(t)\rangle,\ \ x \in E.$$

In particular

$$A\langle S(t,x), \varphi(t)\rangle = -\langle S(t,x), \varphi'(t)\rangle - \varphi(0)x,\ \ x \in E.$$

Remark. We will use also the notation $A\langle S(t,x), \varphi(t)\rangle = \langle AS(t,x), \varphi(t)\rangle$.

P r o o f.

a) Let $n_0 \in \mathbb{N}$ such that $S_{n_0} = S * f_{n_0}$ is an n_0-t.i.e.b.s. Let $\varphi \in \mathcal{D}(\mathbb{R})$ and $x \in D(A)$. Then,

$$\langle S(t,x), \varphi(t)\rangle = (-1)^{n_0}\langle S_{n_0}(t,x), \varphi^{(n_0)}(t)\rangle,\ \ n_0 \in \mathbb{N}.$$

Proposition 3.3 in [2] implies $S_{n_0}(t,x) \in D(A)$ and $AS_{n_0}(t,x) = S_{n_0}(t, Ax)$.

This and the continuity of A imply

$$A\langle S(t,x), \varphi(t)\rangle = (-1)^{n_0} A\langle S_{n_0}(t,x), \varphi^{(n_0)}(t)\rangle$$

$$= (-1)^{n_0} A \int S_{n_0}(t,x)\varphi^{(n_0)}(t)dt = (-1)^{n_0} A \lim_{v\to\infty} \sum_{i=1}^{v} S_{n_0}(t_j, x)\varphi^{(n_0)}(t_j)\Delta t_j$$

$$= (-1)^{n_0} \lim_{v\to\infty} \sum_{j=1}^{v} AS_{n_0}(t_j, x)\varphi^{(n_0)}(t_j)\Delta t_j$$

$$= (-1)^{n_0}\langle AS_{n_0}(t,x), \varphi^{(n_0)}(t)\rangle = \langle S(t, Ax), \varphi(t)\rangle,\ x \in E,\ \varphi \in \mathcal{D}$$

where $\Big(\sum\limits_{j=1}^{v} S_{n_0}(t_j,x)\varphi^{(n_0)}(t_j)\Delta t_j\Big)$, $v\in\mathbb{N}$ is a sequence of integral sums for $\int S_{n_0}(t,x)\varphi^{(n_0)}(t)dt$.

Let $\varphi\in\mathcal{K}_1$ and φ_v be a sequence in $\mathcal{D}(\mathbb{R})$ which converges to φ in $\mathcal{K}_1$. Then,

$$A\langle S(t,x),\varphi(t)\rangle = \lim_{v\to\infty}\langle S(t,Ax),\varphi_v(t)\rangle = \langle S(t,Ax),\varphi(t)\rangle.$$

This implies the assertion.

b) By Proposition 3.3 in [2], we have $\int\limits_0^t S_{n_0}(s,x)ds\in D(A)$ for every $x\in E$. Thus, $\Big\langle \int\limits_0^t S_{n_0}(s,x)ds,\varphi(t)\Big\rangle\in D(A)$ for every $\varphi\in\mathcal{K}_1$ and $x\in E$. By putting $\varphi^{(n_0+1)}$ instead of φ, we obtain $\langle S(\cdot,x),\varphi\rangle\in D(A)$ for every $\varphi\in\mathcal{K}_1$.

c) Note, for $x\in D(A)$, $t\ge 0$, $S_{n_0}(t,x)\in D(A)$, $AS_{n_0}(t,x)=S_{n_0}(t,Ax)$, $S_{n_0}(t,x)=\dfrac{t^{n_0}}{n_0!}x+\displaystyle\int_0^t S_{n_0}(s,Ax)ds$ (Proposition 3.3. in [2]). This implies

$$\langle S(t,x),\varphi(t)\rangle = (-1)^{n_0}\langle S_{n_0}(t,x),\varphi^{(n_0)}(t)\rangle$$

$$(-1)^{n_0}\langle f_{n_0+1}(t,x),\varphi^{(n_0)}(t)\rangle + (-1)^{n_0}\langle f_1 * S_{n_0})(t,Ax),\varphi^{(n_0)}(t)\rangle$$

$$= \langle f_1(t,x),\varphi(t)\rangle + \langle (f_1 * S_{n_0}^{(n_0)})(t,Ax),\varphi(t)\rangle$$

$$= \langle f_1(t,x),\varphi(t)\rangle + \langle (f_1 * S)(t,Ax),\varphi(t)\rangle,\quad x\in D(A),\ \varphi\in\mathcal{K}_1,$$

which gives the first assertion.

Again by using the quoted Proposition 3.3 in [2], it follows

$$A\langle (f_1 * S)(t,x),\varphi(t)\rangle = (-1)^{n_0}\langle A(f_1 * S_{n_0})(t,x),\varphi^{(n_0)}(t)\rangle$$

$$= (-1)^{n_0}\langle S_{n_0}(t,x),\varphi^{(n_0)}(t)\rangle - (-1)^{n_0}\langle f_{n_0+1}(t,x),\varphi^{(n_0)}(t)\rangle$$

$$= \langle S_{n_0}^{(n_0)}(t,x),\varphi(t)\rangle - \langle f_1(t,x),\varphi(t)\rangle = \langle S(t,x),\varphi(t)\rangle - \langle f_1(t,x),\varphi(t)\rangle,$$

which gives the second assertion. The particular case follows by putting φ' in the second relation in assertion c). $\square$

3. *Relations with distribution semigroup of Lions*

We follow Definition 6.1 in [18] of a distribution semigroup. Note, instead of $\mathcal{S}(\mathbb{R})$ which is used as the basic space in [18], we use in this paper the space $\mathcal{K}_1(\mathbb{R})$.

Let us recall conditions for a $\mathcal{G} \in \mathcal{D}'_+(L(E))$ to be a *distribution semigroup.*

(D.1) $\mathcal{G}(\varphi * \psi, \cdot) = \mathcal{G}(\varphi, \mathcal{G}(\psi, \cdot)), \quad \varphi, \psi \in \mathcal{D}_0$;

(D.2) $\bigcap\limits_{\varphi \in \mathcal{D}_0} N(\mathcal{G}(\varphi, \cdot)) = \{0\}$;

(D.3) The linear hull $\mathcal{R}$ of $\bigcup\limits_{\varphi \in \mathcal{D}_0} R(\mathcal{G}(\varphi, \cdot))$ is dense in E;

(D.4) for every $x \in \mathcal{R}$ there exists a function $u : \mathbb{R} \to E$ such that $\mathrm{supp} u \subset [0, \infty)$, $u(0) = x$ and u is continuous for $t \geq 0$ and $\mathcal{G}(\varphi, x) = \int\limits_0^\infty \varphi(t) u(t) dt$ for any $\varphi \in \mathcal{D}_0$.

If, in addition, there exists $\xi_0 \in \mathbb{R}$ such that

(D.5) $e^{-\xi t}\mathcal{G} \in \mathcal{S}'_+(L(E))$, for $\xi > \xi_0$, then it is called *an exponentially bounded distribution semigroup,* EDSG *in short.*

With another convolution $\varphi \otimes \psi = \int\limits_0^t \varphi(x-t)\psi(t) dt$, $\varphi, \psi \in \mathcal{D}$, Wang [35] and Kunstmann [17] have introduced a quasi-distribution semigroup, QDSG in short, as an element of $\mathcal{D}'(L(E))$ satisfying

(Q.D.1) $\mathcal{G}(\varphi \otimes \psi, \cdot) = \mathcal{G}(\varphi, \mathcal{G}(\psi, \cdot))$, $\varphi, \psi \in \mathcal{D}$,

(Q.D.2) = (D.2).

One can simply show that (Q.D.1) implies $\mathcal{G} \in \mathcal{D}'_+(L(E))$. If, moreover, (D.5) holds, than it is called exponentially bounded quasi-distribution semigroup, EQDSG in short. In fact in [35] is used another definition of exponential boundedness.

Let $(S_0(t))_{t \geq 0}$ be a 0-EDSG with an infinitesimal generator which is not densely defined. We recall:

$$S_0(\varphi, x) = 0 \quad \text{for every} \quad \varphi \in \mathcal{D}_0 \Rightarrow x = 0. \tag{12}$$

As in [18], we extend $(S_0(t))_{t \geq 0}$ on $T \in \mathcal{E}'(\mathbb{R}), \mathrm{supp} T \subset [0, \infty)$ using δ sequences $\{\rho_v\}$ in $\mathcal{D}_0$, $(\rho_v \to \delta)$. We denote by $D(S_0(T))$ the set of $x \in E$ such that:

(i) $S_0(\rho_v, x) \to x$, $v \to \infty$,

(ii) $\lim\limits_{v \to \infty} S_0(T * \rho_v, x)$ exists, and does not depend on ρ_v for which (i) holds.

This means that if $\tilde{\rho}_v$ is another δ sequence in $\mathcal{D}_0$ for which (i) holds, then $S_0(T * \tilde{\rho}_v, x) - S_0(T * \rho_v, x) \to 0$ as $v \to \infty$. This limit defines $S_0(T, x)$.

Because of (12), we can define the closure of $S_0(T, \cdot)$ which will be denoted by $\overline{S_0(T, \cdot)}$.

In the next theorem we summerize basic relations between C_0-semigroups and various distribution exponentially bounded semigroups.

Theorem 4. *a) Let $(T(t))_{t \geq 0}$ be a C_0-semigroup and A its infinitesimal generator. Then by*

$$\mathcal{J}(\varphi, x) = (T * \check{\varphi})(0)(x) = \langle T(t, x), \varphi(t) \rangle, \quad \varphi \in \mathcal{K}_{1+}, \; x \in E$$

is defined an EDSG, $Ax = \overline{\mathcal{J}(-\delta', x)}$, $x \in D(A)$ and

$$A\mathcal{J}(\varphi, x) = -\int_0^\infty T(t, x)\varphi'(t)dt - \langle \delta(t, x), \varphi(t) \rangle =$$

$$= -\int_0^\infty T(t, x)\varphi'(t)dt - \varphi(0)x, \; x \in D(A), \varphi \in \mathcal{K}_{1+}.$$

b) Let $(S_0(t))_{t \geq 0}$ be a 0-EDSG with a densely defined infinitesimal generator A. Then, by

$$\mathcal{J}_0 : \varphi \to \langle S_0(t, \cdot), \varphi(t) \rangle, \quad \varphi \in \mathcal{K}_{1+},$$

is defined an EDSG. Moreover, $A = \mathcal{J}_0(-\delta', \cdot)$.

If A is not densely defined, then by $\mathcal{J}_0$ is defined an element of $\mathcal{K}'_{1+}(L(E))$ which satisfies all the properties for an EDSG except that $\{\mathcal{J}_0(\varphi, x);\ \varphi \in \mathcal{D}_0,\ x \in E\}$ is dense in E.

c) Let $(S_n(t))_{t \geq 0}$, $n \in \mathbb{N}_0$ be an n-t.i.e.b.s. Assume that its infinitesimal generator A is densely defined. Then,

$$\mathcal{J}_n(\varphi, x) = (S_n * \check{\varphi})(0)(x), \quad \varphi \in \mathcal{K}_{1+}, \tag{13}$$

defines an element of $\mathcal{K}'_{1+}(L(E))$ which is an EDSG if and only if $n = 0$.

d) Let $S \in \mathcal{D}'_+(L(E))$. It is a 0-EDSG if and only if it is an EQDSG.

P r o o f.

a) Clearly $\mathcal{J} \in \mathcal{K}'_{1+}(L(E))$. Then,

$$(T * (\varphi \check{*} \psi))(0)(x) = ((T * \check{\psi}) * \check{\varphi})(0)(x), \; \varphi, \psi \in \mathcal{D}_0(\mathbb{R})$$

and this implies $\mathcal{J}(\varphi * \psi, x) = \mathcal{J}(\varphi, \mathcal{J}(\psi, x)),\ x \in E,\ \varphi, \psi \in \mathcal{D}_0(\mathbb{R})$.

Put $y = \mathcal{J}(\psi, x)$, where $\psi \in \mathcal{D}_0(\mathbb{R})$ and $x \in E$ are fixed. Then, a distribution $\varphi \mapsto \mathcal{J}(\varphi, y),\ \varphi \in \mathcal{D}_0(\mathbb{R})$ is determined by a continuous function $u : [0, \infty) \to E,\ u(0) = y$. In fact, we have $u(t) = (T * \check{\psi})(t)$.

Let $x \in E$ be fixed. If $\mathcal{J}(\varphi, x) = 0$ for every $\varphi \in \mathcal{D}_0$, then it follows $x = 0$ or $x \neq 0$ and $T = \sum\limits_{\alpha=0}^{m} a_\alpha \delta^{(\alpha)} \otimes I$, where I is the identity operator and $a_\alpha \in \mathbb{C},\ \alpha = 0, 1, ..., m$. But since T could not be of this form, we have $x = 0$. Let us prove that $\{\langle T(t, x), \varphi(t)\rangle;\ \varphi \in \mathcal{D}_0,\ x \in E\}$ is dense in E. For every $t_0 \in [0, \infty)$ and every $x \in E$,

$$\langle T(t, x), \delta_n(t - t_0)\rangle \text{ converges in } E \text{ to } T(t_0, x) \text{ as } n \to \infty,$$

where $\delta_n(t - t_0)$ is a δ - sequence. For example, we can take $\delta_n(t - t_0) = n\varphi(n(t - t_0))$, where $\varphi \in \mathcal{D}_0(\mathbb{R}),\ \int \varphi = 1$. Since the set $\{T(t_0, x); t_0 \in [0, \infty),\ x \in E\}$ is dense in E, the assertions follows. Thus, we verify all the conditions which $\mathcal{J}$ has to satisfy to be an EDSG.

We have to prove that $Ax = \overline{\mathcal{J}(-\delta', x)}, x \in D(A)$; this implies that A is the infinitesimal generator of semigroup $\mathcal{J}$.

By noting that for $x \in D(A),\ T(\cdot, x)$ is differentiable, we have

$$\overline{\mathcal{J}(-\delta', x)} = (T * \delta')(0)(x) = T'(0)(x) = Ax.$$

Let $x \in D(A)$, then we have

$$A\mathcal{J}(\varphi, x) = \mathcal{J}(-\delta' * \varphi, x) = \mathcal{J}(-\varphi', x) = \int_0^\infty T'(t, x)\varphi(t)dt,\quad \varphi \in \mathcal{K}_{1+}.$$

Now, by partial integration, we obtain

$$A\mathcal{J}(\varphi, x) = T(t, x)\varphi(t)|_0^\infty - \int_0^\infty T(t, x)\varphi'(t)dt = -\int_0^\infty T(t, x)\varphi(t)dt - \varphi(0)x.$$

b) Let $(S_0(t))_{t \geq 0}$ be a 0-EDSG with a densely defined infinitesimal generator A. Let $S_0(\varphi, x),\ \varphi \in \mathcal{K}_{1+},\ x \in E$, be defined by

$$S_0(\varphi, x) = (S_0(\cdot, x) * \check{\varphi})(0) = \langle S_0(t, x), \varphi(t)\rangle. \tag{14}$$

Let $\varphi, \psi \in \mathcal{D}_0$. Then,

$$(S_0(\cdot, x) * (\varphi \check{*} \psi))(0) = \langle S_0(t, x), (\varphi * \psi)(t)\rangle$$

$$= \langle S_0(t+s,x), \varphi(t)\psi(s)\rangle = \langle S_0(t, \langle S_0(s,x), \psi(s)\rangle), \varphi(t)\rangle.$$

This implies

$$S_0(\varphi * \psi, x) = S_0(\varphi, S_0(\psi, x)),\ x \in E,\ \varphi, \psi \in \mathcal{D}_0.$$

Clearly, for fixed $\psi \in \mathcal{D}_0$ and $x \in E$, $\mathcal{D}_0 \ni \varphi \mapsto S_0(\varphi, S_0(\psi, x))$ is determined by $u(t) = (S_0(\cdot, x) * \check{\psi})(t)$ which is continuous and supported by $[0, \infty)$.

Since S_0 is a non-degenerate one and not of the form $\sum\limits_{\alpha=0}^{m} a_\alpha \delta^{(\alpha)} \otimes I$, it follows

$$(S_0(\varphi, x) = 0 \text{ for every } \varphi \in \mathcal{K}_{1_0}) \Rightarrow (x = 0).$$

Similary as in [18], pp. 152-153, we will prove that $\{S_0(\varphi, x); \varphi \in \mathcal{D}_0,\ x \in E\}$ is dense in E. This will imply that (13) defines an EDSG.

The duals of $D(A)$ and E, $D(A)'$ and E' are equal, since $D(A)$ is dense in E. Thus,

$\mathcal{H}(\varphi, \cdot) =^t \langle S_0(t, \cdot), \varphi(t)\rangle \in D(A)',\ \varphi \in \mathcal{K}_{1+}$, where superscript t denotes a transposed operator.

Moreover, $\mathcal{H} \in \mathcal{K}_1'(L(D(A)', E))$, $\operatorname{supp}\mathcal{H} \subset [0, \infty)$. Theorem 3, a) and c) implies

$$-\mathcal{H}(\varphi)\,{}^tA - \mathcal{H}(\varphi') = \varphi(0) I_{E'}$$

$$-{}^tA\mathcal{H}(\varphi) - \mathcal{H}(\varphi') = \varphi(0) I_{D(A)'}.$$

Let $x' \in E'$ and

$$\langle\langle S_0(t,x), \varphi(t)\rangle, x'\rangle = 0, \quad \text{for every} \quad \varphi \in \mathcal{D}_0,\ x \in E. \tag{15}$$

We have to prove that $x' = 0$ which means, by Hahn-Banach theorem, that $\{\langle S_0(t,x), \varphi(t)\rangle;\ \varphi \in \mathcal{D}_0,\ x \in E\}$ is dense in E. Now (15) implies $\langle \mathcal{H}(\varphi, \cdot), x'\rangle = 0$ which implies $x' = 0$.

We define $A_0 x = \overline{S_0(-\delta', x)}$. Then, $A_0 = A$ because the resolvent for both operators are determined by $\mathcal{L}(S_0)(\lambda), Re\lambda > \omega$, (cf. [18]).

Let $\overline{D(A)} \neq E$. Then, except of density of $\{S_0(\varphi, x);\ \varphi \in \mathcal{D}_0,\ x \in E\}$, one can simply prove that S_0 verifies all the other properties which should be satisfied by an EDSG.

c) Let $\mathcal{J}$ be an EDSG. As it is remarked by Arendt, Theorem 4.3 in [2] and Theorem 3.2 in [25] imply that there exists an n-t.i.e.b.s., $(S_n)(t))_{t\geq 0}$, $n \in \mathbb{N}$, such that

$$\mathcal{J}(\varphi, x) = \langle S_n^{(n)}(t,x), \varphi(t)\rangle = (S_n^{(n)}(\cdot, x) * \check{\varphi})(0),\ \varphi \in \mathcal{D},\ x \in E.$$

This implies $S_n^{(n)} = S_n * f_{-n} = S_0 = \mathcal{J}$, where S_0 satisfies Theorem 2.

If for some $n \in \mathbb{N}$, (13) defines an EDSG then $(S_n(t))_{t\geq 0}$ determine exponentially bounded distribution semigroups which is impossible by the uniqueness of an EDSG with the given infinitesimal generator A.

d) The properties of a $0-EDSG$ given in Theorems 1,2, Proposition 1 and in [35], Theorem 3.8, as well as arguments in [35], Section 4, imply the assertion. □

The last part of Theorem 4 shows that our investigations give another approach and insight into the theory of EQDSG.

With the asumptions as in the second part of Theorem 4 b), we have the following theorem.

Theorem 5. *Let $(S_0(t)_{t\geq 0}$ be a $0-EDSG$ with an infinitesimal generator A such that $\overline{D(A)} \neq E$. Denote by E_0 the closure in E of the set $\tilde{E} = \{S_0(\varphi, x); x \in E, \varphi \in \mathcal{D}_0\}$. Then the set $\{S_0(\varphi, x); x \in E_0, \varphi \in \mathcal{D}_0\}$ is dense in E_0 and $S_{0|E_0 \times \mathcal{K}_{1+}}$ defines an EDSG with the infinitesimal generator $A = A_{|D(A)\cap E_0}$.*

P r o o f. Let $y \in E_0$ and $y_n = S_0(\varphi_n, x_n) \in \tilde{E} \subset E_0$ be a sequence which converges to y in E_0. We have $S_0(\varphi_n, x_n) = \lim_{m\to\infty} S_0(\varphi_n * \rho_m, x_n)$, where ρ_m is a δ sequence in $\mathcal{D}_0$. Since

$$S_0(\varphi_n * \rho_m, x_n) = S_0(\rho_m, (S_0(\varphi_n, x_n)) = S_0(\rho_m, y_n), m, n \in \mathbb{N},$$

we obtain that y is an limit point for $\{S_0(\varphi, x); x \in E_0, \varphi \in \mathcal{D}_0\}$ and $y_n \in \tilde{E}$.

Since $\{S_0(\varphi, x), x \in E_0, \varphi \in \mathcal{D}_0\} \subset D(A)$, we have that $D(A) \cap E_0$ is dense in E_0. This completes the proof. □

4. *Applications*

Let A be an operator on E, and $T \in \mathcal{K}_1'$ (E), supp$T \subset [a, \infty), a \geq 0$. Then, $u \in \mathcal{K}_1'$ (E) is a solution to equation

$$u' = Au + T, \text{ in } \mathcal{K}_1'(E) \tag{16}$$

if $\langle u(t), \varphi(t)\rangle \in D(A)$ for every $\varphi \in \mathcal{K}_{1+}'(\mathbb{R})$ and (16) holds.

Let A be an infinitesimal generator of a $0-EDSG(S_0(t))_{t\geq 0}$. We assume that $D(A)$ is supplied with the graph norm $\|x\| + \|Ax\|, x \in D(A)$.

Let $U \in \mathcal{K}_1'(L(E, D(A))), V \in \mathcal{K}_1'(L(D(A), E))$ and supp$U \subset [a, \infty)$, supp $V \subset [b, \infty), a, b \geq 0$. Then, $U * V$ and $V * U$ are elements of $\mathcal{K}_1'(L(D(A)))$

and $\mathcal{K}_1'(L(E))$, respectively and their supports are contained in $[a+b,\infty)$ (cf. [28]).

Theorem 6. *Let A be an infinitesimal generator of a $0-EDSG$ $(S_0(t))_{t\geq 0}$. Then $(S_0(t))_{t\geq 0}$ is an $EDSG$ with an infinitesimal generator A on $E_0\times\mathcal{K}_{1+}'$, where $E=\overline{\{S_0(\varphi,x);\varphi\in\mathcal{D}_0,x\in E\}}$ and*

$$\Big(-A+\frac{\partial}{\partial t}\Big)*S_0=\delta\otimes I_{E_0},S_0*\Big(-A+\frac{\partial}{\partial t}\Big)=\delta\otimes I_{D(A)\cap E_0}.$$

Let $T\in\mathcal{K}_1'(E_0)$ and $\mathrm{supp}T\subset[a,\infty)$ for some $a\geq 0$. Then, $u=S_0*T$ is a unique solution to

$$u'=Au+T\ \mathit{in}\ \mathcal{K}_1'(L(E_0)),\mathrm{supp}u\subset[a,\infty).$$

The proof of Theorem 6 is the same as the proof of Theorem 4.1 in [18].

Example 1. Let $A=\sum_{j=0}^k a_j\Big(\frac{d}{dx}\Big)^j, p(x)=\sum_{j=0}^k a_j(ix)^j\,(i^2=-1)$, $Rep(x)<\infty$, $a_j\in\mathbb{C}$, $j=0,1,...,k$, $E=C_b(\mathbb{R})$ or $E=L^\infty(\mathbb{R})$, $Af=\sum_{j=0}^k a_j\Big(\frac{d}{dx}\Big)^j f$, where

$$D(A)=\{f\in E;\sum_{j=0}^k a_j\Big(\frac{d}{dx}\Big)^j f\in E \text{ distributionally }\}.$$

It is proved in [16] that

$$S_t f=\frac{1}{\sqrt{2\pi}}\mathcal{F}^{-1}(\phi_t)*f,t\geq 0,f\in E,$$

where $(\phi_t(\xi)=f_1(t)*e^{p(\xi)t})_{t\geq 0}$, is an 1-t.i.e.b.s. with the generator A such that $\overline{D(A)}\neq E$. Here $\mathcal{F}$ denotes the Fourier transformation and $\mathcal{F}^{-1}$ denotes its inverse; $\mathcal{F}(f)(\lambda)=\tilde{f}(\lambda)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}e^{-i\lambda t}f(t)dt,\lambda\in\mathbb{R}$. We have

$$\langle f_{-1}(t)*\phi_t(\xi),\alpha(t))\rangle=\langle e^{p(\xi)t},\alpha(t)\rangle,\alpha\in\mathcal{K}_{1+}'(\mathbb{R}).$$

Thus $\mathcal{F}^{-1}(e^{p(\xi)t})=\frac{d}{dt}\Big(\frac{1}{\sqrt{2\pi}}\mathcal{F}^{-1}(\phi_t)\Big)$, in the sense of distributions, defines a $0-EDSG$ which is not an EDSG. If $T\in\mathcal{K}_1'(E_0),\mathrm{supp}T\subset[a,\infty),a\geq 0$, then the solution to equation $u'=Au+T$ is given by $\mathcal{F}^{-1}(e^{p(\xi)t})*T(t)$ (cf. Theorem 6).

Example 2. Let $m>0$. Consider

$$\frac{\partial}{\partial t}u(x,t)-i\Delta^{\frac{m}{2}}u(x,t)=f(x,t),\ \ f\in\mathcal{K}_1'(L^p(\mathbb{R}^n)),\mathrm{supp}f\subset[a,\infty),a\geq 0.$$

With $m = 2$ and $f \in \mathcal{T}'(L^p(\mathbb{R}^n))$ this equation is analyzed in [5] and [6].

We recall the definition of the space M_p of Fourier multipliers on $L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$ (cf. [25]): $a \in M_p$ if and only if a is a tempered distribution and $\|a\|_{M_p} = \sup_{\|f\|_{L^p}=1} \|\mathcal{F}^{-1}(a\tilde{f})\|_{L^p} < \infty$.

Let $m > 0, k > 0$ and $H_m(\xi) = |\xi|^m, \xi \in \mathbb{R}^n$. Then, the following asertion is proved in [25]:

$$a_{H_m,k}(\xi,t) = \frac{k}{t^k}\int_0^t (t-s)^{k-1} e^{-isH_m(\xi)} ds \in M_p, \quad t > 0, \ \xi \in \mathbb{R}^n,$$

if:

$$\text{for } m \neq 1, \ k \geq n\Big|\frac{1}{p} - \frac{1}{2}\Big|, \ \text{ for } \ m = 1, \ k \geq (n-1)\Big|\frac{1}{p} - \frac{1}{2}\Big|. \tag{17}$$

Moreover, $\|a_{H_m,k}(\xi,t)\|_{M_p}$ does not depend on t.

Define, for $\varphi \in \mathcal{K}_1(\mathbb{R})$, $u \in \mathcal{S}(\mathbb{R}^n)$,

$$G_0(\varphi,u)(x) = \mathcal{F}^{-1}\Big(\int_{\mathbb{R}} e^{-itH_m(\xi)}\varphi(t)dt\tilde{u}(\xi)\big)(x), x \in \mathbb{R}^n.$$

Let k be an integer which satisfies (17). Then,

$$G_0(\varphi,u)(x) = \frac{(-1)^k}{k!}\int_{\mathbb{R}} t^k\varphi^{(k)}(t)\mathcal{F}^{-1}(a_{H_m,k}(\xi,t)\tilde{u}(\xi))dt(x), x \in \mathbb{R}^n.$$

It implies that for given $\varphi \in \mathcal{K}_1(\mathbb{R})$ and $\mathcal{S}(\mathbb{R}^n)$ there exists $C_\varphi > 0$ such that

$$\|G_0(\varphi,u)\|_{L^p},$$

which shows that it can be extended on $L^p(\mathbb{R}^n)$ as a continuous linear operator.

Moreover, G_0 is an EDSG with the infinitesimal generator

$$A = iH_m(\partial) = i\Big(\frac{\partial^2}{\partial x_1^2} + ... + \frac{\partial^2}{\partial_n^2}\Big)^{\frac{m}{2}}$$

which is defined on the closure in L^p of the set $\{u \in \mathcal{S}; \mathcal{F}^{-1}(|\xi|^m\tilde{u}) \in L^p\}$.

Thus we have that $u = G_0 * f$ is the solution to equation

$$\frac{\partial}{\partial t}u - iH_m(\partial)u = f, f \in \mathcal{K}_1'(L^p), \mathrm{supp} f \subset [a,\infty).$$

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