Bulletin T.CXXIII de l'Académie Serbe des Sciences et des Arts - 2002
Classe des Sciences mathématiques et naturelles
Sciences mathématiques, № 27

MATRIX TRANSFORMATIONS BETWEEN THE SEQUENCE SPACE BV^P AND CERTAIN BK SPACES

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(Presented at the 4th Meeting, held on May 31, 2002)

A b s t r a c t. *In this paper, we characterize matrix transformations between the sequence space bv^p $(1 < p < \infty)$ and certain BK spaces. Furthermore, we apply the Hausdorff measure of noncompactness to give necessary and sufficient conditions for a linear operator between these spaces to be compact.*

AMS Mathematics Subject Classification (2000): 40H05, 46A45
Key Words: Matrix transformations, measure of noncompactness

1. *Introduction*

We write ω for the set of all complex sequences $x = (x_k)_{k=0}^{\infty}$. Let ϕ, ℓ_∞, c and c_0 denote the set of all finite, bounded, convergent and null sequences, and cs be the set of all convergent series. We write $\ell_p = \{x \in \omega : \sum_{k=0}^{\infty} |x_k|^p < \infty\}$ for $1 \leq p < \infty$, and $bv = \{x \in \omega : \sum_{k=0}^{\infty} |x_k - x_{k-1} < \infty\}$ for the set of all sequences of bounded variation and extend this definition to reals $p \geq 1$ by putting

$$bv^p = \left\{ x \in \omega : \sum_{k=0}^{\infty} |x_k - x_{k-1}|^p < \infty \right\}$$

so that $bv^1 = bv$. The sets bv^p also arise from the sets ℓ_p as the matrix domains of the difference operator in ℓ_p, that is a sequence x is in bv^p, if and only if the sequence $(x_k - x_{k-1})_{k=0}^{\infty}$ is in ℓ_p. It is this concept rather than the first one that plays an important role in our studies.

In this paper, we determine the β–duals of the sets bv^p, characterize some matrix transformations and apply the *Hausdorff measure of noncompactness* to give necessary and sufficient conditions for the entries of an infinite matrix to be a compact operator between the spaces bv^p for $1 < p < \infty$ and certain *BK spaces*.

In this section, we give some notations and recall some definitions and well–known results.

By e and $e^{(n)}$ $(n \in \mathbb{N}_0)$, we denote the sequences such that $e_k = 1$ for $k = 0, 1, \ldots$, and $e_n^{(n)} = 1$ and $e_k^{(n)} = 0$ $(k \neq n)$. For any sequence $x = (x_k)_{k=0}^{\infty}$, let $x^{[n]} = \sum_{k=0}^{n} x_k e^{(k)}$ be its n–section.

A sequence $(b^{(n)})_{n=0}^{\infty}$ in a linear metric space X is called *Schauder basis* if, for every $x \in X$ there is a unique sequence $(\lambda_n)_{n=0}^{\infty}$ of scalars such that $x = \sum_{n=0}^{\infty} \lambda_n b^{(n)}$.

An FK *space* is a complete linear metric sequence space with the property that convergence implies coordinatewise convergence; a BK *space* is a normed FK space. An FK space $X \supset \phi$ is said to have AK if every sequence $x = (x_k)_{k=0}^{\infty} \in X$ has a unique representation $x = \sum_{k=0}^{\infty} x_k e^{(k)}$, that is $x = \lim_{n\to\infty} x^{[n]}$.

Let x and y be sequences, X and Y be subsets of ω and $A = (a_{nk})_{n,k=0}^{\infty}$ be an infinite matrix of complex numbers. We write $xy = (x_k y_k)_{k=0}^{\infty}$, $x^{-1} * Y = \{a \in \omega : ax \in Y\}$ and $M(X,Y) = \bigcap_{x \in X} x^{-1} * Y = \{a \in \omega : ax \in Y \text{ for all } x \in X\}$ for the *multiplier space of* X *and* Y. In the special case of $Y = cs$, we write $x^{\beta} = x^{-1} * cs$ and $X^{\beta} = M(X, cs)$ for the β–*dual of* X. By $A_n = (a_{nk})_{k=0}^{\infty}$ we denote the sequence in the n–th row of A, and we write $A_n(x) = \sum_{k=0}^{\infty} a_{nk} x_k$ $(n = 0, 1, \ldots)$ and $A(x) = (A_n(x))_{n=0}^{\infty}$, provided $A_n \in x^{\beta}$ for all n. The set $X_A = \{x \in \omega : A(x) \in X\}$ is called the *matrix domain of* A *in* X and (X, Y) denotes the class of all matrices that map X into Y, that is $A \in (X, Y)$ if and only if $X_A \subset Y$, or equivalently $A_n \in X^{\beta}$ for all n and $A(x) \in Y$ for all $x \in X$.

Let X and Y be Banach spaces. Then $B(X, Y)$ is the set of all continuous linear operators $L : X \mapsto Y$, a Banach space with the operator norm defined by $\|L\| = \sup\{\|L(x)\| : \|x\| \le 1\}$ $(L \in B(X, Y))$. If $Y = \mathbb{C}$ then we write $X^* = B(X, \mathbb{C})$ for the space of continuous linear functionals on X with its norm defined by $\|f\| = \sup\{|(x)| : \|x\| \le 1\}$ $(f \in X^*)$. We recall that a linear operator $L : X \mapsto Y$ is called *compact* if $D(L) = X$ for the domain of L and if, for every bounded sequence (x_n) in X, the sequence $(L(x_n))$ has a convergent subsequence in Y. It is well known (cf. [10, Theorem 4.2.8, p. 87]) that if X and Y are BK spaces and $A \in (X, Y)$ then $L_A \in B(X, Y)$ where L_A is defined by $L_A(x) = A(x)$ for all $x \in X$; we denote this by $(X, Y) \subset B(X, Y)$.

Let $1 < p < \infty$ and $\mu = (\mu_n)_{n=0}^{\infty}$ be a non–decreasing sequence of positive reals tending to infinity. We define the matrices Σ and Δ by $\Sigma_{nk} = 1$ for $0 \le k \le n$, $\Sigma_{nk} = 0$ for $k > n$, $\Delta_{n,n-1} = -1$, $\Delta_{nn} = 1$ and $\Delta_{nk} = 0$ otherwise, and use the convention that any term with a negative subscript is equal to zero. So $bv^p = (\ell_p)_\Delta$, as has been mentioned above.

Proposition 1.1. *The space bv^p is a BK space with*

$$\|x\|_{bv^p} = \left(\sum_{k=0}^{\infty} |x_k - x_{k-1}|^p \right)^{1/p};$$

the sequence $(b^{(k)})_{k=0}^{\infty}$ with $b^{(k)} = \Sigma(e^{(k)})$, that is $b_j^{(k)} = 0$ for $j < k$ and $b_j^{(k)} = 1$ for $j \ge k$ $(k = 0, 1, \ldots)$, is a Schauder basis of bv^p.

P r o o f. Since ℓ_p is a BK space with $\|x\|_p = (\sum_{k=0}^{\infty} |x_k|^p)^{1/p}$, bv^p is a BK space with $\|\cdot\|_{bv^p}$ by [7, Theorem 3.3, p. 178]. Furthermore ℓ_p has AK. Hence the sequence $(b^{(k)})_{k=0}^{\infty}$ is a Schauder basis of bv^p by [5, Theorem 2.2]. $\square$

2. *The β–dual of the space bv^p*

In this section, we give the β–dual of bv^p for $p \ge 1$. If $X \supset \phi$ is a BK space and $a \in \omega$ then we write

$$\|a\|_X^* = \|a\|^* = \sup \left\{ \left| \sum_{k=0}^{\infty} a_k x_k \right| : \|x\| = 1 \right\},$$

provided the expression on the right is defined and finite which is the case whenever $a \in X^\beta$ (cf. [10, Theorem 7.2.9, p.107]). Let $1 < p < \infty$ and $q = p/(p-1)$. We write $\mathbf{(n+1)^{1/q}} = ((n+1)^{1/q})_{n=0}^{\infty}$.

Theorem 2.1. *Let $1 < p < \infty$. We define the matrix E by $E_{nk} = 0$ for $0 < k < n-1$ and $E_{nk} = 1$ for $k \ge n$ $(n = 0, 1, \ldots)$ and write $M(bv^p) =$*

$((\mathbf{n}+\mathbf{1})^{\mathbf{1/q}})^{-1} * \ell_\infty$.

(a) Then

$$(bv^p)^\beta = (\ell_q \cap M(bv^p))_E.$$

(b) Furthermore

$$\|a\|^*_{bv^p} = \|E(a)\|_q \textit{ for all } a \in (bv^p)^\beta. \tag{2.1}$$

P r o o f. (a) By [5, Theorem 2.5], $(bv^p)^\beta = (\ell_p^\beta \cap M(bv^p,c))_E = (l_q)_E \cap (M(bv^p,c))_E$. We are going to show

$$M(bv^p,c) \subset M(bv^p) \subset M(bv^p,c_0). \tag{2.2}$$

First we assume $a \in M(bv^p,c)$. Then $ax \in c$ for all $x \in bv^p$. Now $x \in bv^p$ if and only if $y = \Delta(x) \in \ell_p$. Then $x = \Sigma(y)$ and $a_n x_n = \sum_{k=0}^n a_n y_k$ $(n = 0,1,\ldots)$ for all $y \in \ell_p$. We define the matrix $C = (c_{nk})_{n,k=0}^\infty$ by $c_{nk} = a_n$ for $0 \le k \le n$ and $c_{nk} = 0$ for $n > k$ $(n = 0,1,\ldots)$. Then $C \in (\ell_p,c)$, and [10, Example 8.4.5B, p. 129] yields

$$\sup_n \sum_{k=0}^\infty |c_{nk}|^q = \sup_n \sum_{k=0}^n |a_n|^q = \sup_n (n+1)|a_n|^q < \infty, \tag{2.3}$$

hence $a(\mathbf{n}+\mathbf{1})^{\mathbf{1/q}} \in \ell_\infty$. This shows

$$M(bv^p,c) \subset M(bv^p). \tag{2.4}$$

Conversely, we assume $a \in M(bv^p)$. Then there exists a constant K such that $(n+1)^{1/q}|a_n| \le K$ for all n, and so $|a_n| \le K/(n+1)^{1/q} \to 0$ $(n \to \infty)$, that is

$$a \in c_0. \tag{2.5}$$

Defining the matrix C as above, we see that (2.3) holds again, and by [10, Example 8.4.5D, p.129], conditions (2.3) and (2.5) yield $C \in (\ell_p,c_0)$, that is $ax \in c_0$ for all $x \in bv^p$. Thus we have shown $M(bv^p,c_0)$, and with (2.4), we obtain (2.2).

(b) Let $a \in (bv^p)^\beta$ be given. We observe that $x \in bv^p$ if and only if $y = \Delta(x) \in \ell_p$. Abel's summation by parts yields, with $R = E(a)$,

$$\sum_{k=0}^n a_k x_k = \sum_{k=0}^{n+1} R_k y_k - R_{n+1}x_{n+1} \ (n = 0,1,\ldots).$$

Since $a \in (bv^p)^\beta$ implies $R \in M(bv^p, c_0)$ by Part (a), it follows that

$$\sum_{k=0}^{\infty} a_k x_k = \sum_{k=0}^{\infty} R_k y_k. \tag{2.6}$$

Now $\|x\|_{bv^p} = \|y\|_p$ implies $\|a\|^*_{bv^p} = \|R\|^*_{\ell_p}$ and (2.1) follows from the fact that ℓ_p^* and ℓ_q are norm isomorphic. □

Remark 1. We observe that neither $\ell_q \subset M(bv^p)$ nor $M(bv^p) \subset \ell_q$. If we define the sequences a and $\tilde{a}$ by

$$a_k = \begin{cases} \dfrac{1}{\nu+1} & (k = 2^\nu) \\ 0 & (k \neq 2^\nu) \end{cases} \quad (\nu = 0, 1, \ldots) \text{ and } \tilde{a}_k = \frac{1}{(k+1)^{1/q}} \ (k = 0, 1, \ldots)$$

then $a \in \ell_q \setminus M(bv^p)$ and $\tilde{a} \in M(bv^p) \setminus \ell_q$, since

$$\sum_{k=0}^{\infty} |a_k|^q = \sum_{\nu=0}^{\infty} \frac{1}{(\nu+1)^q} < \infty \text{ but } |a_{2^\nu}|(2^\nu+1)^{1/q} \geq \frac{2^{\nu/q}}{\nu+1} \to \infty \ (\nu \to \infty) \text{ and}$$
$$\tilde{a}_k (k+1)^{1/q} = 1 \text{ for } k = 0, 1, \ldots \text{ but } \sum_{k=0}^{\infty} \tilde{a}_k = \sum_{k=0}^{\infty} \frac{1}{k+1} = \infty.$$

3. *Matrix Transformations on the spaces bv^p*

In this section we characterize matrix transformations on the spaces bv^p.

Throughout let $1 < p < \infty$ and $q = p/(p-1)$. A subset X of ω is said to be *normal* if $x \in X$ and $y \in \omega$ with $|y_k| \leq |x_k|$ $(k = 0, 1, \ldots)$ together imply $y \in X$. We need the following general results.

Proposition 3.1.([5, Theorem 2.7 (a)]) *Let $X \supset \phi$ be a normal FK space with AK and Y be a linear space. If $M(X_\Delta, c) = M(X_\Delta, c_0)$ then $A \in (X_\Delta, Y)$ if and only if*

$$R^A \in (X, Y) \textit{ where } r^A_{nk} = \sum_{j=k}^{\infty} a_{nk} \ (n, k = 0, 1, \ldots) \tag{3.1}$$

and

$$R^A_n \in (X_\Delta, c) \textit{ for all } n. \tag{3.2}$$

Proposition 3.2.(cf. [7, Theorem 1.23, p. 155]) *Let $X \supset \phi$ and Y be BK spaces.*
(a) Then $A \in (X, \ell_\infty)$ if and only if

$$\|A\|_X^* = \sup_n \|A_n\|_X^* < \infty. \tag{3.3}$$

Furthermore, if $A \in (X, \ell_\infty)$ then $\|L_A\| = \|A\|_X^*$.
(b) If $(b^{(k)})_{k=0}^\infty$ is a Schauder basis of X and Y_1 is a closed BK space in Y, then $A \in (X, Y_1)$ if and only if $A \in (X, Y)$ and $A(b^{(k)}) \in Y_1$ for all k.

First we characterize the classes (bv^p, ℓ_∞), (bv^p, c_0) and (bv^p, c).

Theorem 3.1. *We have*
(a) $A \in (bv^p, \ell_\infty)$ if and only if

$$\|A\|_{(bv^p,\ell_\infty)} = \sup_n \left(\sum_{k=0}^\infty \left| \sum_{j=k}^\infty a_{nj} \right|^q \right)^{1/q} < \infty \tag{3.4}$$

and

$$\sup_k \left(k^{1/q} \left| \sum_{j=k}^\infty a_{nj} \right| \right) < \infty \text{ for all } n; \tag{3.5}$$

(b) $A \in (bv^p, c_0)$ if and only if conditions (3.4) and (3.5) hold and

$$\lim_{n\to\infty} \sum_{j=k}^\infty a_{nj} = 0 \text{ for each } k; \tag{3.6}$$

(c) $A \in (bv^p, c)$ if and only if conditions (3.4) and (3.5) hold and

$$\lim_{n\to\infty} \sum_{j=k}^\infty a_{nj} = \alpha_k \text{ for each } k. \tag{3.7}$$

(d) Let Y denote any of the spaces ℓ_∞, c_0 or c. If $A \in (bv^p, Y)$ then $\|L_A\| = \|A\|_{(bv^p,\ell_\infty)}$.

P r o o f. (a) By Theorem 2.1, $M(bv^p, c) = M(bv^p, c_0)$, so Proposition 3.1. yields that $A \in (bv^p, \ell_\infty)$ if and only if $R \in (\ell_p, \ell_\infty)$ and $R_n \in M(bv^p, c)$ for all n where $r_{nk} = \sum_{j=k}^\infty a_{nj}$ for all n and k. Now $M(bv^p, c) = ((k^{1/q})_{k=0}^\infty)^{-1} * \ell_\infty$, and this is condition (3.5). Furthermore, by [10, Example 8.4.5D, p.

129], $R \in (\ell_p, \ell_\infty)$ if and only if $\sup_n \sum_{k=0}^{\infty} |r_{nk}|^q < \infty$, and this is condition (3.4).

(b) Since $(b^{(k)})_{k=0}^{\infty}$ with $b^{(k)} = \Sigma(e^{(k)})$ for all k is a Schauder basis of bv^p and $b_j^{(k)} = 0$ for $j < k$ and $b_j^{(k)} = 1$ for $j \geq k$ $(k = 0, 1, \dots)$ by Proposition 1.1, we have

$$A_n(b^{(k)}) = \sum_{j=0}^{\infty} a_{nj} b_j^{(k)} = \sum_{j=k}^{\infty} a_{nj} \text{ for each } k.$$

Now Part (b) follows from Part (a) and Proposition 3.2.

(c) Part (c) is proved in exactly the same way as Part (b).

(d) If $A \in (bv^p, \ell_\infty)$ then $\|A\|^*_{bv^p} = \|L_A\|$ by Proposition 3.2. Since $\|A\|^*_{bv^p} = \sup_n \|A_n\|^*_{bv^p}$ for all n, the conclusion follows from (2.1) in Theorem 2.1. Since $(bv^p, c_0) \subset (bv^p, c) \subset (bv^p, \ell_\infty)$, the assertion also follows for $Y = c_0$ or $Y = c$ by what we have just shown and Parts (b) and (c). □

Now we characterize the classes (bv^p, ℓ_1) and (bv^p, bv). We need the following result.

Proposition 3.3. *Let $X \supset \phi$ be a BK space. Then $A \in (X, \ell_1)$ if and only if*

$$\|A\|_{(X,\ell_1)} = \sup_{\substack{N \subset \mathbb{N}_0 \\ N \text{ finite}}} \left\| \sum_{n \in N} A_n \right\| < \infty \text{ (cf. [4, Satz 1]).}$$

Furthermore, if $A \in (X, \ell_1)$ then

$$\|A\|_{(X,\ell_1)} \leq \|L_A\| = 4 \cdot \|A\|_{(X,\ell_1)}. \tag{3.8}$$

P r o o f. We have to show (3.8). Let $A \in (X, \ell_1)$ and $m \in \mathbb{N}_0$ be given. Then, for all $N \subset \{0, \dots, m\}$ and for all $x \in X$ with $\|x\| = 1$,

$$\left| \sum_{n \in N} A_n(x) \right| \leq \sum_{n=0}^{m} |A_n(x)| \leq \|L_A\|,$$

and this implies

$$\|A\|_{(X,\ell_1)} \leq \|L_A\|. \tag{3.9}$$

Furthermore, given $\varepsilon > 0$, there is $x \in X$ with $\|x\| = 1$ such that

$$\|A(x)\|_1 = \sum_{n=0}^{\infty} |A_n(x)| \geq \|L_A\| - \frac{\varepsilon}{2},$$

and there is an integer $m(x)$ such that

$$\sum_{n=0}^{m(x)} |A_n(x)| \geq \|A(x)\|_1 - \frac{\varepsilon}{2}.$$

Consequently $\sum_{n=0}^{m(x)} |A_n(x)| \geq \|L_A\| - \varepsilon$. By [7, Lemma 3.9, p. 181],

$$4 \cdot \max_{N \subset \{0,\dots,m(x)\}} \left| \sum_{n \in N} A_n(x) \right| \geq \sum_{n=0}^{m(x)} |A_n(x)| \geq \|L_A\| - \varepsilon,$$

and so $4 \cdot \|A\|_{(X,\ell_1)} \geq \|L_A\| - \varepsilon$. Since $\varepsilon > 0$ was arbitrary, we have $4 \cdot \|A\|_{(X,\ell_1)} \geq \|L_A\|$, and together with (3.9) this yields (3.8) □

A matrix T is called a *triangle* if $t_{nk} = 0$ $(k > n)$ and $t_{nn} \neq 0$ for all n.

Proposition 3.4.([7, Theorem 3.8, p. 180]) *Let T be a triangle. Then, for arbitrary subsets X and Y of ω, $A \in (X, Y_T)$ if and only if $B = TA \in (X, Y)$. Furthermore, if X and Y are BK spaces and $A \in (X, Y_T)$ then*

$$\|L_A\| = \|L_B\|. \tag{3.10}$$

Theorem 3.2. *We have*
(a) $A \in (bv^p, \ell_1)$ if and only if condition (3.5) holds and

$$\|A\|_{(bv^p,\ell_1)} = \sup_{\substack{N \subset I\!N_0 \\ N \text{ finite}}} \left(\sum_{k=0}^{\infty} \left| \sum_{n \in N} \left(\sum_{j=k}^{\infty} a_{nk} \right) \right|^q \right)^{1/q} < \infty. \tag{3.11}$$

Furthermore, if $A \in (bv^p, \ell_1)$ then

$$\|A\|_{(bv^p,\ell_1)} \leq \|L_A\| \leq 4 \cdot \|A\|_{(bv^p,\ell_1)}. \tag{3.12}$$

(b) $A \in (bv^p, bv)$ if and only if condition (3.5) holds and

$$\|A\|_{(bv^p,bv)} = \sup_{\substack{N \subset I\!N_0 \\ N \text{ finite}}} \left(\sum_{k=0}^{\infty} \left| \sum_{n \in N} \left(\sum_{j=k}^{\infty} (a_{nk} - a_{n-1,k}) \right) \right|^q \right)^{1/q} < \infty. \tag{3.13}$$

Furthermore, if $A \in (bv^p, bv)$ then

$$\|A\|_{(bv^p,bv)} \leq \|L_A\| = 4 \cdot \|A\|_{(bv^p,bv)}. \tag{3.14}$$

P r o o f. (a) Part (a) follows from Proposition 3.3. and Theorem 2.1.
(b) Part (b) follows from Part (a) and Proposition 3.4. □

4. *Measure of noncompactness and transformations*

If X and Y are metric spaces, then $f : X \mapsto Y$ is a compact map if $f(Q)$ is relatively compact (i.e., if the closure of $f(Q)$ is a compact subset of Y) subset of Y for each bounded subset Q of X. In this section, among other things, we investigate when in some special cases the operator L_A is compact. Our investigations use the measure of noncompactness. Recall that if Q is a bounded subset of a metric space X, then the *Hausdorff measure of noncompactness* of Q, is denoted by $\chi(Q)$, and

$$\chi(Q) = \inf\{\epsilon > 0 : Q \quad \text{has a finite } \epsilon - \text{net in } X\}.$$

The function χ is called the *Hausdorff measure of noncompactness*, and for its properties see ([1, 2, 8]). Denote by $\overline{Q}$ the closure of Q. For the convenience of the reader, let us mention that: If Q, Q_1 and Q_2 are bounded subsets of a metric space (X, d), then

$$\begin{aligned}
\chi(Q) = 0 &\iff Q \quad \text{is a totally bounded set,} \\
\chi(Q) &= \chi(\overline{Q}), \\
Q_1 \subset Q_2 &\implies \chi(Q_1) \leq \chi(Q_2), \\
\chi(Q_1 \cup Q_2) &= \max\{\chi(Q_1), \chi(Q_2)\}, \\
\chi(Q_1 \cap Q_2) &\leq \min\{\chi(Q_1), \chi(Q_2)\}.
\end{aligned}$$

If our space X is a normed space, then the function $\chi(Q)$ has some additional properties connected with the linear structure. We have e.g.

$$\begin{aligned}
\chi(Q_1 + Q_2) &\leq \chi(Q_1) + \chi(Q_2), \\
\chi(\lambda Q) &= |\lambda|\chi(Q) \quad \text{for each} \quad \lambda \in \mathbb{C}.
\end{aligned}$$

If X and Y are normed spaces, and $A \in B(X, Y)$, then the Hausdorff measure of noncompactness of A, denoted by $\|A\|_\chi$, is defined by $\|A\|_\chi = \chi(AK)$, where $K = \{x \in X : \|x\| \leq 1\}$ is the unit ball in X. Furthermore, A is compact if and only if $\|A\|_\chi = 0$, and $\|A\|_\chi \leq \|A\|$.

Recall the following well known result (see e.g. [2, Theorem 6.1.1] or [1, 1.8.1]): Let X be a Banach space with a Schauder basis $\{e_1, e_2, \ldots\}$, Q a bounded subset of X, and $P_n : X \mapsto X$ the projector onto the linear span of $\{e_1, e_2, \dots, e_n\}$. Then

$$\begin{aligned} &\frac{1}{a} \limsup_{n\to\infty} \left(\sup_{x\in Q} \|(I-P_n)x\| \right) \leq \chi(Q) \leq \\ &\quad \leq \inf_n \sup_{x\in Q} \|(I-P_n)x\| \leq \limsup_{n\to\infty} \left(\sup_{x\in Q} \|(I-P_n)x\| \right), \end{aligned} \tag{4.1}$$

where $a = \limsup_{n\to\infty} \|I - P_n\|$.

Theorem 4.1. *Let A be an infinite matrix, $1 < p < \infty$, $q = p/(p-1)$ and for any integers n, r, $n > r$, set*

$$\|A\|^{(r)}_{(bv^p,\ell_\infty)} = \sup_{n>r} \left(\sum_{k=0}^{\infty} \left| \sum_{j=k}^{\infty} a_{nj} \right|^q \right)^{1/q}.$$

(a) If $A \in (bv^p, c_0)$, then

$$\|L_A\|_\chi = \lim_{r\to\infty} \|A\|^{(r)}_{(bv^p,\ell_\infty)}. \tag{4.2}$$

(b) If $A \in (bv^p, c)$, then

$$\frac{1}{2} \cdot \lim_{r\to\infty} \|A\|^{(r)}_{(bv^p,\ell_\infty)} \leq \|L_A\|_\chi \leq \lim_{r\to\infty} \|A\|^{(r)}_{(bv^p,\ell_\infty)}. \tag{4.3}$$

(c) If $A \in (bv^p, \ell_\infty)$, then

$$0 \leq \|L_A\|_\chi \leq \lim_{r\to\infty} \|A\|^{(r)}_{(bv^p,\ell_\infty)}. \tag{4.4}$$

P r o o f. Let us remark that the limits in (4.2), (4.3) and (4.4) exist. Set $K = \{x \in bv^p : \|x\| \leq 1\}$. In the case (a) by inequality (4.1) we have

$$\|L_A\|_\chi = \chi(AK) = \lim_{r\to\infty} \left[\sup_{x\in K} \|(I-P_r)Ax\| \right], \tag{4.5}$$

where $P_r : c_0 \mapsto c_0$, $r = 0, 1, \dots$, is the projector on the first $r+1$ coordinates, i.e., $P_r(x) = (x_0, \dots, x_r, 0, 0, \dots)$, for $x = (x_k) \in c_0$; (let us remark that $\|I - P_r\| = 1$, $r = 1, 2, \dots$. Let $A_{(r)} = (\tilde{a}_{nk})$ be infinite matrix defined by $\tilde{a}_{nk} = 0$ if $0 \leq n \leq r$ and $\tilde{a}_{nk} = a_{nk}$ if $r < n$. Now, by Theorem 4.1 (d) we have

$$\sup_{x\in K} \|(I-P_r)Ax\| = \|L_{A_{(r)}}\| = \|A_{(r)}\|_{(bv^p,\ell_\infty)} = \|A\|^{(r)}_{(bv^p,\ell_\infty)}. \tag{4.6}$$

Clearly, by (4.5) and (4.6) we get (4.2).
(b) Let us remark that every sequence $x = (x_k)_{k=0}^{\infty} \in c$ has a unique representation $x = le + \sum_{k=0}^{\infty}(x_k - l)e^{(k)}$ where $l \in \mathbb{C}$ is such that $x - le \in c_0$. Let us define $P_r : c \mapsto c$ by $P_r(x) = le + \sum_{k=0}^{m}(x_k - l)e^{(k)}$, $r = 0, 1, \ldots$. It is known that $\|I - P_r\| = 2$, $r = 0, 1, \ldots$. Now the proof of (b) is similar as in the case (a), and we omit it (it should be borne in mind that now a in (4.1) is 2).
Let us prove (4.4). Now define $P_r : \ell_\infty \mapsto \ell_\infty$, by $P_r(x) = (x_0, x_1, \ldots, x_r, 0, \ldots)$, $x = (x_k) \in \ell_\infty$, $r = 0, 1, \ldots$. It is clear that

$$AK \subset P_r(AK) + (I - P_r)(AK).$$

Now, by the elementary properties of the function χ we have

$$\begin{aligned} \chi(AK) &\le \chi(P_r(AK)) + \chi((I - P_r)(AK)) = \chi((I - P_r)(AK)) \\ &\le \sup_{x \in K} \|(I - P_r)Ax\| = \|L_{A_{(r)}}\|. \end{aligned} \tag{4.7}$$

By (4.7) and Theorem 4.1 (d) we get (4.4). □

Now as a corollary of the above theorem we have

Corollary 4.1. *If either $A \in (bv^p, c_0)$ or $A \in (bv^p, c)$, then*

$$L_A \quad \text{is compact if and only if} \quad \lim_{r \to \infty} \|A\|^{(r)}_{(bv^p, \ell_\infty)} = 0. \tag{4.8}$$

If $A \in (bv^p, \ell_\infty)$, then

$$L_A \quad \text{is compact if} \quad \lim_{r \to \infty} \|A\|^{(r)}_{(bv^p, l_\infty)} = 0. \tag{4.9}$$

The following example will show that it is possible for L_A in (4.9) to be compact in the case $\lim_{r \to \infty} \|A\|^{(r)}_{(bv^p, \ell_\infty)} > 0$, and hence in general in (4.9) we have just " if".

Example 4.1. *Let the matrix A be defined by $A_n = e^{(0)}$ $(n = 0, 1, \ldots)$. Then $\sup_n \left(\sum_{k=0}^{\infty} \left|\sum_{j=k}^{\infty} a_{nj}\right|^q\right)^{1/q} = 1 < \infty$ and $\sup_k \left(k^{1/q} \left|\sum_{j=k}^{\infty} a_{nj}\right|\right) = 0 < \infty$ for all n. By Theorem 4.1 (a) it follows $A \in (bv^p, \ell_\infty)$. Further,*

$$\|A\|^{(r)}_{(bv^p, \ell_\infty)} = \sup_{n > r} \left(\sum_{k=0}^{\infty} \left|\sum_{j=k}^{\infty} a_{nj}\right|^q\right)^{1/q} = \sup_{n > r} \left|\sum_{j=0}^{\infty} a_{nj}\right| = 1 \textit{ for all } r,$$

whence

$$\lim_{r\to\infty} \|A\|^{(r)}_{(bv^p,\ell_\infty)} = 1 > 0.$$

Since $L_A(x) = x_0 e$ for all $x \in bv^p$, L_A is a compact operator.

Theorem 4.2. *Let A be an infinite matrix, $1 < p < \infty$, $q = p/(p-1)$ and for any integer r, set*

$$\|A\|^{(r)}_{(bv^p,\ell_1)} = \sup_{\substack{N\subset IN_0\setminus\{0,1,\dots,r\}\\ N \text{ finite}}} \left(\sum_{k=0}^{\infty} \left| \sum_{n\in N} \left(\sum_{j=k}^{\infty} a_{nj} \right) \right|^q \right)^{1/q}.$$

If $A \in (bv^p, \ell_1)$, then

$$\lim_{r\to\infty} \|A\|^{(r)}_{(bv^p,\ell_1)} \le \|L_A\|_\chi \le 4 \lim_{r\to\infty} \|A\|^{(r)}_{(bv^p,\ell_1)}. \tag{4.10}$$

P r o o f. Every sequence $x = (x_k)_{k=0}^{\infty} \in \ell_1$ has a unique representation

$$x = \sum_{k=0}^{\infty} x_k e^{(k)}.$$

We define $P_r : \ell_1 \mapsto \ell_1$ by $P_r(x) = (x_0, x_1, \dots, x_r, 0, 0, \dots)$, $r = 0, 1, \dots$. Since $\|I - P_r\| = 1$, $r = 0, 1, \dots$, by Theorem 4.2 (a) and (4.1) we get (4.10) (the proof is similar as in the case (4.2)). □

Corollary 4.2.*Let A be as in Theorem 5.2. If $A \in (bv^p, \ell_1)$, then*

$$L_A \text{ is compact if end only if } \lim_{r\to\infty} \|A\|^{(r)}_{(bv^p,\ell_1)} = 0.$$

Theorem 4.3.*Let A be an infinite matrix, $1 < p < \infty$, $q = p/(p-1)$ and for any integer r, set*

$$\|A\|^{(r)}_{(bv^p,bv)} = \sup_{\substack{N\subset IN_0\setminus\{0,1,\dots,r\}\\ N \text{ finite}}} \left(\sum_{k=0}^{\infty} \left| \sum_{n\in N} \left(\sum_{j=k}^{\infty} (a_{nj} - a_{n-1,j}) \right) \right|^q \right)^{1/q}.$$

If $A \in (bv^p, bv)$, then

$$\lim_{r\to\infty} \|A\|^{(r)}_{(bv^p,bv)} \le \|L_A\|_\chi \le 4 \lim_{r\to\infty} \|A\|^{(r)}_{(bv^p,bv)}. \tag{4.11}$$

P r o o f. Let $b^{(k)}$ $k = 0, 1, \ldots$, be as in Proposition 2.1. $(b^{(k)})_{k=0}^{\infty}$ is Schauder basis of bv and it holds

$$x = \sum_{k=0}^{\infty} (x_k - x_{k-1}) b^{(k)}, \quad x \in bv.$$

Now let us define $P_r : bv \mapsto bv$ by

$$P_r(x) = \sum_{k=0}^{r} (x_k - x_{k-1}) b^{(k)}, \quad r = 0, 1, \ldots.$$

Therefore $(I - P_r)(x) = (0, \ldots, 0, x_{r+1} - x_r, x_{r+2} - x_r, \ldots)$. By

$$\begin{aligned}
&\|(I - P_r)(x)\|_{bv} = \\
&\quad = |x_{r+1} - x_r| + |x_{r+2} - x_r - (x_{r+1} - x_r)| + |x_{r+3} - x_r - (x_{r+2} - x_r)| + \ldots \\
&\quad = |x_{r+1} - x_r| + |x_{r+2} - x_{r+1}| + |x_{r+3} - x_{r+2}| + \ldots \\
&\quad \leq \|x\|_{bv},
\end{aligned} \tag{4.12}$$

we get $\|I - P_r\| \leq 1$. Since $I - P_r$ is a projector, we have $\|I - P_r\| \geq 1$. Therefore $\|I - P_r\| = 1$. Now, by Theorem 4.2 (b) and (4.1) we get (4.11). □

Now as a corollary of the above theorem we have

Corollary 4.3. *Let A be as in Theorem 5.3. If $A \in (bv^p, bv)$, then*

$$L_A \textit{ is compact if and only if } \lim_{r\to\infty} \|A\|_{(bv^p, bv)}^{(r)} = 0.$$

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