Bulletin T.CXXII de l'Académie Serbe des Sciences et des Arts - 2001
Classe des Sciences mathématiques et naturelles
Sciences mathématiques, № 26

ON THE MICROLOCAL DECOMPOSITION OF ULTRADISTRIBUTIONS AND ULTRADIFFERENTIABLE FUNCTIONS

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(Presented at the 9th Meeting, held on December 22, 2000)

A b s t r a c t. The microlocal decomposition for ultradistributions and ultradifferentiable functions is derived by Bengel-Schapira's method and these classes of functions are microlocalized as subsheaves $\mathcal{C}_M^*$, $\mathcal{C}_M^{d,*}$ of the sheaf $\mathcal{C}_M$ of Sato's microfunctions on a real analytic manifold M. Moreover, the exactness of the sequences

$$0 \longrightarrow \mathcal{A}_M \longrightarrow \mathcal{D}b_M^* \longrightarrow \pi_*\mathcal{C}_M^* \longrightarrow 0$$

and

$$0 \longrightarrow \mathcal{A}_M \longrightarrow \mathcal{D}f_M^* \longrightarrow \pi_*\mathcal{C}_M^{d,*} \longrightarrow 0$$

is shown and some fundamental properties on $\mathcal{C}_M^*$, $\mathcal{C}_M^{d,*}$ are described. Here $\mathcal{D}b_M^*$ is a sheaf of ultradistributions and $\mathcal{D}f_M^*$ is a sheaf of ultradifferentiable functions. We give some solvability conditions applicable to partial differential equations by operating Aoki's classes of microdifferential operators of infinite order on these sheaves.

AMS Mathematics Subject Classification (2000): 58J15
Key words: hyperfunctions, ultradistributions, ultradifferentiable functions, microdifferential operators

0. *Introduction*

The foundation of the microlocal decomposition for distributions dates back to the time when A. Martineau [30](1964) showed that any distribution is represented as a sum of boundary values of holomorphic functions with bounds. This work was generalized by H. Komatsu [24](1973) for ultradistributions. On the other hand, the microlocal point of view was introduced by M. Sato [36] as the notion of singular spectrum. Moreover, the notion of wave front set due to L. Hörmander and that of essential support of Bros-Iagolnitzer came to being at the beginning of 1970's. Bros-Iagolnitzer [11] conjectured at the same time that analytic singular support for distributions is decomposable. Bengel-Schapira [5](1979) solved it positively by considering Cousin's problem with bounds in a tuboid. They constructed the subsheaves $\mathcal{C}^f$, $\mathcal{C}^d$ of the sheaf $\mathcal{C}$ of microfunctions as J. M. Bony [8] did and initiated the algebro-analytical treatment of distributions. P. Laubin [27](1983) decomposed distributions microlocally by J. Sjöstrand[41]'s method, and J. W. de Roever [34](1984) showed the microlocal decomposition for ultradistributions by a special integral representation. E. Andronikoff [1](1986) reconstructed the sheaf $\mathcal{C}^f$ by microlocalizing the functor $TH(\cdot)$ by which M. Kashiwara [20] solved the Riemann-Hilbert problem of the holonomic $\mathcal{D}_X$-modules.

We study in this paper the microlocalized sheaves for ultradistributions and ultradifferentiable functions by Bengel-Schapira's methods, and give some applications to algebraic analysis.

1. *Ultradistributions and ultradifferentiable functions*

1.1. *Supple sheaves*

First of all we recall the notion of suppleness according to Bengel-Schapira [5].

Definition 1.1.1. *Let $\mathcal{F}$ be a sheaf of Abelian groups on a topological space X. We say that $\mathcal{F}$ is supple if for any open Ω of X, any closed Z, Z_1, Z_2 of Ω with $Z = Z_1 \cup Z_2$, and any section $u \in \Gamma_Z(\Omega; \mathcal{F})$, there exists $u_i \in \Gamma_{Z_i}(\Omega; \mathcal{F})$ $(i = 1, 2)$ with $u = u_1 + u_2$.*

Proposition 1.1.2. *(a) The flabby sheaves are supple. (b) The supple sheaves are soft on a totally paracompact Hausdorff space.*

P r o o f. (a) Let $\mathcal{F}$ be a flabby sheaf on a topological space X. Let

$u \in \Gamma(\Omega; \mathcal{F})$ be a section whose support is the union of two closed sets Z_1 and Z_2. Define u_1 by

$$u_1 = \begin{cases} u & \text{on } \Omega \setminus Z_2, \\ 0 & \text{on } \Omega \setminus Z_1. \end{cases}$$

Note that this gives a well-defined value 0 on the intersection $(\Omega \setminus Z_1) \cap (\Omega \setminus Z_2) = \Omega \setminus (Z_1 \cup Z_2)$, and hence determines a well-defined section on the union $(\Omega \setminus Z_1) \cup (\Omega \setminus Z_2) = \Omega \setminus (Z_1 \cap Z_2)$. By the flabbiness of $\mathcal{F}$, extend this section to Ω, and denote it by u_1 again. Then $\mathrm{supp}(u_1) \subset Z_1$. Moreover, defining $u_2 := u - u_1$, we see that $\mathrm{supp}(u_2) \subset Z_2$ also holds, and hence the result.
(b) Let $\mathcal{F}$ be a supple sheaf on a totally paracompact Hausdorff space X. Let Z be a closed set of X. Since we have

$$\Gamma(Z; \mathcal{F}) = \varinjlim_{\Omega \supset Z} \Gamma(\Omega; \mathcal{F}),$$

for any section $u \in \Gamma(Z; \mathcal{F})$, there exists a section $\tilde{u} \in \Gamma(\Omega; \mathcal{F})$ which represents u. Here the limit on the right hand side is taken with respect to all open sets containing Z. Take an open V of X such that $\Omega \supset \overline{V} \supset V \supset Z$. By the suppleness of $\mathcal{F}$, there exist $\tilde{u}_1 \in \Gamma_{\overline{V}}(\Omega; \mathcal{F})$ and $\tilde{u}_2 \in \Gamma_{\Omega \setminus V}(\Omega; \mathcal{F})$ with $\tilde{u} = \tilde{u}_1 + \tilde{u}_2$. Extend $\tilde{u}_1$ to X by 0, and then we see that the restriction of this section to Z coincides with u again. This shows the proposition.

Now we quote the following two theorems due to Bengel-Schapira [5].

Theorem 1.1.3. *Let X be a locally compact and σ-compact Hausdorff topological space, and let $\mathcal{F}$ be a sheaf of Abelian groups on X. Let A be a subgroup of $\Gamma_c(X; \mathcal{F})$, the space of sections with compact support of $\mathcal{F}$. Assume A to be stable by the decomposition of support, i.e., for any $u \in A$ and for any closed subsets Z, Z_1, Z_2 of X with i) $Z = Z_1 \cup Z_2$, and ii) supp$(u) \subset Z$, there exists $u_i \in A$ $(i = 1, 2)$ with $u = u_1 + u_2$ and supp$(u_i) \subset Z_i$. Then there exists a unique subsheaf $\mathcal{G}$ of $\mathcal{F}$ such that*
a) $\Gamma_c(X; \mathcal{G}) = A$,
b) $\mathcal{G}$ is supple.
Moreover, for any $x \in X$, we have $\mathcal{G}_x = A_x$.

Theorem 1.1.4. *Let $\mathcal{F}$ be a sheaf of Abelian groups on a locally compact and σ-compact Hausdorff space X. Let $X = \bigcup_i \Omega_i$ be an open covering of X.*

Suppose that for any i the sheaf $\mathcal{F} \mid \Omega_i$ on Ω_i is supple. Then $\mathcal{F}$ is supple.

Finally we show

Theorem 1.1.5. *Let X be as in Theorem 1.1.4, and let $\mathcal{F}, \mathcal{F}', \mathcal{F}''$ be sheaves of Abelian groups on X. Let $\mathcal{F}, \mathcal{F}'$ be supple and assume the sequence*

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F}'' \longrightarrow 0$$

to be exact. Then $\mathcal{F}''$ is supple.

P r o o f. Recall the fact that

$$0 \longrightarrow \Gamma_c(X, \mathcal{F}) \longrightarrow \Gamma_c(X, \mathcal{F}') \longrightarrow \Gamma_c(X, \mathcal{F}'') \longrightarrow 0$$

holds under the assumption.

1.2. *Ultradistributions and ultradifferentiable functions*

For the theory of ultradistributions and ultradifferentiable functions we review some conditions according to H. Komatsu [24] and others. Note that these conditions were first studied by C. Roumieu [35].

Let M_p ($p = 0, 1, 2, \ldots$) be a sequence of positive numbers. We shall assume the following conditions (M.0) and (M.1):

(M.0)

$$M_0 = 1,$$

(M.1)(*logarithmic convexity*)

$$M_p^2 \leq M_{p-1} M_{p+1}, \quad p = 1, 2, \ldots.$$

Two logarithmic convex sequences M_p and N_p is defined to be equivalent, $M_p \sim N_p$, if there are numbers $A \geq 1$ and $H \geq 1$ with

$$\frac{N_p}{AH^p} \leq M_p \leq AH^p N_p, \quad p = 0, 1, 2, \ldots$$

Moreover, we shall consider the conditions (M.2) and (M.3) for M_p ($p = 0, 1, 2, \ldots$):

(M.2)(*stability under ultradifferential operators*)

$$M_p \sim \min_{0 \leq q \leq p} M_q M_{p-q}, \quad p = 0, 1, 2, \ldots;$$

(M.3)($strong\ non-quasi-analyticity$) There are a sequence $\widetilde{M}_p$ and a constant $\delta > 0$ such that

$$M_p \sim \widetilde{M}_p \qquad \text{with} \quad \frac{\widetilde{M}_{p-1}}{\widetilde{M}_p} p^{1+\delta} \downarrow 0, \quad p = 1, 2, \ldots .$$

We shall possibly replace (M.2) and (M.3) by the following weaker conditions:
(M.2)$'$($stability\ under\ differential\ operators$)

$$M_{p+1} \sim M_p, \quad p = 0, 1, 2, \ldots ;$$

(M.3)$'$ ($non-quasi-analyticity$)

$$\sum_{p=1}^{\infty} \frac{M_{p-1}}{M_p} < \infty.$$

We note that if $s > 1$ the Gevrey sequence

$$M_p = (p!)^s \text{ or } p^{ps} \text{ or } \Gamma(1 + ps)$$

satisfies the above conditions (M.0), (M.1), (M.2), and (M.3).

Definition 1.2.1. *We assume that M_p satisfies (M.1) and (M.3)$'$. Then $*$ will stand for either (M_p) or $\{M_p\}$. Let M be a real analytic manifold, U be an open subset of M, and $\varphi(x) \in C^\infty(U)$. Then we set*

$$\|\varphi\|_{K,h,M_p} = \sup_{x \in K, \alpha \geq 0} \frac{| D^\alpha \varphi(x) |}{h^{|\alpha|} M_{|\alpha|}}$$

for a compact subset K of U and a positive number h. We define the space $\mathcal{D}f^*_M(U)$ of ultradifferential functions of class $*$ on U by
i) in case $* = (M_p)$,

$$\mathcal{D}f^{(M_p)}_M(U) = \{\varphi \in C^\infty(U) \mid \textit{ for } \forall K \subset\subset U \textit{ and } \forall h > 0 \quad \|\varphi\|_{K,h,M_p} < \infty\},$$

ii) in case $* = \{M_p\}$,

$$\mathcal{D}f^{\{M_p\}}_M(U) = \{\varphi \in C^\infty(U) \mid \textit{ for } \forall K \subset\subset U \quad \exists h > 0 \quad \|\varphi\|_{K,h,M_p} < \infty\}$$

We denote by $\mathcal{D}f^*_{M,c}(U)$ the space of ultradifferentiable functions of class $*$ with compact support on U. Refer to H. Komatsu [24] for its topology. We *also define the space $\mathcal{D}b_M^*$ of ultradistributions of class $*$ by the strong dual space of $\mathcal{D}f_{M,c}^*(U)$. Note that the presheaves*

$$U \longmapsto \mathcal{D}f_M^*(U),$$

$$U \longmapsto \mathcal{D}b_M^*(U)$$

become sheaves. We denote thus by $\mathcal{D}f_M^*$ the sheaf of ultradifferentiable functions of class $*$ on M, and by $\mathcal{D}b_M^*$ the sheaf of ultradistributions of class $*$ on M. We always consider $\mathcal{D}f_M^*$ and $\mathcal{D}b_M^*$ as subsheaves of the sheaf $\mathcal{B}_M$ of Sato's hyperfunctions on M.

We introduce an order to the set of all (M_p)'s and $\{M_p\}$'s satisfying the conditions as follows.
i) $(M_p) < (N_p), \{M_p\} < \{N_p\}$ if there are constants L and C such that

$$M_p \leq CL^p N_p, \quad p = 0, 1, 2, \dots.$$

ii) $\{M_p\} < (N_p)$ if for any $\epsilon > 0$ there is a constant C_ϵ such that

$$M_p \leq C_\epsilon \epsilon^p N_p, \quad p = 0, 1, 2, \dots.$$

iii) $(M_p) < \{M_p\}$.

We may consider $\mathcal{D}b_M^{(\infty)}$ and $\mathcal{D}f_M^{(\infty)}$ as the sheaves of distributions and differentiable functions, respectively. The order is considered as
iv) $(M_p), \{M_p\} < (\infty)$.

Definition 1.2.2. *Let M_p satisfy (M.1), (M.2) and (M.3). Then we define the associated function $M(\rho)$ (resp. the growth function $M^\star(\rho)$) of M_p by*

$$M(\rho) = \sup_p \log \frac{\rho^p M_0}{M_p} \quad \left(\textit{resp. } M^\star(\rho) = \sup_p \log \frac{\rho^p p! M_0}{M_p} \right).$$

Next we recall the following theorem by H. Komatsu [24].

Theorem 1.2.3. *Suppose that the sequence M_p $p = 0, 1, 2, \dots$ satisfy (M.1), (M.2) and (M.3). Let V be a Stein open set in $\mathbb{C}^n$ and Γ be an open convex cone in $\mathbb{R}^n$. We set*

$$V_\Gamma = (\mathbb{R}^n + \sqrt{-1}\Gamma) \cap V, \textit{ and } U = \mathbb{R}^n \cap V.$$

Let $F(x+\sqrt{-1}y)$ *be a holomorphic function on* V_Γ *and assume the condition:*

For any compact set K *in* U *and closed subcone* $\Gamma' \subset \Gamma$, *there are constants* L *and* C *(resp. for any* $L > 0$ *there is a constant* C*) satisfying the estimate*

$$\sup_{x \in K} \mid F(x+\sqrt{-1}y) \mid \leq C \exp M^\star \left(\frac{L}{\mid y \mid} \right)$$

for $y \in \Gamma$ *with* $\mid y \mid$ *sufficiently small.*

Then there is an ultradistribution $F(x+\sqrt{-1}\Gamma 0) \in \mathcal{D}b^{(M_p)}(U)$ *(resp.* $\mathcal{D}b^{\{M_p\}}(U)$*) which is the boundary value of* $F(x+\sqrt{-1}y)$ *in the sense that*

$$F(x+\sqrt{-1}y) \longrightarrow F(x+\sqrt{-1}\Gamma 0) \quad in\ \mathcal{D}b^{(M_p)}(U) \quad (resp.\ \mathcal{D}b^{\{M_p\}}(U))$$

as y *tends to 0 in a closed subcone* Γ' *of* Γ. *Moreover, the boundary value* $F(x+\sqrt{-1}\Gamma 0)$ *coincides with that in the sense of hyperfunction.*

1.3. $*^{-1}$-*Singular spectrum and* $*$-*singular spectrum*

Hereafter $*$ stands for either (M_p) or $\{M_p\}$. In this section we will give the definition of singular spectrum of various classes.

Definition 1.3.1. *Let* $u(x)$ *be a hyperfunction with compact support. we define the Fourier transform* $\tilde{u}(\xi)$ *of* u *by*

$$\tilde{u}(\xi) = \int u(x) e^{-\sqrt{-1}x\xi}\, dx.$$

Let us give a well-known theorem about the estimate of $\tilde{u}(\xi)$. Refer to H. Komatsu [24] for more details.

Theorem 1.3.2. *Suppose that* M_p *satisfies (M.1), (M.2)′ and (M.3)′. Let* $u(x)$ *be a hyperfunction with compact support on M. Then* $u(x)$ *is in* $\mathcal{D}b^*_M(M)$ $(\mathcal{D}f^*_M(M))$ *if and only if*
i) in case $\mathcal{D}b^{(M_p)}$, *there exist positive numbers* L *and* C *with*

$$\mid \tilde{u}(\xi) \mid \leq C \exp M(L \mid \xi \mid),$$

ii) in case $\mathcal{D}b^{\{M_p\}}$, *for any positive* ϵ *there exists a positive number* C_ϵ *with*

$$\mid \tilde{u}(\xi) \mid \leq C_\epsilon \exp M(\epsilon \mid \xi \mid),$$

iii) in case $\mathcal{D}f^{(M_p)}$, *for any positive* ϵ *there exists a positive number* C_ϵ *with*

$$\mid \tilde{u}(\xi) \mid \leq C \exp \{-M(\epsilon \mid \xi \mid)\},$$

iv) in case $\mathcal{D}f^{\{M_p\}}$*, there exist positive numbers* L *and* C *with*

$$|\,\tilde{u}(\xi)\,| \leq C \exp\{-M(L\,|\,\xi\,|)\}.$$

Definition 1.3.3. *Let* M *be an open set of* $\mathbb{R}^n$*, and denote by* $\sqrt{-1}S^*M$ *the pure imaginary cotangential sphere bundle of* M*. Let* $u(x) \in \mathcal{D}b^{\dagger}_M(M)$ *with compact support. We introduce* $*^{-1}$*-singular spectrum (resp.* $*$*-singular spectrum)* $\mathrm{SS}^{*^{-1}}(u)$ *(resp.* $\mathrm{SS}^*(u)$*) of* u *in* $\sqrt{-1}S^*M$ *when* $\dagger < *$*. Let* $\mathring{q} = (\mathring{x}, \sqrt{-1}\,\mathring{\xi}\infty) \in \sqrt{-1}S^*M$*. Then* $\mathring{q} \notin \mathrm{SS}^{*^{-1}}(u)$ *(resp.* $\mathrm{SS}^*(u)$*) if*
i) in case $* = (M_p)$*, there exist a neighborhood* U *of* $\mathring{x}$ *and a conic neighborhood* $\varXi$ *of* $\mathring{\xi}$ *such that for any* $\phi \in \mathcal{D}f^{\dagger}(U)$ *there exist* $L > 0$ *and* $C > 0$ *(resp. for any* $\phi \in \mathcal{D}f^{\dagger}(U)$ *and any* $\epsilon > 0$ *there exists* $C_\epsilon > 0$*) satisfying*

$$|\,\widetilde{\phi u}(\xi)\,| \leq C \exp M(L\,|\,\xi\,|), \quad \xi \in \varXi,$$

$$\Big(\textit{resp.} \quad |\,\widetilde{\phi u}(\xi)\,| \leq C_\epsilon \exp\{-M(\epsilon\,|\,\xi\,|)\}, \quad \xi \in \varXi,\Big)$$

ii) in case $* = \{M_p\}$*, there exist a neighborhood* U *of* $\mathring{x}$ *and a conic neighborhood* $\varXi$ *of* $\mathring{\xi}$ *such that for any* $\phi \in \mathcal{D}f^{\dagger}(U)$ *and any* $\epsilon > 0$ *there exists* $C_\epsilon > 0$ *(resp. for any* $\phi \in \mathcal{D}f^{\dagger}(U)$ *there exist* $L > 0$ *and* $C > 0$*) satisfying*

$$|\,\widetilde{\phi u}(\xi)\,| \leq C_\epsilon \exp M(\epsilon\,|\,\xi\,|), \quad \xi \in \varXi,$$

$$\Big(\textit{resp.} \quad |\,\widetilde{\phi u}(\xi)\,| \leq C \exp\{-M(L\,|\,\xi\,|)\}, \quad \xi \in \varXi.\Big)$$

Note that if $* = \{p!\}$, SS^* coincides with the singular spectrum S.S. that M. Sato et al. [37] introduced. We denote it by SS hereafter. We remark that L. Hörmander [18] introduced the wave front set of several classes WF_L which is the original idea of our SS^*.

1.4. *Operations for ultradistributions*

Hereafter we always assume that the sequence M_p satisfies (M.1), (M.2) and (M.3). We devote this section to product, integration, and restriction of ultradistributions.

We recall some theorems by M. Sato et al. [37]. Note that we define $\widehat{\mathrm{SS}}$ as follows.

$$\begin{aligned}\widehat{\mathrm{SS}}(u) &= \{(x, \sqrt{-1}\xi) \in \sqrt{-1}T^*M \mid x \in \mathrm{supp}u \text{ and } \xi = 0\} \\ &\cup \{(x, \sqrt{-1}\xi) \in \sqrt{-1}T^*M \mid \xi \neq 0 \text{ and } (x, \sqrt{-1}\xi\infty) \in \mathrm{SS}(u)\}.\end{aligned}$$

Theorem 1.4.1. *Let M_1 and M_2 be real analytic manifolds, and set $M = M_1 \times M_2$. For $u_1(x_1) \in \mathcal{D}b^*(M_1)$ (resp. $\mathcal{D}f^*(M_1)$) and $u_2(x_2) \in \mathcal{D}b^*(M_2)$ (resp. $\mathcal{D}f^*(M_2)$), we can define canonically the product $u(x_1, x_2) = u_1(x_1)u_2(x_2)$ so that*

$$\widehat{\mathrm{SS}}(u) \subset \widehat{\mathrm{SS}}(u_1) \times \widehat{\mathrm{SS}}(u_2).$$

P r o o f. If we consult Theorem 2.4.1 in Chapter I of M. Sato et al. [37], we obtain the desired result.

Theorem 1.4.2. *Let $u(x)$ and $v(x)$ be ultradistributions of class $*$ on a real analytic manifold M such that $\mathrm{SS}^*(u) \cap (\mathrm{SS}^*(v))^a = \emptyset$. Then the product $u(x)v(x) \in \mathcal{D}b^*(M)$ exists with the property*

$$\begin{aligned}\mathrm{SS}^*(uv) \subset \{&(x, \sqrt{-1}(\theta\xi_1 + (1-\theta)\xi_2)\infty) \mid \\ &(x, \sqrt{-1}\xi_1\infty) \in \mathrm{SS}^*(u), (x, \sqrt{-1}\xi_2\infty) \in \mathrm{SS}^*(v),\ 0 \leq \theta \leq 1\},\end{aligned}$$

where $a : \sqrt{-1}S^*M \longrightarrow \sqrt{-1}S^*M$ is the antipodal mapping (i.e., $a(x, \sqrt{-1}\xi\infty) = (x, -\sqrt{-1}\xi\infty)$), and $(\mathrm{SS}^*(v))^a$ denotes the image of $\mathrm{SS}^*(v)$ by a.

P r o o f. The theorem can be reduced to the case where u and v have small compact support in $\mathbb{R}^n$. Moreover we may assume from the beginning that there exist an open set U of $\mathbb{R}^n$ and closed convex cones Ξ_1 and Ξ_2 of $\mathbb{R}^n$ such that

$$\mathrm{SS}^*(u_1) \subset U \times \sqrt{-1}\widehat{\Xi_1}\infty,$$
$$\mathrm{SS}^*(u_2) \subset U \times \sqrt{-1}\widehat{\Xi_2}\infty,$$

where $\widehat{\Xi_i}$ is the image of Ξ_i by the natural projection from $\mathbb{R}^n \setminus \{0\}$ to S^{n-1} $(i = 1, 2)$. Take a closed cone Ξ of $\mathbb{R}^n$ with $\Xi \cap \mathrm{ch}(\Xi_1 \cup \Xi_2) = \emptyset$. Here $\mathrm{ch}(\cdot)$ means the convex hull. Then

$$\Xi \cap \Xi_2 = \emptyset, \quad (\Xi - \Xi_2) \cap \Xi_1 = \emptyset.$$

Let $\xi \in \Xi$, $\eta \in \Xi_2$. Since we have $\mid \xi - \eta \mid \geq c(\mid \xi \mid + \mid \eta \mid)$ for some $c > 0$, we obtain for any $\epsilon > 0$ and for some $L > 0$ the estimate

$$\begin{aligned}\mid \tilde{u}(\xi - \eta)\tilde{v}(\eta) \mid \ &\leq C_\epsilon \exp\{-M(\epsilon \mid \xi - \eta \mid)\} C \exp M(L \mid \eta \mid) \\ &\leq C_\epsilon C \exp\{-M(c\epsilon \mid \xi \mid + c\epsilon \mid \eta \mid) + M(L \mid \eta \mid)\}.\end{aligned}$$

Here, by Proposition 3.6 of H. Komatsu [24],

$$-M(\rho_1 + \rho_2) \leq -M(\frac{\rho_1}{H}) - M(\frac{\rho_2}{H}) + \log(AM_0)$$

holds for some $H \geq 1$. We obtain thereby

$$| \tilde{u}(\xi-\eta)\tilde{v}(\eta) | \leq C_\epsilon C A \exp\left\{-M(\frac{c\epsilon}{H} \mid \xi \mid)\right\} \exp\left\{-M(\frac{c\epsilon}{H} \mid \eta \mid) + M(L \mid \eta \mid)\right\}.$$

Hence we have

$$| \int_{\Xi_2} \tilde{u}(\xi-\eta)\tilde{v}(\eta)\, d\eta | \leq C_\epsilon \exp\{-M(\epsilon \mid \xi \mid)\}$$

for any $\epsilon > 0$ and $\xi \in \Xi$.

Next let $\xi \in \Xi$, $\eta \notin \Xi_2$. Take a small $\epsilon' > 0$ and suppose that

$$| \xi | \geq 1, \quad | \eta | \leq \epsilon' \mid \xi \mid .$$

Then we have $\xi - \eta \notin \Xi_1$ and $| \xi - \eta | \geq (1-\epsilon') \mid \xi \mid \geq (1-2\epsilon') \mid \xi \mid + \mid \eta \mid$, which leads to the estimate

$$\begin{aligned} | \tilde{u}(\xi-\eta)\tilde{v}(\eta) | \leq \quad & C_{\epsilon,\epsilon''} \exp\left\{-M(\frac{\epsilon(1-2\epsilon')}{H} \mid \xi \mid)\right\} \\ & \cdot \exp\left\{-M(\frac{\epsilon}{H} \mid \eta \mid) - M(\epsilon'' \mid \eta \mid)\right\} \end{aligned}$$

for any $\epsilon > 0$ and $\epsilon'' > 0$.

Finally let $\xi \in \Xi$, $\eta \notin \Xi_2$ and suppose that

$$| \eta | \geq \epsilon' \mid \xi \mid .$$

Then

$$| \eta | \geq (| \eta | + \epsilon' \mid \xi \mid)/2,$$

$$\begin{aligned} | \tilde{u}(\xi-\eta)\tilde{v}(\eta) | \quad & \leq C_{\epsilon''} \exp\left\{M(L \mid \xi - \ eta \mid) - M(\frac{\epsilon''}{2} \mid \eta \mid + \frac{\epsilon'\epsilon''}{2} \mid \xi \mid)\right\} \\ & \leq C_{\epsilon''} A \exp\left\{-M(\frac{\epsilon'\epsilon''}{2H} \mid \xi \mid)\right\} \exp\left\{M((1+\frac{1}{\epsilon'})L \mid \eta \mid) - M(\frac{\epsilon''}{2H} \mid \eta \mid)\right\} \end{aligned}$$

hold for any $\epsilon'' > 0$. Hence we deduce

$$| \int_{\mathbb{R}^n \setminus \Xi_2} \tilde{u}(\xi-\eta)\tilde{v}(\eta)\, d\eta | \leq C_\epsilon \exp\{-M(\epsilon \mid \xi \mid)\}$$

for any $\epsilon > 0$ and $\xi \in \Xi$. Thus if we define the product of u and v by

$$u(x)v(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n \times \mathbb{R}^n} \tilde{u}(\xi-\eta)\tilde{v}(\eta) e^{\sqrt{-1}x\xi}\, d\eta d\xi,$$

we will obtain the desired result. Note that it is easy to show $uv \in \mathcal{D}b^*(\mathbb{R}^n)$.

Theorem 1.4.3. *Let M and N be real analytic manifolds, and let $f : M \times N \longrightarrow N$ be the natural projection. If $f\mid_{\mathrm{supp}u}$ is a proper map for $u(t,x)$ an ultradistribution of class $*$ on $M \times N$, then the integration of $u(t,x)$ along the fiber*

$$v(x) = \int_{f^{-1}(x)} u(t,x)\,dt$$

can be defined. Moreover, we have

$$\mathrm{SS}^*(v) \subset \varpi(\mathrm{SS}^*(u) \cap M \times \sqrt{-1}S^*N),$$

where ϖ denotes the natural projection from $M \times \sqrt{-1}S^*N$ to $\sqrt{-1}S^*N$.

P r o o f. We define the integration of ultradistributions as that of hyperfunctions. Refer to M. Sato et al. [37] for more details. We can obtain the estimate of SS^* immediately from Fourier transform of v.

Corollary 1.4.4. *Let $u(x)$ and $v(x)$ be ultradistributions of class $*$ on a real analytic manifold M, either of which has compact support. Then the convolution $u(x) * v(x) \in \mathcal{D}b^*(M)$ exists with the property*

$$\mathrm{SS}^*(u*v) \subset \{(x+y, \sqrt{-1}\xi\infty) \mid (x, \sqrt{-1}\xi\infty) \in \mathrm{SS}^*(u), (y, \sqrt{-1}\xi\infty) \in \mathrm{SS}^*(v)\}.$$

Let u be an ultradistribution of class $*$ on an open set U of $\mathbb{R}^n$ with $(x, \pm\sqrt{-1}(1,0,\dots,0)\infty) \notin \mathrm{SS}^*(u)$ for any $x \in U$. Then we say u contains x_1 as $*$-ultradifferentiable parameter on U.

Corollary 1.4.5. *Let $u(x)$ be an ultradistribution of class $*$ in a neighborhood U of the origin in $\mathbb{R}^n$, and assume that it contains x_1 as $*$-ultradifferentiable parameter. Then the restriction $u\mid_{x_1=0}$ can be defined as*

$$u\mid_{x_1=0} = \int \delta(x_1)u(x)\,dx_1.$$

Moreover, we have the estimate

$$\mathrm{SS}^*(u\mid_{x_1=0}) \subset \pi(\mathrm{SS}^*(u) \cap \{x_1 = 0\} \times \sqrt{-1}S^{n-1}\infty),$$

where π is the natural projection from $\sqrt{-1}S^*U$ to $\{\xi_1 = 0\}$.

2. *Microlocal decomposition*

2.1. *The sheaves $\mathcal{C}_M^*$ and $\mathcal{C}_M^{d,*}$*

We construct the sheaves $\mathcal{C}_M^*$ and $\mathcal{C}_M^{d,*}$ in order to decompose analytic microlocal singularities of ultradistributions and ultradifferentiable functions. As we know well, the sheaf $\mathcal{C}_M$ of microfunctions on a real analytic manifold M enjoys the exact sequence

$$0 \longrightarrow \mathcal{A}_M \longrightarrow \mathcal{B}_M \longrightarrow \pi_*\mathcal{C}_M \longrightarrow 0,$$

which is due to M. Sato et al. [37]. Here $\mathcal{A}_M$ denotes the sheaf of real analytic functions on M, and π is the natural projection from $\sqrt{-1}S^*M$ to M. Moreover there exists a canonical surjective spectrum map

$$\mathrm{Sp}_M : \pi^{-1}\mathcal{B}_M \longrightarrow \mathcal{C}_M,$$

by which we define the singular spectrum $\mathrm{SS}(u)$ of $u \in \mathcal{B}_M$ by

$$\mathrm{SS}(u) = \mathrm{supp}(\mathrm{Sp}_M(u)).$$

The injection

$$\mathcal{D}b_M^* \hookrightarrow \mathcal{B}_M$$

$$(\text{resp.} \quad \mathcal{D}f_M^* \hookrightarrow \mathcal{B}_M)$$

induces a sheaf homomorphism

$$\pi^{-1}\mathcal{D}b_M^* \longrightarrow \mathcal{C}_M$$

$$(\text{resp.} \quad \pi^{-1}\mathcal{D}f_M^* \longrightarrow \mathcal{C}_M).$$

Then we define a subsheaf $\mathcal{C}_M^*$ (resp. $\mathcal{C}_M^{d,*}$) of $\mathcal{C}_M$ on $\sqrt{-1}S^*M$ as the image of the above morphism and call it the sheaf of microfunctions of class $*$ (resp. $d,*$). Explicitly the sheaf $\mathcal{C}_M^*$ (resp. $\mathcal{C}_M^{d,*}$) coincides with the sheaf associated to the presheaf

$$\Omega \longmapsto \Gamma(\pi(\Omega); \mathcal{D}b_M^*)/\{u \in \Gamma(\pi(\Omega); \mathcal{D}b_M^*) \mid \mathrm{SS}(u) \cap \Omega = \emptyset\}.$$

$$(\text{resp. } \Omega \longmapsto \Gamma(\pi(\Omega); \mathcal{D}f_M^*)/\{u \in \Gamma(\pi(\Omega); \mathcal{D}f_M^*) \mid \mathrm{SS}(u) \cap \Omega = \emptyset\}.)$$

Furthermore, we have a canonical exact sequence

$$0 \longrightarrow \mathcal{A}_M \longrightarrow \mathcal{D}b_M^* \longrightarrow \pi_*\mathcal{C}_M^* \longrightarrow 0$$

$$(\text{resp. } 0 \longrightarrow \mathcal{A}_M \longrightarrow \mathcal{D}f^*_M \longrightarrow \pi_*\mathcal{C}^{d,*}_M \longrightarrow 0).$$

Note that if $\dagger < *$, we have canonical injections

$$\mathcal{C}^{d,\dagger}_M \hookrightarrow \mathcal{C}^{d,*}_M \hookrightarrow \mathcal{C}^{d,(\infty)}_M \hookrightarrow \mathcal{C}^{(\infty)}_M \hookrightarrow \mathcal{C}^*_M \hookrightarrow \mathcal{C}^\dagger_M \hookrightarrow \mathcal{C}_M.$$

2.2. *Curvilinear expansion*

As we know well, the δ-function on $\mathbb{R}^n$ can be written as

$$\delta(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{\sqrt{-1}\xi x}\, d\xi.$$

If we change the variables ξ into $\xi + \sqrt{-1} \mid \xi \mid x$, we derive the following representation due to Bros-Iagolnitzer:

$$\delta(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\left(1+\sqrt{-1}\frac{\xi}{\mid \xi \mid}x\right) e^{\sqrt{-1}\xi x - |\xi| x^2}\, d\xi.$$

However if we introduce an extra variable u as in Melin-Sjöstrand [32], we also have

$$\begin{aligned}\delta(x-x') &= \frac{1}{(2\pi\sqrt{\pi})^n}\int_{\mathbb{R}^n\times\mathbb{R}^n}(2\mid \xi \mid)^{\frac{n}{2}}\left\{1+\frac{1}{2}\frac{\xi}{\mid \xi \mid}(x-x')\right\} \\ &\qquad \times e^{\sqrt{-1}\xi(x-x') - |\xi|\{(x-u)^2+(x'-u)^2\}}\, du d\xi.\end{aligned}$$

If we introduce the polar coordinates $\xi = r\omega$ here and integrate it with respect to the radius r, we naturally deduce the formula

$$\begin{aligned}\delta(x-x') = &\ \frac{2^n\Gamma(n+\frac{n}{2})}{(-2\pi\sqrt{-1})^{n+\frac{n}{2}}}\cdot \\ &\cdot\int_{\mathbb{R}^n\times S^{n-1}} \frac{\left\{1+\frac{1}{2}\sqrt{-1}\omega(x-x')\right\} du d\omega}{[\omega(x-x')+\sqrt{-1}\{(x-u)^2+(x'-u)^2\}]^{n+\frac{n}{2}}}.\end{aligned}$$

This formula was obtained by P. Laubin [27]. In this situation we have the following theorem.

Theorem 2.2.1. *Let Z be a closed subset of $\mathbb{R}^n \times S^{n-1}$ and let*

$$W_Z(x,x') = \frac{2^n\Gamma(n+\frac{n}{2})}{(-2\pi\sqrt{-1})^{n+\frac{n}{2}}}\int_Z \frac{\left\{1+\frac{1}{2}\sqrt{-1}\omega(x-x')\right\} du d\omega}{[\omega(x-x')+\sqrt{-1}\{(x-u)^2+(x'-u)^2\}]^{n+\frac{n}{2}}}.$$

Then W_Z is a distribution and we have the estimate

$$\mathrm{SS}(W_Z) \subset \{(x, x', \sqrt{-1}(\xi, \xi')\infty) \mid x = x', \xi = -\xi', (x, \xi) \in Z\}.$$

Refer to M. Kashiwara et al. [21] for details about the above estimate for $\mathrm{SS}(W_Z)$.

2.3. *Microlocal decomposition of singularities*

Now we give our main theorem. Let M be a real analytic manifold in this section.

Theorem 2.3.1. *The sheaf $\mathcal{C}_M^*$ (resp. $\mathcal{C}_M^{d,*}$) is supple.*

Before entering the proof, we prepare the following lemma.

Lemma 2.3.2. *Let G be a proper open convex subset of the sphere S^{n-1}. Let u be an element of $(\mathcal{D}b^*/\mathcal{A})(\mathbb{R}^n)$ (resp. $(\mathcal{D}f^*/\mathcal{A})(\mathbb{R}^n)$) whose singular spectrum $\mathrm{SS}(u)$ is a compact set K of $\mathbb{R}^n \times \sqrt{-1}G\infty$. Let K_1 and K_2 be two compact sets of $\mathbb{R}^n \times \sqrt{-1}G\infty$ with $K = K_1 \cup K_2$. Then there exists $u_i \in (\mathcal{D}b^*/\mathcal{A})(\mathbb{R}^n)$ (resp. $(\mathcal{D}f^*/\mathcal{A})(\mathbb{R}^n)$) $(i = 1, 2)$ with $u = u_1 + u_2$ and $\mathrm{SS}(u_i) \subset K_i$ $(i = 1, 2)$.*

P r o o f. We identify $\mathbb{R}^n \times \sqrt{-1}G\infty$ with $\mathbb{R}^n \times G$.
a) Let u be an ultradistribution of class $*$ on $\mathbb{R}^n$ whose singular spectrum $\mathrm{SS}(u)$ is a compact set of K of $\mathbb{R}^n \times G$. We take closed sets Z_1, Z_2 of $\mathbb{R}^n \times S^{n-1}$ such that $Z_1 \cup Z_2 = \mathbb{R}^n \times S^{n-1}$, $K \cap Z_1 \subset K_1$, $K \cap Z_2 \subset K_2$ and $m(Z_1 \cap Z_2) = 0$ where m implies $(2n-1)$ dimensional Lebesgue measure on $\mathbb{R}^n \times S^{n-1}$. Then, by Theorems 1.4.2 and 1.4.3, $U_{Z_1}(x)$ and $U_{Z_2}(x)$ defined by

$$U_{Z_i}(x) = \int_{\mathbb{R}^n} W_{Z_i}(x, x') u(x')\, dx' \quad (i = 1, 2)$$

are ultradistributions of class $*$ on $\mathbb{R}^n$ with the estimates

$$\mathrm{SS}(U_{Z_i}) \subset Z_i \cap \mathrm{SS}(u) \subset K_i \quad (i = 1, 2).$$

Finally remark that

$$u(x) = U_{Z_1}(x) + U_{Z_2}(x).$$

b) Let u be an ultradifferentiable function of class $*$ on $\mathbb{R}^n$ whose singular spectrum $\mathrm{SS}(u)$ is a compact set K of $\mathbb{R}^n \times G$. Then we can define U_{Z_1} and

U_{Z_2} in the same way as in the part a) and we obtain the same estimates. Moreover, by Theorems 1.4.2 and 1.4.3, we have

$$\mathrm{SS}^*(U_{Z_1}) = \mathrm{SS}^*(U_{Z_2}) = \emptyset,$$

which shows the lemma.

Proof of Theorem 2.3.1. Put

$$A = \{u \in (\mathcal{D}b^*/\mathcal{A})(\mathbb{R}^n) \mid \mathrm{SS}(u) \subset \mathbb{R}^n \times \sqrt{-1}G\infty,\ \ \mathrm{SS}(u) \text{ is compact}\}$$

(resp. $A = \{u \in (\mathcal{D}f^*/\mathcal{A})(\mathbb{R}^n) \mid \mathrm{SS}(u) \subset \mathbb{R}^n \times \sqrt{-1}G\infty,\ \ \mathrm{SS}(u) \text{ is compact}\}$). By Lemma 2.3.2 and Theorem 1.1.3 there exists a unique subsheaf $\mathcal{G}$ of the sheaf $\mathcal{C}\mid_{\mathbb{R}^n\times\sqrt{-1}G\infty}$ such that

$$\begin{aligned} &\text{a)} \quad \Gamma_c(\mathbb{R}^n \times \sqrt{-1}G\infty, \mathcal{G}) = A,\\ &\text{b)} \quad \mathcal{G} \text{ is supple.}\end{aligned}$$

Then it is easy to show that

$$\mathcal{G} = \mathcal{C}^* \mid_{\mathbb{R}^n\times\sqrt{-1}G\infty} \quad (\text{resp. } \mathcal{G} = \mathcal{C}^{d,*} \mid_{\mathbb{R}^n\times\sqrt{-1}G\infty}).$$

Thus, by Theorem 1.1.4, the theorem is proved.

Note that the suppleness of $\mathcal{C}_M^*$ was shown by J. W. de Roever [34] for the first time.

Corollary 2.3.3. *Let $u \in \mathcal{D}b^*(M)$ (resp. $\mathcal{D}f^*(M)$) with $\mathrm{SS}(u) = F$. Let $F = \bigcup_{i=1}^{p} F_i$ be a closed covering of F in $\sqrt{-1}S^*M$. Then there exists $u_i \in \mathcal{D}b^*(M)$ (resp. $\mathcal{D}f^*(M)$) with $\mathrm{SS}(u_i) \subset F_i$ and $u = \sum_{i=1}^{p} u_i$.*

Corollary 2.3.4 *("Edge of the Wedge theorem") Let u_i $(i = 1, \ldots, p)$ be ultradistributions of class $*$ (resp. ultradifferentiable functions of class $*$) on M with $\sum_{i=1}^{p} u_i = 0$. Let $F_i = \mathrm{SS}(u_i) \subset \sqrt{-1}S^*M$. Then there exist ultradistributions of class $*$ (resp. ultradifferentiable functions of class $*$) $u_{ij} (i, j = 1, \ldots, p)$ such that*

$$u_i = \sum_{j\neq i} u_{ij} \quad \textit{for any } i,$$

$$\mathrm{SS}(u_{ij}) \subset F_i \cap F_j \quad \textit{for any } i, j.$$

P r o o f. We prove this by induction on p. When $p = 2$, this corollary is trivial. Now suppose that the statement is true in the case of $p-1$. Since

$$u_p = -\sum_{j=1}^{p-1} u_j,$$

we have

$$\mathrm{SS}(u_p) \subset \bigcup_{j=1}^{p-1} (F_p \cap F_j).$$

By Corollary 2.3.3 there exist $v_{pj} \in \mathcal{D}b^*(M)$ (resp. $\mathcal{D}f^*(M)$) $(j = 1, \ldots, p-1)$ such that

$$u_p = \sum_{j=1}^{p-1} v_{pj}, \quad \mathrm{SS}(v_{pj}) \subset F_p \cap F_j.$$

Then

$$\sum_{j=1}^{p-1} (u_j + v_{pj}) = 0, \quad \mathrm{SS}(u_j + v_{pj}) \subset F_j.$$

Thus by the hypothesis of the induction there exist $v_{jk} \in \mathcal{D}b^*(M)$ (resp. $\mathcal{D}f^*(M)$) such that

$$u_j + v_{pj} = \sum_{\substack{k=1 \\ k \neq j}}^{p-1} v_{jk} \quad j = 1, \ldots, p-1,$$

$$\mathrm{SS}(v_{jk}) \subset F_j \cap F_k \quad j, k = 1, \ldots, p-1, j \neq k.$$

Putting $v_{jp} = -v_{pj}$ and transposing this to the right-hand side, we complete the proof for the case of p.

Definition 2.3.5. *We define sheaves $\mathcal{C}_M^{\dagger,*}$, $\mathcal{C}_M^{\dagger/*}$, $\mathcal{C}_M^{d,\dagger,*}$ on $\sqrt{-1}S^*M$ by the following exact sequences.*
i)

$$0 \longrightarrow \mathcal{C}_M^{d,*} \longrightarrow \mathcal{C}_M^{\dagger} \longrightarrow \mathcal{C}_M^{\dagger,*} \longrightarrow 0.$$

ii)

$$0 \longrightarrow \mathcal{C}_M^{*} \longrightarrow \mathcal{C}_M^{\dagger} \longrightarrow \mathcal{C}_M^{\dagger/*} \longrightarrow 0$$

when $* > \dagger$.

iii)

$$0 \longrightarrow \mathcal{C}_M^{d,*} \longrightarrow \mathcal{C}_M^{d,\dagger} \longrightarrow \mathcal{C}_M^{d,\dagger,*} \longrightarrow 0$$

when $* < \dagger$.

Theorem 2.3.6. *i) The sheaf* $\mathcal{C}_M^{\dagger,*}$ *is supple and we have the exact sequence*

$$0 \longrightarrow \mathcal{D}f_M^* \longrightarrow \mathcal{D}b_M^\dagger \longrightarrow \pi_*\mathcal{C}_M^{\dagger,*} \longrightarrow 0.$$

ii) The sheaf $\mathcal{C}_M^{\dagger/*}$ *is supple and we have the exact sequence*

$$0 \longrightarrow \mathcal{D}b_M^* \longrightarrow \mathcal{D}b_M^\dagger \longrightarrow \pi_*\mathcal{C}_M^{\dagger/*} \longrightarrow 0$$

when $* > \dagger$.
iii) The sheaf $\mathcal{C}_M^{d,\dagger,*}$ *is supple and we have the exact sequence*

$$0 \longrightarrow \mathcal{D}f_M^* \longrightarrow \mathcal{D}f_M^\dagger \longrightarrow \pi_*\mathcal{C}_M^{d,\dagger,*} \longrightarrow 0$$

when $* < \dagger$.

P r o o f. i) The suppleness of $\mathcal{C}_M^{\dagger,*}$ is directly derived from the definition and Theorem 1.1.5. When we construct a diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & \mathcal{A}_M & \to & \mathcal{A}_M & \to & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{D}f_M^* & \to & \mathcal{D}b_M^\dagger & \to & \pi_*\mathcal{C}_M^{\dagger,*} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \pi_*\mathcal{C}_M^{d,*} & \to & \pi_*\mathcal{C}_M^{\dagger} & \to & \pi_*\mathcal{C}_M^{\dagger,*} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

we can easily see the proof by the nine lemmata.
The parts ii) and iii) follow from a similar argument as in i).

Theorem 2.3.7. *Let* Ω *be an open subset in* $\sqrt{-1}S^*M$ *and let* $u \in \mathcal{D}b_M^*(\pi(\Omega))$. *Then the following conditions i), ii) are equivalent:*
i) $\mathrm{SS}^*(u) \cap \Omega = \emptyset$,
ii) $\mathrm{Sp}(u)\ |_\Omega \in \mathcal{C}_M^{d,*}(\Omega)$.

P r o o f. Let i) hold. For any closed set Z in Ω, we put

$$v_Z(x) = \int_{\pi(\Omega)} W_Z(x,x')u(x')\,dx',$$

where W_Z is the same as that of Theorem 2.2.1. Then, by Theorems 1.4.2 and 1.4.3, $\mathrm{SS}^*(v_Z) = \emptyset$. Hence $v_Z \in \mathcal{D}f^*_M(\pi(\Omega))$. Moreover,

$$\mathrm{Sp}(u - v_Z) \mid_{\overset{\circ}{Z}} = 0.$$

Then we have

$$\mathrm{Sp}(u) \mid_{\overset{\circ}{Z}} \in \mathcal{C}^{d,*}_M(\overset{\circ}{Z})$$

for any closed Z in Ω. Therefore we obtain ii).
Next let ii) hold. Take $\overset{\circ}{q} \in \mathrm{SS}^*(u) \cap \Omega$. By the softness of $\mathcal{C}^{d,*}_M$ and the exact sequence

$$0 \longrightarrow \mathcal{A}_M(\pi(\Omega)) \longrightarrow \mathcal{D}f^*_M(\pi(\Omega)) \longrightarrow \mathcal{C}^{d,*}_M(\pi^{-1}(\pi(\Omega))) \longrightarrow 0,$$

for any closed set Z in Ω, there exists $v_Z \in \mathcal{D}f^*_M(\pi(\Omega))$ so that

$$\mathrm{Sp}(u - v_Z) \mid_Z = 0.$$

Take a closed set Z so that $\overset{\circ}{q} \in Z$. Since $\overset{\circ}{q} \notin \mathrm{SS}(u - v_Z)$, we have $\overset{\circ}{q} \notin \mathrm{SS}^*(u - v_Z)$. Moreover, $\overset{\circ}{q} \notin \mathrm{SS}^*(v_Z)$, which leads to $\overset{\circ}{q} \notin \mathrm{SS}^*(u)$.

Moreover, we have the following theorem in the same way as above.

Theorem 2.3.8. *Let Ω be an open subset in $\sqrt{-1}S^*M$ and let $u \in \mathcal{D}b^\dagger_M(\pi(\Omega))$ $(\dagger < *)$. Then the following conditions i), ii) are equivalent:*
i) $\mathrm{SS}^{*^{-1}}(u) \cap \Omega = \emptyset$,
ii) $\mathrm{Sp}(u) \mid_\Omega \in \mathcal{C}^*_M(\Omega)$.

These theorems imply that we can generalize the definitions of SS^* and $\mathrm{SS}^{*^{-1}}$ even for hyperfunctions. Therefore we have

Dfinition 2.3.9. *Let $u \in \mathcal{B}_M(M)$. We define $\mathrm{SS}^*(u)$ (resp. $\mathrm{SS}^{*^{-1}}(u)$) of u in $\sqrt{-1}S^*M$. Let $\overset{\circ}{q} = (\overset{\circ}{x}, \sqrt{-1}\,\overset{\circ}{\xi}\infty) \in \sqrt{-1}S^*M$. Then $\overset{\circ}{q} \notin \mathrm{SS}^*(u)$ (resp. $\mathrm{SS}^{*^{-1}}(u)$) if $\mathrm{Sp}(u)_{\overset{\circ}{q}} \in \mathcal{C}^{d,*}_{M,\overset{\circ}{q}}$ (resp. $\mathcal{C}^*_{M,\overset{\circ}{q}}$).*

2.4. *Operations for the sheaves $\mathcal{C}^*_M$ and $\mathcal{C}^{d,*}_M$*

We study in this section some operations of $\mathcal{C}^*_M$ and $\mathcal{C}^{d,*}_M$ induced from those of Sato's microfunctions.

Definition 2.4.1. *Let M_1 and M_2 be real analytic manifolds and let $M = M_1 \times M_2$. Let $(\sqrt{-1}S^*M)' = \sqrt{-1}S^*M - \sqrt{-1}S^*M_1 \times M_2 - M_1 \times \sqrt{-1}S^*M_2$,* *and define*

$$p_1 : (\sqrt{-1}S^*M)' \longrightarrow \sqrt{-1}S^*M_1 \text{ and } p_2 : (\sqrt{-1}S^*M)' \longrightarrow \sqrt{-1}S^*M_2$$

by

$$p_1((x_1,x_2),\sqrt{-1}(\xi_1,\xi_2)\infty) = (x_1,\sqrt{-1}\xi_1\infty)$$

and

$$p_2((x_1,x_2),\sqrt{-1}(\xi_1,\xi_2)\infty) = (x_2,\sqrt{-1}\xi_2\infty),$$

respectively.

Theorem 2.4.2. *There exist canonical sheaf homomorphisms*

$$p_1^{-1}\mathcal{C}^*_{M_1} \times p_2^{-1}\mathcal{C}^*_{M_2} \longrightarrow \mathcal{C}^*_M \mid_{(\sqrt{-1}S^*M)'},$$

$$p_1^{-1}\mathcal{C}^{d,*}_{M_1} \times p_2^{-1}\mathcal{C}^{d,*}_{M_2} \longrightarrow \mathcal{C}^{d,*}_M \mid_{(\sqrt{-1}S^*M)'} .$$

$$(u_1(x_1),u_2(x_2)) \longmapsto u_1(x_1)u_2(x_2)$$

P r o o f. We only treat the case of $\mathcal{C}^*$ because the other case is the same. Let $(x_\nu,\sqrt{-1}\xi_\nu\infty) \in \sqrt{-1}S^*M_\nu$ and let $u_\nu \in \mathcal{C}^*_{M_\nu,(x_\nu,\sqrt{-1}\xi_\nu\infty)}$ $(\nu = 1,2)$. There are $f_1 \in \mathcal{D}b^*_{M_1,x_1}$ and $f_2 \in \mathcal{D}b^*_{M_2,x_2}$ such that $u_1 = \mathrm{Sp}(f_1)$ and $u_2 = \mathrm{Sp}(f_2)$ since we have

$$\pi^{-1}\mathcal{D}b^*_{M_\nu} \longrightarrow \mathcal{C}^*_{M_\nu} \longrightarrow 0.$$

Then, by Theorem 1.4.1, we can define $u_1(x_1)u_2(x_2) = \mathrm{Sp}(f_1f_2) \in \mathcal{C}^*_{M,((x_1,x_2),\sqrt{-1}(\xi_1,\xi_2)\infty)}$. We can show that this definition is independent of the choice of f_1 and f_2 in the same way as in the case of $\mathcal{C}_M$.

Definition 2.4.3. *Let N and M be real analytic manifolds, and let $f : N \longrightarrow M$ be a real analytic map. We associate to f natural maps*

$$\sqrt{-1}S^*N \xleftarrow{\rho} N \underset{M}{\times} \sqrt{-1}S^*M \setminus \sqrt{-1}S^*_NM \xrightarrow{\varpi} \sqrt{-1}S^*M$$

where

$$\rho((y,\sqrt{-1}\xi\infty)) = (y,\sqrt{-1}f^*(\xi)\infty),$$

$$\varpi((y,\sqrt{-1}\xi\infty)) = (f(y),\sqrt{-1}\xi\infty).$$

We have the following theorems. Refer to M. Kashiwara et al. [21] for more details.

Theorem 2.4.4. *Let N be a submanifold of M. Then there exist sheaf homomorphisms*

$$\rho_! \varpi^{-1} \mathcal{C}_M^* \longrightarrow \mathcal{C}_N^*,$$
$$\rho_! \varpi^{-1} \mathcal{C}_M^{d,*} \longrightarrow \mathcal{C}_N^{d,*}.$$

Let M be a real analytic manifold, and let Δ_M be a diagonal set of $M \times M$. Then define

$$N = \Delta_M \underset{M \times M}{\times} (\sqrt{-1}S^*(M \times M)) - \Delta_M \underset{M \times M}{\times} (M \times \sqrt{-1}S^*M)$$

$$-\Delta_M \underset{M \times M}{\times} (\sqrt{-1}S_M^*M \times M) - \sqrt{-1}S_M^*(M \times M).$$

For a point $x^\star = (x, x, \sqrt{-1}(\xi_1, \xi_2)\infty) \in N$, with $\xi_1 \neq 0$, $\xi_2 \neq 0$, and $\xi_1+\xi_2 \neq 0$, we let $p_1(x^\star) = (x, \sqrt{-1}\xi_1\infty) \in \sqrt{-1}S^*M$, $p_2(x^\star) = (x, \sqrt{-1}\xi_2\infty) \in \sqrt{-1}S^*M$, and $q(x^\star) = (x, \sqrt{-1}(\xi_1 + \xi_2)\infty) \in \sqrt{-1}S^*M$.

Theorem 2.4.5. *Under the above notation there exist sheaf homomorphisms*

$$q_!(p_1^{-1}\mathcal{C}_M^* \times p_2^{-1}\mathcal{C}_M^*) \longrightarrow \mathcal{C}_M^*,$$
$$q_!(p_1^{-1}\mathcal{C}_M^{d,*} \times p_2^{-1}\mathcal{C}_M^{d,*}) \longrightarrow \mathcal{C}_M^{d,*}.$$
$$(u(x), v(x)) \longmapsto u(x)v(x)$$

Theorem 2.4.6. *There exists a sheaf homomorphism*

$$Fq_!(p_1^{-1}\mathcal{C}_M^* \times p_2^{-1}\mathcal{C}_M^{d,*}) \longrightarrow \mathcal{C}_M^{d,*}.$$
$$(u(x), v(x)) \longmapsto u(x)v(x)$$

P r o o f. If we recall the exact sequences

$$\pi^{-1}\mathcal{D}f_M^* \longrightarrow \mathcal{C}_M^{d,*} \longrightarrow 0, \quad \pi^{-1}\mathcal{D}b_M^* \longrightarrow \mathcal{C}_M^* \longrightarrow 0$$

and Theorem 1.4.2, the compatibility between the product of hyperfunctions and that of microfunctions shows the theorem.

Theorem 2.4.7. *Let M and N be real analytic manifolds, and let $\tilde{\varpi}$ be the natural projection from $M \times \sqrt{-1}S^*N$ to $\sqrt{-1}S^*N$, and let U be an open subset of $\sqrt{-1}S^*N$. If for $u(t,x) \in \mathcal{C}^*_{M\times N}(\tilde{\varpi}^{-1}(U))$ (resp. $\mathcal{C}^{d,*}_{M\times N}(\tilde{\varpi}^{-1}(U))$) $\tilde{\varpi}\mid_{\mathrm{supp} u(t,x)}$ is a proper map, then the integration $v(x) = \int_{f^{-1}(x)} u(t,x)\,dt$ is well-defined as a microfunction of class $*$ (resp. $d,*$). Thus there exists a homomorphism*

$$\tilde{\varpi}_!(\mathcal{C}^*_{M\times N}\mid_{M\times\sqrt{-1}S^*N} \otimes v_M) \longrightarrow \mathcal{C}^*_N$$

$$(\textit{resp.}\quad \tilde{\varpi}_!(\mathcal{C}^{d,*}_{M\times N}\mid_{M\times\sqrt{-1}S^*N} \otimes v_M) \longrightarrow \mathcal{C}^{d,*}_N)$$

where v_M is the sheaf of real analytic volume elements on M.

2.5. *Some classes of microdifferential operators*

Let X be an open set in $\mathbb{C}^n$ and $\mathring{q}$ be a point in the cotangent bundle T^*X of X. Let us denote by $\mathcal{E}^\infty_X$ (resp. $\mathcal{E}_X$) the sheaf on T^*X of rings of microdifferential operators of infinite (resp. finite) order constructed by M. Sato et al. [37]. A microdifferential operator of infinite order P is represented as the infinite formal sum

$$P = \sum_{j\in\mathbb{Z}} P_j(z, D_z),$$

where $P_j(z,\zeta)$ is a holomorphic function defined in a neighborhood Ω of $\mathring{q}$, homogeneous of degree j with respect to ζ, and $\{P_j(z,\zeta)\}$ satisfies the following conditions i), ii).
i) For any $\epsilon > 0$ and any compact set K of Ω, there is a constant $C_{\epsilon,K} > 0$ such that

$$\mid P_j(z,\zeta) \mid \leq C_{\epsilon,K}\frac{\epsilon^j}{j!} \quad \text{for } j \geq 0, (z,\zeta) \in K.$$

ii) For any compact set K of Ω, there is a constant $R_K > 0$ such that

$$\mid P_j(z,\zeta) \mid \leq (-j)! R_K^{-j} \quad \text{for } j < 0, (z,\zeta) \in K.$$

Definition 2.5.1. *Let Ω be an open subset of T^*X and let $P \in \mathcal{E}^\infty_X(\Omega)$. Then we say that P belongs to $\mathcal{E}^{\infty,(M_p)}_X(\Omega)$ (resp. $\mathcal{E}^{\infty,\{M_p\}}_X(\Omega)$) if and only* *if, for any compact set K of Ω, there are constants $h > 0$, $C > 0$ (resp. for any $h > 0$ and any compact set K of Ω, there is a constant $C > 0$) such that*

$$| P_j(z,\zeta) | \leq C \frac{h^j}{M_j} \quad \text{for } j > 0, (z,\zeta) \in K.$$

Note that T. Aoki [3] constructed $\mathcal{E}^{\infty}_{(\rho)}$ and $\mathcal{E}^{\infty}_{\{\rho\}}$ which coincide with our $\mathcal{E}_X^{\infty,(p!^{\frac{1}{\rho}})}$ and $\mathcal{E}_X^{\infty,\{p!^{\frac{1}{\rho}}\}}$, respectively.

We can prove the following theorem in the same way as in T. Aoki [3].

Theorem 2.5.2. *Let $* = (M_p)$ or $\{M_p\}$.*
i) $\mathcal{E}_X^{\infty,*}(\Omega)$ forms a subring of $\mathcal{E}_X^{\infty}(\Omega)$.
ii) $\mathcal{E}_X^{\infty,*}(\Omega)$ is closed under the adjoint operation.

Note that if $* < \dagger$

$$\mathcal{E}_X^{\infty,\dagger} \hookrightarrow \mathcal{E}_X^{\infty,*}.$$

Now let M be an open

$$\mathcal{E}_X^{\infty,\dagger} \hookrightarrow \mathcal{E}_X^{\infty,*}.$$

Now let M be an open subset of $\mathbb{R}^n$ with complexification X. Then we have

Theorem 2.5.3. *$\mathcal{C}_M^*$ and $\mathcal{C}_M^{d,*}$ are left $\mathcal{E}_X^{\infty,\dagger}$ modules if $* < \dagger$.*

P r o o f. First we study the case of $\mathcal{C}_M^{(M_p)}$. Let $P(z,D) = \sum_{j\in\mathbb{Z}} p_j(z,D) \in \mathcal{E}_X^{\infty,(M_p)}$. For any $u \in \mathcal{C}_M^{(M_p)}$, $P(x,D)u(x)$ is defined by

$$\begin{cases} P(x,D)u(x) &= \int K(x,x')u(x')\,dx', \\ K(x,x') &= \int K(x,x',\xi)\omega(\xi), \end{cases}$$

where $K(x,x',\xi)$ is a boundary value of

$$\sum_{j\in\mathbb{Z}} P_j(z,\sqrt{-1}\zeta)\Phi_{j+n}(\sqrt{-1} < z - z', \zeta >)$$

from $\mathrm{Im}(< z - z', \zeta >) > 0$ and

$$\omega(\xi) = \sum_{i=1}^{n} (-1)^{i-1}\xi_i d\xi_1 \wedge \ldots \wedge d\xi_{i-1} \wedge d\xi_{i+1} \wedge \ldots \wedge d\xi_n.$$

Here

$$\Phi_m(\tau) = \frac{\Gamma(m)}{(-\tau)^m} \quad (m \geq 1),$$
$$\Phi_{-m}(\tau) = \frac{-1}{m!}\tau^m\{\log(-\tau) + (\gamma - 1 - \frac{1}{2} - \ldots - \frac{1}{m})\} \ \ (m \geq 0),$$

where γ is the Euler constant. Then we have only to show $K(x, x', \xi) \in \mathcal{C}^{(M_p)}_{M\times M\times S^{n-1}}$ in order to prove $P(x, D)u(x) \in \mathcal{C}^{(M_p)}_M$ since we know $\mathcal{C}^{(M_p)}$ is closed under integration by Theorem 2.4.7. Thus we estimate $\sum_{j=0}^{\infty} P_j(z, \sqrt{-1}\zeta)\Phi_{n+j}(\sqrt{-1}\tau)$. For any compact set K there exist $C_K > 0$ and $h > 0$ such that

$$| P_j(z, \sqrt{-1}\zeta) | \leq \frac{C_K h^j}{M_j} \quad (j \geq 0).$$

Hence

$$| P_j(z, \sqrt{-1}\zeta)\Gamma(n + j)(-\sqrt{-1}\tau)^{-(n+j)} | \leq \frac{C}{| \tau |^n}\frac{\Gamma(n + j)}{M_j}\left(\frac{h}{| \tau |}\right)^j,$$

$$| \sum_{j=0}^{\infty} P_j(z, \sqrt{-1}\zeta)\Phi_{n+j}(\sqrt{-1}\tau) | \leq C' \exp M^\star\left(\frac{h'}{| \tau |}\right).$$

for positive h' and C'. Therefore, by Theorem 1.2.3, $K(x, x', \xi) \in \mathcal{C}^{(M_p)}_{M\times M\times S^{n-1}}$ is shown. Next we study the case of $\mathcal{C}^{d,(M_p)}_M$. Since we have

$$\begin{aligned} \mathrm{SS}^{(M_p)}(K(x, x')) &\subset \mathrm{SS}(K(x, x')) \\ &\subset \{(x, x', \sqrt{-1}(\xi, \eta)\infty) \mid x = x',\ \ \xi = -\eta\}, \end{aligned}$$

we obtain

$$\mathrm{SS}^{(M_p)}(Pu) \subset \mathrm{SS}^{(M_p)}(u)$$

by Theorems 2.4.5 and 2.4.7. Therefore $P(x, D)u(x) \in \mathcal{C}^{d,(M_p)}_M$ if $u \in \mathcal{C}^{d,(M_p)}_M$.

The other cases are almost the same.

3. *Microlocal solvability*

3.1. *Quantized contact transformations*

We give some preliminaries to state our theorems. We have the following theorem due to M. Sato et al. [37].

Theorem 3.1.1. *Let M and N be real analytic manifolds of dimension n. Assume that a real-valued real analytic function $\Omega(x,y)$ defined on $M\times N$ satisfies the conditions*
i) $H=\{(x,y)\in M\times N \mid \Omega(x,y)=0\}$ is non-singular; i.e., $d_{(x,y)}\Omega(x,y)\neq 0$ holds on H, ii) We have on H,

$$\det\begin{pmatrix} 0 & d_y\Omega \\ d_x\Omega & d_xd_y\Omega \end{pmatrix}\neq 0.$$

Then, for any $P(x,D_x)\in\mathcal{E}_M^\infty$, $Q(y,D_y)\in\mathcal{E}_N^\infty$ is uniquely determined such that

$$\int P(x,D_x)\delta(\Omega(x,y))u(y)\,dy=\int\delta(\Omega(x,y))Q(y,D_y)u(y)\,dy$$

holds for any microfunction $u(y)$. Conversely, if Q is given, then P is uniquely determined so that the above holds.
Moreover, the correspondence $P\longmapsto Q^*$ gives rise to the sheaf homomorphism, the quantized contact transformation,

$$q\circ p^{-1}\mathcal{E}_M^\infty \xrightarrow{\sim}\mathcal{E}_N^\infty$$

where

$$p:\sqrt{-1}S_M^*(M\times N)\longrightarrow\sqrt{-1}S^*M,$$
$$q:\sqrt{-1}S_M^*(M\times N)\longrightarrow\sqrt{-1}S^*N$$

are locally isomorphic projections.

Note that this theorem is proved by showing

$$p^{-1}\mathcal{E}_M^\infty\xrightarrow{\sim}\mathcal{E}_{M\times N}^\infty\delta(\Omega(x,y))=\mathcal{E}_{M\times N}^\infty/\mathcal{I}$$

where

$$\begin{aligned}\mathcal{I}=\{P(x,y,D_x,D_y)\ &\in\mathcal{E}_{M\times N}^\infty\mid \\ &P(x,y,D_x,D_y)\delta(\Omega(x,y))=0\}.\end{aligned}$$

The following division theorem is proved in the same way as in T. Aoki [3].

Theorem 3.1.2. *Let $P(x, D_x)$ be a microdifferential operator of finite order m defined in a neighborhood ω of $(\overset{\circ}{x}, \sqrt{-1}\,\overset{\circ}{\xi}dx) = (0, \sqrt{-1}(1, 0, \ldots, 0)dx) \in \sqrt{-1}T^*M$. Denote its principal symbol by $\sigma(P)(x, \sqrt{-1}\xi)$ and assume that $\sigma(P)(0, \sqrt{-1}(1, 0, \ldots, \xi_n))/\xi_n^p$ is holomorphic and never vanishes in a neighborhood of $\xi_n = 0$. Then we can find a neighborhood ω' of $(\overset{\circ}{x}, \sqrt{-1}\,\overset{\circ}{\xi}dx)$ such that any $S(x, D) \in \mathcal{E}_M^{\infty,*}(\omega')$ can be written uniquely in the form*

$$S(x, D) = Q(x, D)P(x, D) + R(x, D)$$

where

$$Q(x, D), R(x, D) \in \mathcal{E}_M^{\infty,*}(\omega')$$

and $R(x, D)$ has the form

$$\sum_{j=0}^{p-1} R^{(j)}(x, D')D_n^j.$$

The above theorem implies that, for an arbitrary $A(x, y, D_x, D_y) \in \mathcal{E}_{M\times N}^{\infty,*}$, $Q_j \in \mathcal{E}_{M\times N}^{\infty,*}$ and $\tilde{A}(x, D_x) \in \mathcal{E}_M^{\infty,*}$ can be chosen so that, determining $\tilde{A}$ uniquely,

$$A = \sum_{j=1}^{2n} Q_j R_j + \tilde{A}$$

may hold, where $R_j(x, y, D_x, D_y)$ are properly chosen generators for

$$\begin{aligned}\mathcal{J} = \{P(x, y, D_x, D_y) \quad &\in \mathcal{E}_{M\times N}^{\infty,*} \mid \\ &P(x, y, D_x, D_y)\delta(\Omega(x, y)) = 0\}.\end{aligned}$$

This shows that we have an isomorphism

$$\mathcal{E}_{M\times N}^{\infty,*}/\mathcal{J} \xrightarrow{\sim} p^{-1}\mathcal{E}_M^{\infty,*}.$$

Therefore we have

Theorem 3.1.3 *Under the same assumptions in Theorem 3.1.1. Then we have a quantized contact transformation*

$$q \circ p^{-1}\mathcal{E}_M^{\infty,*} \xrightarrow{\sim} \mathcal{E}_N^{\infty,*}.$$

Moreover, we have the following theorem.

Theorem 3.1.4. *Under the same assumptions of Theorem 3.1.1, we have isomorphisms*

$$\mathcal{C}_M^* \longrightarrow \mathcal{C}_N^*,$$

$$\mathcal{C}_M^{d,*} \longrightarrow \mathcal{C}_N^{d,*}.$$

$$u(y) \longmapsto \int \delta(\Omega(x,y))u(y)\,dy$$

P r o o f. It is easy to see

$$\delta(\Omega(x,y)) \in \mathcal{C}_{M\times N}^{(\infty)}$$

and

$$\begin{aligned} \mathrm{SS}(\delta(\Omega(x,y))) \subset \{(x,y, \quad & \sqrt{-1}(\xi,\eta)\infty) \mid \Omega(x,y)=0, \\ & (\xi,\eta) = cd_{(x,y)}\Omega(x,y) \ \ (c>0)\}. \end{aligned}$$

In the case of $\mathcal{C}^*$, by Theorems 2.4.5 and 2.4.7, we have

$$\int \delta(\Omega(x,y))u(y)\,dy \in \mathcal{C}_N^*$$

if $u(y) \in \mathcal{C}_M^*$. In the case of $\mathcal{C}^{d,*}$, we first remark that for $u \in \mathcal{C}_M^*$, we have

$$\mathrm{SS}^*(\int \delta(\Omega(x,y))u(y)\,dy) \subset \{(x,\sqrt{-1}\xi\infty) \mid \ \exists\eta_1\ \exists\eta_2\ \ 0\le\exists\theta\le 1$$
$$(x,y,\sqrt{-1}(\xi,\eta_1)\infty) \in \mathrm{SS}^*(\delta(\Omega(x,y))),\ (y,\sqrt{-1}\eta_2\infty)\in\mathrm{SS}^*(u),\ \theta\eta_1{+}(1{-}\theta)\eta_2{=}0\}.$$

Thus if $u \in \mathcal{C}_M^{d,*}$, we deduce

$$\mathrm{SS}^*(\int \delta(\Omega(x,y)u(y)\,dy) = \emptyset.$$

This shows

$$\int \delta(\Omega(x,y))u(y)\,dy \in \mathcal{C}_N^{d,*}.$$

The bijectivity directly results from the isomorphism from $\mathcal{C}_M$ to $\mathcal{C}_N$.

3.2. *Irregularity for microdifferential operators*

Let us recall the definition of the irregularity for a microdifferential operator of finite order due to T. Aoki [4]. Let X be an open subset of $\mathbb{C}^n$, and $\mathring{q} = (z, \zeta dz) \in T^*X$. Let P be a microdifferential operator of finite order in a neighborhood of $\mathring{q}$ with principal symbol $p = \sigma(P)$ satisfying $p(\mathring{q}) = 0$. Assume that P has constant multiplicity d in the neighborhood of $\mathring{q}$ in the characteristic variety $\mathrm{char}(P)$ of P. Then there are holomorphic functions l and k in the neighborhood of $\mathring{q}$, homogeneous with respect to ζ such that

$$\begin{gathered} p(z,\zeta) = l(z,\zeta)k(z,\zeta)^d, \\ \mathrm{char}(P) = \{k(z,\zeta) = 0\}, \\ d_{(z,\zeta)}k(z,\zeta) \neq 0 \quad \text{ on } \mathrm{char}(P) \cap \{\zeta \neq 0\} \\ l(z,\zeta) \neq 0 \quad \text{ on } \mathrm{char}(P) \cap \{\zeta \neq 0\}. \end{gathered}$$

Let K be a microdifferential operator whose principal symbol is k. Then P can be written in the form

$$P(z,D) = \sum_{i \in I} L_i(z,D)K(z,D)^i,$$

where
i) I is a subset of $\{0, 1, 2, \ldots, d\}$,
ii) L_i is a microdifferential operator whose principal symbol $\sigma(L_i)$ does not vanish identically on $\mathrm{char}(P)$,
iii) the function $\omega : I \longrightarrow \mathbb{Z}$ defined by $\omega(i) = ord(L_i K^i)$ strictly increases with i.

Definition 3.2.1. *The rational number*

$$\sigma = \max_{i \in I \setminus \{d\}} \left\{ 1, \frac{d-i}{\omega(d) - \omega(i)} \right\}$$

is said to be the irregularity of the microdifferential operator P at $\mathring{q}$.

T. Aoki [4] showed that the function $\omega : I \longrightarrow \mathbb{Z}$ does not depend on the choice of K and that a quantized contact transformation preserves the irregularity. Then we have the following theorem due to T. Aoki [3].

Theorem 3.2.2. *Let $\mathring{q} = (0, dz_n) \in T^*X$, and let P be a microdifferential operator of finite order which satisfies the conditions*

i) $p = \sigma(P) = \zeta_1^d$,
ii) the irregularity of P *at* $\mathring{q}$ *is equal to* σ,
Then there exists an invertible operator $Q \in \mathcal{E}_X^{\infty,(p!^{\frac{\sigma}{\sigma-1}})}$ *satisfying*

$$QPQ^{-1} = D_1^d.$$

Note that in the above theorem $Q^{-1} \in \mathcal{E}_X^{\infty,(p!^{\frac{\sigma}{\sigma-1}})}$.

3.3. *Microlocal solvability*

Let M be an open subset in $\mathbb{R}^n$, and X be a complexification of M in $\mathbb{C}^n$. We take a coordinate system of T_M^*X as $(x, \sqrt{-1}\xi dx)$. We set $\mathring{q} = (0, \sqrt{-1}dx_n)$. Let P be a microdifferential operator of finite order in a neighborhood Ω of $\mathring{q}$. We shall assume
i) p can be written as

$$p(x,\xi) = l(x,\xi)k(x,\xi)^d$$

with holomorphic functions l and k in Ω homogeneous with respect to ξ,
ii) $l(\mathring{q}) \neq 0$, $k(\mathring{q}) = 0$,
iii) k is real valued on T_M^*X, homogeneous of order 1 with ξ, $dk(\mathring{q}) \neq 0$,
iv) $\{k(x,\xi) = 0\}$ is regular involutive near $\mathring{q}$ in Ω,
v) the irregularity of P at $\mathring{q}$ is σ,
vi) $* < (p!^{\frac{\sigma}{\sigma-1}})$.

Theorem 3.3.1. *We assume the above conditions i)-vi). Then*

$$P : \mathcal{C}^*_{M,\,\mathring{q}} \longrightarrow \mathcal{C}^*_{M,\,\mathring{q}},$$

$$P : \mathcal{C}^{d,*}_{M,\,\mathring{q}} \longrightarrow \mathcal{C}^{d,*}_{M,\,\mathring{q}}$$

are surjective.

P r o o f. We find a real contact transformation ϕ satisfying

$$k \circ \phi = \xi_1, \;\; \phi(\mathring{q}) = \mathring{q}.$$

Then we have real quantized transformations

$$\mathcal{E}_{X,\,\mathring{q}} \xrightarrow{\sim} \mathcal{E}_{X,\,\mathring{q}},$$

$$\mathcal{C}^*_{M,\,\mathring{q}} \xrightarrow{\sim} \mathcal{C}^*_{M,\,\mathring{q}},$$

$$\mathcal{C}^{d,*}_{M,\,\overset{\circ}{q}} \overset{\sim}{\longrightarrow} \mathcal{C}^{d,*}_{M,\,\overset{\circ}{q}}$$

associated with ϕ. Since a quantized transformation preserves the irregularity of P, we may assume from the beginning

$$k = \xi_1.$$

Moreover by deviding an elliptic factor of P, we have only to study the operator

$$P = D_1^d + (\text{lower}).$$

Then by Theorem 3.2.2, we can find an invertible $Q \in \mathcal{E}_X^{\infty,(p!^{\frac{\sigma}{\sigma-1}})}$ which satisfies

$$QPQ^{-1} = D_1^d.$$

We remark Q and Q^{-1} act on $\mathcal{C}^{*}_{M,\,\overset{\circ}{q}}$ and $\mathcal{C}^{d,*}_{M,\,\overset{\circ}{q}}$. Thus the surjectivity of

$$P : \mathcal{C}^{*}_{M,\,\overset{\circ}{q}} \longrightarrow \mathcal{C}^{*}_{M,\,\overset{\circ}{q}},$$

$$P : \mathcal{C}^{d,*}_{M,\,\overset{\circ}{q}} \longrightarrow \mathcal{C}^{d,*}_{M,\,\overset{\circ}{q}}$$

follows from that of D_1^d.

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