Bulletin T.CXXIII de l'Académie Serbe des Sciences et des Arts - 2002
Classe des Sciences mathématiques et naturelles
Sciences mathématiques, № 27

SOLUTION OF MATHEMATICAL MODELS BY LOCALIZATION

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(Presented at the 9th Meeting, held on December 28, 2001)

A b s t r a c t. A definition of the Laplace transform of elements of $\mathcal{D}'_*(P)$ of a subspace of distributions is given which can successfully be applied to solve in a prescribed domain linear equations with derivatives, partial derivatives, fractional derivatives and convolutions, all with initial or boundary conditions, regardless of the existence of classical or generalized solutions.

AMS Mathematics Subject Classification (2000): 46F12
Key Words: Distributions, tempered distributions, Laplace transform.

1. *Introduction*

Laplace transform of numerical functions has been elaborated as a powerful mathematical theory very useful in practice and many a time applied by engineers. Although it has been belived to have two important shortcomings. First, application of the Laplace transform (In short, LT) (not only to functions but to distributions, ultra distributions, Laplace hyperfunctions,...) calls always for some growth conditions of them ([4], [8], [11], [14], [16], [17], [18] and [19]). Secondly, there is no simple characterisation of the functions which are LT of the numerical functions. Hence, we are not always sure whether or not an obtained function $f(s)$ is the LT of a function $g(t)$ of exponential type.

To overcome these difficulties mathematiciens defined LT of functions as classes ([2], [3]) without a rich repercussions, or used algebraic approaches to the Heaviside calculus ([11], [13]).

Recently H.Komatsu [7] overcame successfully all defects of the classical LT. He defined the LT of Laplace hyperfunctions and of hyperfunctions, as well, but in one dimensional case. Since it is a very abstract theory, it cannot be easily accepted by the greater part of people working in applications.

In [10] a definition was developed of the LT applicable to locally Bochner integrable, Banach space valued functions with arbitrary growth at infinity based on old ideas (cf. [2], [3]). For $f \subset \mathbb{L}_{loc}$ this LT coincides with the LT defined by Komatsu.

The aim of this paper is to define and to elaborate the LT of a subset of distributions which contains also the space $\mathbb{L}_{loc}(\mathbb{R})$, distributions with compact supports and tempered distributions, elements which are sufficient for a wide class of applications, if it is a question of classical or generalized solutions to mathematical models. It can be also used to analyse reasons of the nonexistence of classical solutions. Such a definition does not assume any growth condition for functions belonging to $\mathbb{L}^1_{loc}(\mathbb{R})$ and there is a simple characterisation of the LT images. From generalized functions we use only the space S' of tempered distributions which is becoming a workaday tool of mathematicians, physicists and engineers.

So defined LT can be successfully applied to solve linear equations with derivatives, partial derivatives, fractional derivatives and convolutions all with initial or boundary conditions, regardless of the existence of classical or generalized solutions.

2. *The space of tempered distributions and the Laplace transform*

We repeat some definitions and facts related to the space S' of tempered distributions and to the Laplace transform (in short LT) of them (cf. [17] and [18]).

Let Γ be a closed convex acute cone in $\mathbb{R}^n$, $\Gamma^* = \{t \in \mathbb{R}^n, tx \equiv t_1x_1 + ... + t_nx_n \geq 0,\ \forall x \in \Gamma\}$ and $C = int\Gamma^*$. Let K be a compact set in $\mathbb{R}^n$.

By $\mathcal{S}'(\Gamma + K)$ is denoted the space of tempered distributions defined on the close set $\Gamma + K \subset \mathbb{R}^n$. Then $\mathcal{S}'(\Gamma+)$ is defined by way of

$$\mathcal{S}'(\Gamma+) = \bigcup_{K\subset\mathbb{R}^n} \mathcal{S}'(\Gamma + K). \tag{1}$$

The set $\mathcal{S}'(\Gamma+)$ forms an algebra that is associative and commutative if for the operation of multiplication one takes the convolution, denoted by $*$.

If $\Gamma + K$ is convex, as it will be in our case, then the LT of $f \in \mathcal{S}'(\Gamma+)$ is defined by

$$\widehat{f}(z) = \mathcal{L}(f)(z) = \langle f(t), e^{-zt}\rangle,\ z \in C + i\mathbb{R}^n, \tag{2}$$

where $t = (t_1, ..., t_n),\ z = (z_1, ..., z_n)$ and $zt = z_1t_1 + ... + z_nt_n$. It is one to one operation.

For the properties of so defined LT one can consult [17]. We shall cite only some of them which we use in the sequel:

1) $\mathcal{L}\Big(\dfrac{\partial^m}{\partial t_i^m} f\Big)(z) = (z_i)^m \mathcal{L}(f)(z)$.

2) If $f \in \mathcal{S}'(\Gamma_1+)$ and $g \in \mathcal{S}'(\Gamma_2+)$, then $\mathcal{L}(f\times g)(z,s) = \mathcal{L}(f)(z)\mathcal{L}(g)(s)$, $z \in C_1 + i\mathbb{R}^n,\ s \in C_2 + i\mathbb{R}^n$.

3) If $f, g \in \mathcal{S}'(\Gamma+)$, then $f * g \in \mathcal{S}'(\Gamma+)$ and $\mathcal{L}(f * g)(z) = \mathcal{L}(f)(z)\mathcal{L}(g)(z),\ z \in C + i\mathbb{R}^n$.

4) If $f \in \mathbb{L}_{loc}([0,\infty))$ and bounded in a neighbourhood of zero, $0 < \beta < 1,\ n = 1$, then $\mathcal{L}(f^{(\beta)})(z) = z^{\beta}\mathcal{L}(f)(z)$.

5) $\mathcal{L}(\delta(t - t_0))(z) = e^{-zt_0}$.

6) $\mathcal{L}(f)(z + a) = \mathcal{L}(e^{-at}f)(z),\ \mathbb{R}ea > 0$.

7) If $f \in \mathbb{L}_{loc}(\mathbb{R}^n_+)$ and $|f(x)| \leq Ce^{qx},\ x \geq x_0 > 0$, then $f(x)e^{-qx} \in \mathcal{S}'(\overline{\mathbb{R}}^n_+)$ and

$$\int_{\mathbb{R}^n_+} e^{-(z+q)t} f(t)dt = \int_{\mathbb{R}^n_+} e^{-zt}e^{-qt}f(t)dt = \mathcal{L}(e^{-qt}f)(z).$$

Let $\mathcal{H}_a^{(\alpha,\beta)}(C),\ \alpha \geq 0,\ \beta \geq 0,\ a \geq 0$, denote the sets of holomorphic functions on $C + i\mathbb{R}^n$ which satisfy the following growth condition

$$|f(z)| \leq Me^{a|x|}(1 + |z|^2)^{\alpha/2}(1 + \Delta^{-\beta}(x, \partial C)),\ z = x + iy \in C + i\mathbb{R}^n, \tag{3}$$

where ∂C is the boundary of C and $\Delta(x, \partial C)$ is the distance between x and ∂C. We set

$$\mathcal{H}_a(C) = \bigcup_{\alpha\geq 0,\beta\geq 0} \mathcal{H}_a^{(\alpha,\beta)}(C) \text{ and } \mathcal{H}_+(C) = \bigcup_{a\geq 0} \mathcal{H}_a(C).$$

Proposition A. *([17] p.191). The algebras $\mathcal{H}_+(C)$ and $\mathcal{S}'(C^*+)$ and also their subalgebras $\mathcal{H}_0(C)$ and $\mathcal{S}'(C^*)$ are isomorphic. This isomorphism is accomplished via the LT.*

A property of the defined LT which can be used in a practical way is the following:

Let $f \in \mathcal{S}'(\overline{\mathbb{R}}^n_+ + \overline{P})$. The LT of f, $\mathcal{L}(f)$, can be obtained by one after the other applications of the LT-s $\mathcal{L}_1(f), ..., \mathcal{L}_n(f)$, $\quad \mathcal{L}(f) = \mathcal{L}_1(f) \circ ... \circ \mathcal{L}_n(f)$.

If $\sigma \geq 0$, $f \in \mathcal{S}'(C^*+)$ and $g = e^{\sigma t} f$ then bydefinition $\mathcal{L}(g)(s) = \langle f(t), e^{-(s-\sigma)t}\rangle$, $Res > \sigma$.

Let $F(s)$ be a function holomorphic for $Res > \sigma$. The function $F(\xi+\sigma)$ is holomorphic for $Re\xi > 0$. If $F(\xi+\sigma) \in \mathcal{H}(\mathbb{R}_+)$, then there exists $f \in \mathcal{S}'(\overline{\mathbb{R}_+})$ such that $\mathcal{L}(e^{\sigma t} f)(s) = F(s)$.

3. *The space $\mathcal{D}'_*(P)$*

Let P be the set $\prod_{i=1}^{n} [a_i, b_i), 0 \leq a_i < b_i,\ i = 1, ..., n$. $\overline{P}$ is compact. Since $\overline{\mathbb{R}}^n_+$ is a closed convex and acute cone, $\mathcal{S}'(\overline{\mathbb{R}}^n_+ + \overline{P})$ is well defined.

Let $\mathcal{A}$ be the vector space:

$$\mathcal{A} = \{T \in e^{\omega t}\mathcal{S}'(\overline{\mathbb{R}}^n_+ + \overline{P});\ \mathrm{supp} T \subset \{(\overline{\mathbb{R}}^n_+ + \overline{P}) \setminus P\}\},\ \omega \in \mathbb{R},$$

where $e^{\omega t} = e^{\omega t_1} ... e^{\omega t_n}$. $\mathcal{A}$ is a subspace of $e^{\omega t}\mathcal{S}'(\overline{\mathbb{R}}^n_+ \overline{P})$.

Now we can define an equivalence relation in $e^{\omega t}\mathcal{S}'(\overline{\mathbb{R}}^n_+ + \overline{P})$: $f \sim g \iff f - g \in \mathcal{A}$. Let us denote $\mathcal{B}$ by:

$$\mathcal{B} = e^{\omega t}\mathcal{S}'(\overline{\mathbb{R}}^n_+ + \overline{P})/\mathcal{A},\ b \in \mathcal{B} \iff b = class(T) \equiv cl(T), T \in e^{\omega t}\mathcal{S}'(\overline{\mathbb{R}}^n_+ + \overline{P}). \tag{4}$$

Definition 1. *Let $\mathcal{D}'(P)$ denote the space of distributions defined on P. Then*

$$\mathcal{D}'_*(P) = \{T \in \mathcal{D}'(P); \exists \overline{T} \in e^{\omega t}\mathcal{S}'((\overline{\mathbb{R}}^n_+ + \overline{P})), \overline{T}|_P = T\}, \tag{5}$$

where $\overline{T}|_P$ *is the restriction of* $\overline{T}$ *on* P. *Since* $\mathcal{D}'$ *is not a flabby sheaf,* $\mathcal{D}'_*(P) \neq \mathcal{D}'(P)$.

Proposition 1. $\mathcal{D}'_*(P)$ *is algebraically isomorphic to* $\mathcal{B}$.

P r o o f. If $T_* \in \mathcal{D}'_*(P)$, then there exists $\overline{T} \in e^{\omega t}\mathcal{S}'(\overline{\mathbb{R}}^n_+ + \overline{P})$ such that $\overline{T}|_P = T_*$. We can define the mapping $\lambda : \mathcal{D}'_*(P) \to \mathcal{B}$, for $T_* \in \mathcal{D}'_*(P)$, $\lambda(T_*) = cl(\overline{T}) \in \mathcal{B}$. The inverse mapping λ^{-1} exists and $\lambda^{-1}(cl(\overline{T})) = \overline{T}|_P = T_* \in \mathcal{D}'_*(P)$. T_* does not tend on the chosen element from $cl(\overline{T})$. If we take an other representative T_1 of the $cl(\overline{T})$, then $T_1 = \overline{T} + S$, $S \in \mathcal{A}$. Then $T_1|_P = \overline{T}|_P$. Now it is easily seen that λ is an algebraic isomorphism of two vector spaces.

Definition 2. *The LT of elements in* $\mathcal{D}'_*(P)$ *is defined by*

$$\mathcal{L}(\mathcal{D}'_*(P)) = \mathcal{L}(e^{\omega t}\mathcal{S}'(\overline{\mathbb{R}}^n_+ + \overline{P}))/\mathcal{L}(\mathcal{A}).$$

If $T_* \in \mathcal{D}'_*(P)$, *then* $\mathcal{L}(T_*) = cl(\mathcal{L}\overline{T})$, *where* $\overline{T}$ *is such that* $\overline{T}|_P = T_*$.

Remark 1. a) Let H_P denote the function $H_P(t) = 1,\ t \in P,\ H(t) = 0,\ t \in \mathbb{R}^n \setminus P$.

b) If $f \in \mathbb{L}_{loc}(\mathbb{R}^n)$, then the regular distribution $[H_P f]$ defined by $H_P f$ belongs to $\mathcal{D}'_*(P)$ for every $P \subset \mathbb{R}^n$ and f has the LT in the sense of Definition 2.

c) If $f \in e^{\omega t}\mathcal{S}'(\overline{\mathbb{R}_+} + \overline{P})$ and $g \in \mathcal{A}$, then $f * g \in \mathcal{A}$, as well.

4. *Localization of functions*

4.1. *One dimensional case*

Let $f \in \mathcal{C}^{(p)}((-\infty, b))$, $p \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, and H_a be a function such that $H_a(x) = 0,\ -\infty < x < a < b$; $H_a(x) = 1,\ a \leq x < b$. Denote by $[H_a f]$ the regular distribution defined by $H_a f$. Hence, $[H_a f] \in \mathcal{D}'((-\infty, b))$, $\operatorname{supp}[H_a f] \subset [a, b)$ or $[H_a f] \in \mathcal{D}'([a, b))$, as well (cf. [17]). By $[f_a^{(p)}]$, $p \in \mathbb{N}$, we denote the distribution defined by the function $f_a^{(p)}$ that equals to $f^{(p)}(x)$, $x \in (a, b)$ and equals to zero for $x \in (-\infty, a)$ and is not defined for $x = a$.

Since the function $(H_a f)^{(k)}$ has in general a discontinuity of the first kind in $x = a$, $k = 0, 1, ..., p$, by the well-known formula (cf. [14])

$$
\begin{aligned}
D^p[H_a f] &= [f_a^{(p)}] + f^{(p-1)}(a)\delta(x-a) + \dots + f(a)\delta^{(p-1)}(x-a) \\
&= [f_a^{(p)}] + R_{p,a}(f) = [H_a f^{(p)}] + R_{p,a}(f),
\end{aligned}
\tag{6}
$$

where $D^p[H_a f]$ is the derivative of order p in the sense of distributions, and

$$
R_{p,a}(f) = f^{(p-1)}(a)\delta(x-a) + \dots + f(a)\delta^{(p-1)}(x-a). \tag{7}
$$

Proposition 2.*For $f, g \in \mathbb{L}_{loc}(\mathbb{R})$ and θ the Heaviside function*

$$
[H_0(\theta f * \theta g)] = [(H_0 f * \theta g)].
$$

P r o o f. For $\varphi \in \mathcal{D}([0,b))$

$$
\begin{aligned}
\langle [H_0(\theta f * \theta g)], \varphi\rangle &= \int_0^{x_2} H_0(x) \int_0^x f(t)g(x-t)dt\varphi(x)dx \\
&= \int_0^{x_2}\int_0^x H_0(t)f(t)g(x-t)dt\,\varphi(x)dx \\
&= \int_0^{x_2}((H_0 f * \theta g)(x)\varphi(x)dx \\
&= \langle [(H_0 f * \theta g)], \varphi\rangle.
\end{aligned}
$$

This completes the proof.

Definition 3. *([6]) Let α be a positive real number such that $m-1 < \alpha < m$ for a fixed $m \in \mathbb{N}$. The α-fractional derivative of a function $f \in \mathcal{C}([0,\infty))$ is defined by*

$$
f^{(\alpha)}(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^m}{dx^m} \int_0^x f(x-t)t^{m-1-\alpha}dt, \quad x > 0, \tag{8}
$$

if this derivative exists.

Proposition 3. *Let α be a real number such that $m-1 < \alpha < m$ and $f \in \mathcal{C}^{(m)}((0,b))$.*

1) If $f^{(k)}$ *is bounded in* $[0,\eta]$ *for an* $\eta > 0,\ k = 0,1,...,m,$ *then* $f^{(\alpha)} \in \mathcal{C}((0,b))$.

2) If in addition $f^{(k)}(0) = 0,\ k = 0,...,m-1,$ *then* $f^{(\alpha)} \in \mathcal{C}([0,b))$.

3) If $f^{(k)}$ *is bounded in* $[0,\eta]$ *for any* $\eta > 0,\ k = 0,1,...,m-1,$ *and for* $m \geq 2,\ f^{(j)}(0) = 0,\ j = 0,1,...,m-2,$ *then*

$$[H_0 f^{(\alpha)}] = \frac{1}{\Gamma(m-\alpha)} D_x^m [((H_0 f) * (\theta(t) t^{m-1-\alpha}))(x)]. \tag{9}$$

P r o o f. Since $0 > m-1-\alpha > -1$ and

$$\frac{d^m}{dx^m}\int_0^x f(x-t)t^{m-1-\alpha}dt = f(0)\frac{d^{m-1}}{dx^{m-1}}x^{m-1-\alpha} + ...+$$

$$+f^{(m-1)}(0)x^{m-1-\alpha} + \int_0^x f^{(m)}(x-t)t^{m-1-\alpha}dt,\ \ 0 < x < b,$$

it follows the first part of Proposition 3.

With the additional assumptions we have

$$\frac{d^m}{dx^m}\int_0^x f(x-t)t^{m-1-\alpha}dt = \int_0^x f^{(m)}(x-t)t^{m-1-\alpha}dt,\ 0 < x < b, \tag{10}$$

and

$$|\int_0^x f^{(m)}(x-t)t^{m-1-\alpha}dt| \leq M x^{m-\alpha},\ x \in [0,\eta].$$

Hence, $f^{(\alpha)} \in \mathcal{C}([0,b))$.

By Definition 3 and by (6)

$$\begin{aligned} [H_0 f^{(\alpha)}(x)|_{(0,b)}] &= \frac{1}{\Gamma(m-\alpha)}[H_0(x)\frac{d^m}{dx^m}\int_0^x f(x-t)t^{m-1-\alpha}dt|_{(0,b)}] \\ &= \frac{1}{\Gamma(m-\alpha)}D_x^m[H_0(x)((\theta f) * (\theta(t)t^{m-1-\alpha}))(x)] \\ &-R_{m.0}((\theta f) * (\theta(t)t^{m-1-\alpha})). \end{aligned}$$

By assumption in 3) it follows that

$$\frac{d^k}{dx^k}\int_0^x f(x-t)t^{m-1-\alpha}dt|_{x=0} = 0, \quad k = 0,...,m-2, \ m \geq 2.$$

Therefore, $R_{m,0}((\theta f) * (\theta(t)t^{m-1-\alpha})) = 0$. Now, by Proposition 2 it follows

$$[H_0 f_0^{(\alpha)}] = \frac{1}{\Gamma(m-\alpha)} D^m[(H_0 f * \theta(t)t^{m-1-\alpha})].$$

If $m = 1$

$$[H_0 f^{(\alpha)}] = \frac{1}{\Gamma(1-\alpha)} D[H_0 f * \theta(\tau)\tau^{-\alpha}].$$

4.2. *n-dimensional case*

We keep the following notation:

$P = \prod_{i=1}^{n} [a_i, b_i), \ \ 0 \leq a_i < b_i, \ \ i = 1,...,n;$

$\Omega = \overline{\mathbb{R}}_-^n + P$, then $P \subset \Omega$;

$H_a^n(x) = H_{a_1}(x_1)...H_{a_n}(x_n), \ H_{a_i}(x_i) = 1, \ \ a_i \leq x_i < b_i; \ \ H_{a_i}(x_i) = 0, \ x_i < a_i, \ i = 1,...,n.$

f is a function with continuous partial derivatives on Ω; $[H_a^n f]$ is the distribution, defined by $H_a^n f$, belonging to $\mathcal{D}'(\Omega)$ and to $\mathcal{D}'(P)$, as well.

$\Big(\frac{\partial^p}{\partial x_i^p} f\Big)_{a_i}$ is the function equal to $\frac{\partial^p}{\partial x_i^p} f$ on the $intP$ and equal to zero on $\Omega \setminus P$, but is not defined for $x \in P \setminus intP$.

Proposition 4. *With the notation as above we have*

$$D_{x_i}^p [H_a^n f] = \Big[H_a^n \Big(\frac{\partial^p}{\partial x_i^p} f\Big)_{a_i}\Big] + R_{p,a_i}(f), \ \ p \in \mathbb{N}, \tag{11}$$

where

$$\begin{aligned} R_{p,a_i}(f) \ &= \Big[H_a^n \frac{\partial^{p-1}}{\partial x_i^{p-1}} f(x)|_{x_i=a_i}\Big] \times \delta(x_i - a_i) + ... \\ &+ [H_a^n f(x)|_{x_i=a_i}] \times \delta^{(p-1)}(x_i - a_i). \end{aligned}$$

P r o o f. The method of the proof is just the same as for (6).

Proposition 5. *With the notation as in Proposition 5.*

$$
\begin{aligned}
D^q_{x_j} D^p_{x_i}[H^n_a f] &= \Big[H^n_a \frac{\partial^q}{\partial x^q_j}\Big(\Big(\frac{\partial^p}{\partial x^p_i} f\Big)_{a_i}\Big)_{a_j}\Big] \\
&+\Big[H^n_a \frac{\partial^{q-1}}{\partial x^{q-1}_j}\Big(\frac{\partial^p}{\partial x^p_i} f\Big)_{a_i}(x)|_{x_j=a_j}\Big] \times \delta(x_j - a_j) + \\
&+\Big[H^n_a \Big(\frac{\partial^p}{\partial x^p_i} f\Big)_{a_i}(x)|_{x_j=a_j}\Big] \\
&\times \delta^{(q-1)}(x_j - a_j) + D^q_{x_j} R_{p,a_i}(f).
\end{aligned} \tag{12}
$$

P r o o f. We have only to apply $D^q_{x_j}$ to (11).

Remark 2. To realize

$$D^q_{x_j} R_{p,a_i}$$

we have to use (11).

We illustrate Proposition 5 by calculating

$$D_{x_2} D_{x_1}[H^2_a f],\ D^2_{x_1} D^2_{x_2}[H^2_a f] \ \text{ and }\ D^{\alpha}_{x_1} D^2_{x_2}[H^2_a f],$$

1) $D_{x_2} D_{x_1}[H^2_a f]$.

Let us start with the first derivatives.

$$D_{x_1}[H^2_a f] = \Big[H^2_a \Big(\frac{\partial}{\partial x_1} f\Big)_{a_1}\Big] + \delta(x_1 - a_1) \times [H_{a_2}(x_2) f(a_1, x_2)]$$

$$D_{x_2}[H^2_a f] = \Big[H^2_a \Big(\frac{\partial}{\partial x_2} f\Big)_{a_2}\Big] + [H_{a_1}(x_1) f(x_1, a_2)] \times \delta(x_2 - a_2)$$

$$D_{x_2} D_{x_1}[H^2_a f] = D_{x_2}\Big[H^2_a \Big(\frac{\partial}{\partial x_1} f\Big)_{a_1}\Big] + \delta(x_1 - a_1) \times$$

$$D_{x_2}[H_{a_2}(x_2) f(a_1, x_2)] = \Big[H^2_a \Big(\frac{\partial^2}{\partial x_1 \partial x_1} f\Big)_{a_1,a_2}\Big] + D_{x_1}[H_{a_1}(x_1) f(x_1, a_2)]$$

$$\times \delta(x_2 - a_2) - f(a_1, a_2)\delta(x_1 - a_1) \times \delta(x_2 - a_2) + \delta(x_1 - a_1) \times D_{x_2}[H_{a_2}(x_2) f(a_1 x_2)],$$

where

$$\Big(\frac{\partial^2}{\partial x_2 \partial x_1} f\Big)_{a_1,a_2} = \frac{\partial^2}{\partial x_2 \partial x_1} f(x, y),\ (x, y) \in (a_1, b_1) \times (a_2, b_2).$$

Remark 3. a) This formula is derived by supposing:

$$\Big(\frac{\partial f}{\partial x_1}f\Big)_{a_1}(x_1,x_2)\Big|_{x_2=a_2} = \Big(\frac{\partial}{\partial x_1}f(x_1,a_2)\Big)_{a_1}$$

and

$$\Big(\frac{\partial f}{\partial x_2}f\Big)_{a_2}(x_1,x_2)\Big|_{x_1=a_2} = \Big(\frac{\partial}{\partial x_2}f(a_1,x_2)\Big)_{a_2}$$

b) It follows that $D_{x_1}D_{x_2}[H_a^2 f] = D_{x_2}D_{x_1}[H_a^2 f]$.

2) $D_{x_1}^2 D_{x_2}^2[H_a^2 f]$.
By a similar procedure as in 1) we have

$$D_{x_1}^2 D_{x_2}^2[H_a^2 f] = \Big[H_a^2\Big(\frac{\partial^4}{\partial x_1^2 \partial x_2^2}f\Big)_{a_1,a_2}\Big] + D_{x_1}^2[H_{a_1}f(x_1,a_2)] \times \delta^{(1)}(x_2-a_2)$$

$$+\delta^{(1)}(x_1-a_1) \times D_{x_2}^2[H_{a_2}f(a_1,x_2)] + D_{x_1}^2[H_{a_1}\frac{\partial}{\partial x_2}f(x_1,a_2)] \times \delta(x_2-a_2)$$

$$+\delta^{(1)}(x_1-a_1)\times D_{x_2}^2[H_{a_2}\frac{\partial}{\partial x_1}f(a_1,x_2)] - f(a_1,a_2)(\delta^{(1)}(x_1-a_1)\times\delta^{(2)}(x_2-a_2))$$

$$-\frac{\partial}{\partial x_2}f(a_1,a_2)(\delta^{(1)}(x_1-a_1)\times\delta(x_2-a_2)) - \frac{\partial}{\partial x_1}f(a_1,a_2)(\delta(x_1-a_1)\times\delta^{(1)}(x_2-a_2))$$

$$-\frac{\partial^2}{\partial x_1 \partial x_2}f(a_1,a_2)(\delta(x_1-a_1)\times\delta(x_2-a_2)).$$

3) $D_{x_1}^{\alpha} D_{x_2}^2[H_a^2 f]$, $a_1 = 0$, $b_1 = \infty$.

$$\Big[D_{x_1}^{\alpha}\Big(\frac{\partial^2}{\partial x_2^2}f\Big)_{a_2}\Big] = \frac{1}{\Gamma(1-\alpha)}D_{x_1}\Big[H_a^2\Big(\Big(\frac{\partial^2}{\partial x_2^2}f\Big)_{a_2} \ast_1 \theta(x_1)x_1^{-\alpha}\Big)\Big]$$

$$= \frac{1}{\Gamma(1-\alpha)}D_{x_1}D_{x_2}^2[(H_a^2 f \ast_1 \theta(x_1)x_1^{-\alpha})]$$

$$-\frac{1}{\Gamma(1-\alpha)}D_{x_1}[(H_a^2\frac{\partial}{\partial x_2}f(x_1,x_2)\Big|_{x_2=a_2} \ast_1 \theta(x_1)x_1^{-\alpha})] \times \delta(x_2)$$

$$-\frac{1}{\Gamma(1-\alpha)}D_{x_1}[(H_a^2 f(x_1,x_2)\Big|_{x_2=a_2} \ast_1 \theta(x_1)x_1^{-\alpha})] \times \delta^{(1)}(x_2)$$

5. Applications

The LT defined in Section 2 can be successfully applied to linear equations with derivatives, partial derivatives, fractional derivatives and convolutions to find classical and generalized solutions.

The mode of proceeding to solve such an equation is the following. First we have to localize it on P; of course P and H_a depend on additional conditions (initial, boundary,...). The second step is to find the corresponding equation in $\mathcal{D}'_*(P)$ using Propositions in Section 4. The third step is to solve this equation in $\mathcal{D}'_*(P)$ applying the LT. Since the LT of a $T_* \in \mathcal{D}'_*(P)$ is a class, after applying the LT to the obtained equation in $\mathcal{D}'_*(P)$, we have a class of equations. To construct this class of equations and to work with it we use some properties of the vector space $\mathcal{A}$, specially the following: $\mathcal{A}$ is closely related to derivatives and to convolution by $g \in e^{\omega t}\mathcal{S}'(\mathbb{R}^n_+ + \overline{P})$.

Also we can realize the n-dimensional LT by applying the LT in one dimension successively n-times. The same procedure can be used to realize the inverse LT. Property 7) of the LT (cf. Part 2) can often help giving possibility to use tables for classical LT. At least, Proposition A establishes the existence of a g such that $\mathcal{L}(g)$ equals to the found function $f(z)$ which is the solution to the transformed equation by LT. The solution to the starting equation is given by $g|_P$. If this works for every $a, b \in \Omega \subset \bar{\mathbb{R}}^n$, then we have a solution to the starting equation in Ω. In case we have some condition in a point $x \in \partial\Omega$, we realize it by taking a limit of the found solution.

We give two examples to illustrate the exposed method and nothing else.

5.1. *A differencial equation with fractional derivative*

We consider a type of equations which describes the dynamics of a rod made of generalized Kelvin-Voight Viscoelastic material (cf. [1], [15]) in which the force $F = \omega + A\delta(t - t_0)$, $t_0 > 0$, where ω is a constant and δ is the Dirac distribution, $\gamma \in \mathbb{R}_+$ and $Q \in \mathbb{L}_{loc}(\mathbb{R})$.

This equation is the following:

$$T^{(2)}(t) + \gamma T^{(\alpha)}(t) + \omega T(t) + A\delta(t - t_0)T(t) = Q(t), \ t > 0. \tag{13}$$

With initial condition $T(0) \equiv T_0$, $T^{(1)}(0) \equiv T'_0$.

We look for a solution to (13) belonging to $\mathcal{C}([0, b)) \subset \mathcal{D}'_*([0, b))$.

Since δ can be treated as a measure, (13) has a meaning if $T \in \mathcal{C}([0, \infty))$. Then $\delta(t - t_0)T(t) = T(t_0)\delta(t - t_0)$ (cf. [14] Vol. I, Chapter V).

Firts we have to localize the suposed solution T (cf. 4.1.). In this case $a = 0$ and b is any positive number; the corresponding function H_a, we denote in short by H, and by $\overline{HT}$ is denoted the distribution belonging to $e^{\omega t}\mathcal{S}'(\overline{\mathbb{R}}_+)$ such that $\overline{HT}|_{[0,b)} = [HT] \in \mathcal{D}'_*([0,b))$ defined by the function HT. Let us multiply (13) by H

$$\begin{aligned} (HT^{(2)})(t) \quad &+\gamma(HT^{(\alpha)})(t) + \omega(HT)(t) + \\ &+A\delta(t-t_0)(HT)(t_0) = (HQ)(t), \ \ 0 \le t < b. \end{aligned} \tag{14}$$

By (6) and Proposition 3, to (13) it corresponds in $\mathcal{D}'_*([0,b))$

$$\begin{aligned} D^2[HT] \quad &+\gamma D^\alpha[HT] + \omega[HT] = \\ &= T_0 D^1\delta(t) + T'_0\delta(t) - AT(t_0)\delta(t-t_0) + [HQ]. \end{aligned} \tag{15}$$

Applying LT to (15) we have

$$(z^2 + \gamma z^\alpha + \omega)\mathcal{L}(\overline{HT}) = zT_0 + T'_0 - AT(t_0)e^{-t_0 z} + \mathcal{L}(\overline{HQ}) + \mathcal{L}(W),$$

where $W \in \mathcal{A}$. This gives

$$\begin{aligned} \mathcal{L}(\overline{HT})(z) \quad &= \frac{zT_0 + T'_0}{z^2 + \gamma z^\alpha + \omega} - \frac{AT(t_0)e^{it_0 z}}{z^2 + \gamma z^\alpha + \omega} + \\ &+ \frac{\mathcal{L}(\overline{HQ})(z)}{z^2 + \gamma z^\alpha + \omega} + \mathcal{L}(W_1)(z), \ W_1 \in \mathcal{A}, \text{ as well }. \end{aligned} \tag{16}$$

Now, we can analyse the solution to (13) in $[0, t_0)$ or in $[0, b)$, $b > t_0$. If we seek a solution only in $[0, t_0)$, then in (16) we can take $T(t_0) = 0$.

If $\omega > 0$, it is easy to check by Proposition A that $\overline{HT}$ exists. Namely the function $\Delta(z) = (z^2 + \gamma z^\alpha + \omega)^{-1}$ is holomorphic in $\mathbb{R}_+ + i\mathbb{R}$. We know that z^α denotes the principal value and there is no $z_0 \in \mathbb{R}_+ + i\mathbb{R}$ such that $z_0^2 + \gamma z_0^\alpha + \omega = 0$. Suppose this was false. Then we could have $z_0 = \rho e^{\theta i}$, $|\theta| < \pi/2$, $\theta \neq 0$ such that $\rho^2 \sin 2\theta + \gamma \sin \alpha\theta = 0$, but such a θ does not exist.

If $\omega = -q^2 < 0$, then there exists one and only one $z_0 = \rho_0 > 0$ such that $\rho_0^2 + \gamma\rho_0^\alpha - q^2 = 0$. If we introduce the new variable $s = z - \rho_0$ in (16), then we will also have $\Delta(s + \rho_0) = ((s+\rho_0)^2 + \gamma(s+\rho_0)^\alpha - q^2)^{-1}$ holomorphic in $\mathbb{R}_+ + i\mathbb{R}$. Thus we can apply once again Theorem A.

$$\mathcal{L}^{-1}(\Delta^{-1}(z))(t) = e^{\rho_0 t}\mathcal{L}^{-1}(\Delta^{-1}(s + \rho_0))(t).$$

Property 7) enable us to establish the analytic form of HT (cf. [4], [17]).

5.2. *A partial differential equation*

$$\frac{\partial^4}{\partial x^4}u(x,t)+A^2\frac{\partial^2}{\partial t^2}u(x,t)=f(x,t),\ t>0,\ 0<x<1, \tag{17}$$

with possible additional conditions (cf. [1]):

a) Conditions in $x=0$ and $t=0$

$$\frac{\partial^k}{\partial x^k}u(x,t)|_{x=0}=A_k(t),\ A_k\in\mathbb{L}_{loc}([0,\infty)),\ k=0,1,2,3;$$

$$u(x,0)=B_0(x),\ \frac{\partial}{\partial t}u(x,t)|_{t=0}=B_1(x);B_0,\ \ B_1\in\mathbb{L}_{loc}([0,\infty)).$$

b) Conditions in $x=l$

$$\frac{\partial^k}{\partial x^k}u(x,t)|_{x=l}=C_k(t),\ \ k=0,1,2,3.$$

We are interested in finding a classical solution or a solution belonging to $\mathbb{L}_{loc}(\mathbb{R}^2)$.

If $f(x,t)\equiv 0,\ A_0(t)=$ a sint, $A_1(t)=0,\ C_2(t)=0$ and $C_3(t)=0$, then equation (17) is the mathematical model of a flag pole. This model describes the lateral variations of the amplitude $u(x,t)$ of a beam of length l when one end, $x=0$ (the ground), is forced to move with a prescribed periodic motion of frequency ω (period T) and amplitude a; the other end is free to move (cf. [5]).

This special case can be solved by taking $u(x,t)=U(x)V(t)$ (cf. [1]). In general case solution is much more complicated.

To localize $u(x,t)$ we take $P=[0,l)\times[0,b)$, where b is any positive number. Then we choose the corresponding function $H_0(x,t)=H_0(x)H_0(t)$.

Let us multiply (17) by $H_0(x,t)$

$$H_0(x,t)\frac{\partial^4}{\partial x^4}u(x,t)+A^2H_0(x,t)\frac{\partial^2}{\partial t^2}u(x,t)=H_0(x,t)f(x,t),$$
$$b>t>0,\ \ 0<x<l.$$

We denote by $U(x,t)=H_0(x,t)u(x,t)$ and by $F(x,t)=H_0(x,t)f(x,t)$. By Proposition 5 we have

$$\frac{\partial^4}{\partial x^4}[U(x,t)]+A^2\frac{\partial^2}{\partial t^2}[U(x,t)]=\sum_{k=0}^{3}[H_0(t)A_k(t)]\times\delta^{(3-k)}(x)$$
$$+[H_0(x)B_0(x)]\times\delta^{(1)}(t)+[H_0(x)B_1(x)]\times\delta(t)+[F(x,t)]. \tag{19}$$

This is an equation in $\mathcal{D}'_*(P)$.

Now, we apply the LT to (19) (cf. Section 2)

$$\begin{aligned}(z^4 + A^2s^2)(\mathcal{L}(\overline{U})(z,s) \quad &+\mathcal{L}(W)(z,s)) = \sum_{k=0}^{3} \mathcal{L}(\overline{A}_k)(s)z^{3-k} \\ &+\mathcal{L}(\overline{B}_0)(z)s + \mathcal{L}(\overline{B}_1)(z) + \mathcal{L}(\overline{F})(z,s)\end{aligned} \tag{20}$$

where $W \in \mathcal{A}$ and $\overline{U} \in e^{\omega t}\mathcal{S}'(\overline{\mathbb{R}}^2 + \overline{P})$, $\overline{U}|_P = U(x,t)$. Thus,

$$\mathcal{L}(\overline{U})(z,s) = \frac{Q(z,s)}{z^4 + A^2s^2} - L(W)(z,s),\ z = x + iy,\ s = \xi + i\eta, \tag{21}$$

where

$$Q(z,s) = \sum_{k+1}^{3} \mathcal{L}(\overline{A}_k)(s)z^{3-k} + \mathcal{L}(\overline{B}_0)(z)s + \mathcal{L}(\overline{B}_1)(z) + \mathcal{L}(\overline{F})(z,s).$$

Let us consider the function $M(z,s)$,

$$M(z,s) = \frac{Q(z,s)}{z^4 + A^2s^2} \tag{22}$$

By Theorem A there exists a function $g \in \mathcal{S}'(\overline{\mathbb{R}}^2_+)$ such that

$$M(z,s) = \mathcal{L}(g)(z,s) \text{ if and only if } M(z,s) \in \mathcal{H}_0(\overline{\mathbb{R}}^2_+).$$

Conseqquently, $M(z,s)$ has to be a holomorphic function on $R^2_+ + i\mathbb{R}^2$ and to satisfy the inequality

$$|M(z,s)| \leq c(1+|z|^2+|s|^2)^{\alpha/2}(1+\Delta^{-\beta}((y,\eta),\partial C)), (z,s) \in \mathbb{R}^2_+ + iR^2 \tag{23}$$

for an $\alpha \geq 0,\ \beta \geq 0$ and $c > 0$.

We start with the condition that $M(z,s)$ has to be holomorphic in $\mathbb{R}^2_+ + iR^2$.

We can decompose the denominator in $M(z,s), s \in \mathbb{R}_+ + i\mathbb{R}:$

$$\begin{aligned}z^4 + A^2s^2 \quad &= (z^2 + iAs)(z^2 - iAs) \\ &= (z + \sqrt{e^{-\pi i/2}As})(z - \sqrt{e^{-\pi i/2}As})(z + \sqrt{e^{\pi i/2}As})(z - \sqrt{e^{\pi i/2}As}),\end{aligned}$$

where the root takes the principal branch. Then

$$
\begin{aligned}
M(z,s) = \frac{Q(z,s)}{4Asi}\Big(&\frac{1}{\sqrt{e^{-\pi i/2}As}}\frac{1}{z+\sqrt{e^{-\pi i/2}As}} \\
-\frac{1}{\sqrt{e^{\pi i/2}As}}\frac{1}{z+\sqrt{e^{\pi i/2}As}}\Big) &+ \frac{Q(z,s)}{4Asi}\Big(\frac{1}{\sqrt{e^{\pi i/2}As}}\frac{1}{z-\sqrt{e^{\pi i/2}As}} \\
-\frac{1}{\sqrt{e^{-\pi i/2}As}}\frac{1}{z-\sqrt{e^{-\pi i/2}As}}\Big) &= M_1(z,s)+M_2(z,s), \quad s\in\mathbb{R}_+ + i\mathbb{R}.
\end{aligned}
\tag{24}
$$

The first addend in (24), denoted by $M_1(z,s)$, is holomorphic in $\mathbb{R}^2_+ + i\mathbb{R}^2$. In the second one, denoted by $M_2(z,s)$, we have to fix additional conditions a) in such a manner that $M_2(z,s)$ becomes holomorphic, as well. A necessary condition is that there exist

$$
\lim_{z\to\sqrt{e^{\pi i/2}As}} M_2(z,s) \quad \text{and} \quad \lim_{z\to\sqrt{e^{-\pi i/2}As}} M_2(z,s).
$$

After we obtained $M_2(z,s)$ to be holomorphic in $\mathbb{R}^2_+ + i\mathbb{R}^2$, we can use the inverse LT, $\mathcal{L}^{-1}(M(z,s)) = \mathcal{L}^{-1}(M_1(z,s)) + \mathcal{L}^{-1}(M_2(z,s))$.

Let us consider how to realize $\mathcal{L}^{-}(M_1(z,s))$. We can do it by applying one after the other the inverse LT in one dimension using Tables of the classical LT:

$$
\mathcal{L}^{-1}(M_1(z,s)) = \mathcal{L}_z^{-1}\circ\mathcal{L}_s^{-1}(M_1(z,s)).
$$

So, for example,

$$
\begin{aligned}
\mathcal{L}^{-1} \quad &\Big(\frac{1}{\sqrt{As}}\frac{1}{z+\sqrt{e^{-\pi i/2}As}}\Big)(x,t) \\
&= \mathcal{L}_s^{1}\circ\mathcal{L}_z^{-1}\Big(\frac{1}{\sqrt{A}\sqrt{s}}\frac{1}{z+e^{\pi i/4}\sqrt{A}\sqrt{s}}\Big)(x,t) \\
&= \mathcal{L}_s^{-1}\Big(\frac{\theta(x)}{\sqrt{A}\sqrt{s}}e^{-\frac{\sqrt{2}}{2}\sqrt{A}\,x\sqrt{s}}(\cos\frac{\sqrt{2}}{2}\sqrt{A}\,x\sqrt{s}+i\sin\frac{\sqrt{2}}{2}\sqrt{A}x\sqrt{s})\Big) \\
&= \frac{\theta(t)\theta(x)}{\sqrt{\pi At}}\theta(t)\Big(\cos\frac{x^2A}{t}+i\sin\frac{x^2A}{t}\Big).
\end{aligned}
$$

When we find a $\overline{U}(x,t)$, solution to (21), then $u(x,t)=\overline{U}(x,t)/_P$, where $P=[0,\ell)\times[0,b)$ for every $b\in\mathbb{R}_+; u(x,t)$ is a solution to (17). To satisfy condition b), we only have to take $\lim\limits_{x\to\ell} u(x,\ell)$.

REFERENCES

[1] T. M. A t a n a c k o v i ć, *Stability Theory of Elastic Rods*, World Scientific, Singapore, 1997.

[2] L. B e r g, *Asymptotische Auffasung der Operatorenrechnung,* Studia Math. XX (1962), 215-229.

[3] V. A. D i t k i n, *Operator calculus,* Uspehi Mat. Nauk, T. II, Vip. 6 (22) (1947), 72-158 (in Russian).

[4] G. D o e t s c h, *Handbuch der Laplace-Transformation,* I-III Basel - Stuttgart, 1950-1956.

[5] N. D. F o w k e s and J. J. M a h o n y, *An Introduction to Mathematical Modeling*, John Wiley and Sons, New York, 1996.

[6] R. G o r e n f l o and F. M a i n a r d i, *Fractional Calculus*, Integral and Differential Equations of Fractional Order, p. 223-276 in Fractional Calculus in Continuum Mechanics, Springer, Wien, 1997.

[7] T. K a w a i, *On the theory of Fourier hyperfunctions and it applications to partial differential equations with constant coefficients,* J. Fac. Sci. Univ. Tokyo, Sect. IA, 17 (1970), 467-517.

[8] H. K o m a t s u, *Laplace transform of hyperfunctions - A new fondation of the Heaviside calculus,* J. Fac. Sci. Univ. Tokyo, IA, 34 (1987), 805-820.

[9] H. K o m a t s u, *Solution of differential equations by means of Laplace hyperfunctions,* Structure of solutions of differential equations (Katata / Kyoto, 1995), World Sci. Publishing, River Edge, NJ (1996), 227-252.

[10] G. L u m e r and F. N e u b r a n d e r, *Asymptotic Laplace transforms and evolution equations,* Advances in Partial Differential Equations, Math. Topics Vol. 16, Wiley - VCH, 1999, 37-57.

[11] J. M i k u s i ń s k i, *Sur les fondaments du calcul opératoire,* Studia Math. XI, (1950), 41-70.

[12] S. P i l i p o v i ć and B. S t a n k o v i ć, *Tauberian theorems for integral transform of distributions,* Acta. Math. Hungar. 74 (1-2), (1997), 135-153.

[13] E. E. R o s i n g e r, *Non-Linear Partial Differential Equations,* An Algebraic View of Generalized Solutions, North - Holland, Amsterdam, 1990.

[14] L. S c h w a r t z, *Théorie des Distributions,* 3 éd., Hermann, Paris, 1966.

[15] B. S t a n k o v i ć and T. M. A t a n a c k o v i ć, *Dynamics of a rod made of generalized Kelvin - Voight Viscoelastic material,* Journal of Math. Anal. and Appl. (to be published).

[16] V. S. V l a d i m i r o v, Yu. N. D r o z h i n o v, B. I. Z a v y a l o v, *Tauberian Theorems of Generalized Functions,* Kluwer Academic Press, Dordrecht, 1988.

[17] V. S. V l a d i m i r o v, *Generalized functions in mathematical Physics,* Mir Publishers, Moscow, 1979.

[18] V. S. V l a d i m i r o v, *Equations of Mathematical physics,* "Nauka", Moscow, 1988.

[19] V. V. Z h a r i n o v, *Laplace transform of Fourier hyperfunctions and other close classes of analytic functionals,* Teor. Math. Phys., T. 33:3, (1977), 291-309 (in Russian).

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