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AN EXTENSION ON TRILINEAR HILBERT TRANSFORM

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A b s t r a c t. The Trilinear Hilbert transform $H : L^p \times L^q \times A \to L^r$ is extended to $\mathcal{D}_{L^p} \times \mathcal{D}_{L^q} \times \mathcal{D}_A \to \mathcal{D}_{L^r}$ as a continuous mapping.

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1. *Introduction*

In several papers Lacey and Thiele ([3]-[5]) had studied the continuity of the bilinear Hilbert transform

$$H_\alpha(f,g)(x) = p.v.\int f(x-y)g(x+\alpha y)\frac{dy}{y}, \quad \alpha \in \mathbf{R} \setminus \{0,-1\} \ ,$$

where $f \in L^2(\mathbf{R})$ and $g \in L^\infty(\mathbf{R})$, respectively $f \in L^{p_1}(\mathbf{R})$ and $g \in L^{p_2}(\mathbf{R})$,

$$2/3 < p = \frac{p_1 p_2}{p_1 + p_2}, \ 1 < p_1, p_2 < \infty, \ q = p/(p-1), \ q_1 = p_1/(p_1-1). \quad (1)$$

Their main result is the affirmative answer on the Calderon conjecture, first for $p_1 = 2, p_2 = \infty$ ([3]), then for p_1 and p_2 with the assumptions given above ([5]). Their main result is

$$||H_\alpha(f,a)||_{L^p} \leq C||f||_{L^{p_1}}||a||_{L^{p_2}}, \ f \in L^{p_1}, \ a \in L^{p_2}, \quad (2)$$

where $C > 0$ depends on α, p_1, p_2 (for $p_1 = 2, p_2 = \infty$, p equals 2).

In [1] the bilinear Hilbert transform $H_\alpha : L^2 \times L^\infty \to L^2$ respectively, $H_\alpha : L^{p_1} \times L^{p_2} \to L^p$, was extended to $\mathcal{D}'_{L^2} \times \mathcal{D}_{L^\infty} \to \mathcal{D}'_{L^2}$, respectively, $\mathcal{D}'_{L^q} \times \mathcal{D}_{L^{p_2}} \to \mathcal{D}'_{q_1}$, (with suitable parameters) as a hypocontinuous, respectively, continuous mapping and in [2] the Bilinear Hilbert transform of ultradistributions was defined.

In this paper we extend the trilinear Hilbert transform on $L^p \times L^q \times A$ to L^r whenever $1 < p, q \leq \infty$, $\frac{1}{p} + \frac{1}{q} = \frac{1}{p}$, and $\frac{2}{3} < r < \infty$ to $\mathcal{D}_{L^p} \times \mathcal{D}_{L^q} \times \mathcal{D}_A \to \mathcal{D}_{L^r}$ as a continuous mapping.

2. *Preliminaries*

Denote by A the Wiener algebra: the space of functions whose Fourier transform is in L^1. Denote by $\mathcal{D}_A$ a space of smooth functions φ on $\mathbf{R}$ such that for every $\alpha \in \mathbf{N_0}$ hold

$$\mathcal{F}(\varphi^{(\alpha)}) = \xi^\alpha \hat{\varphi} \in L^1.$$

Define $||\varphi||_k = sup||\xi^\alpha \hat{\varphi}||_{L^1}, k \in \mathbf{N_0}$. In [7] it is proved that the mapping

$$T : L^{p_1} \times \cdots \times L^{p_{n-s-1}} \times A \times \cdots \times A \to L^{p'_{n-s}}$$

is continuous, where $\frac{1}{p_1} + \cdots + \frac{1}{p_{n-1}}$, $1 < p_i \leq \infty$, for $i = 1, 2, ..., n - s - 1$ and

$$\frac{1}{p_{i_1}} + \cdots + \frac{1}{p_{i_r}} < \frac{(n-s) - 2(k-s) + r}{2}$$

for all $1 \leq i_1 < \cdots < i_r \leq n - s$, $1 \leq r \leq n - s$.

Thus, for instance Trilinear Hilbert Transform

$$T(f, g, h) = \int f(x-t) g(x+t) h(x+2t) \frac{dt}{t}$$

maps $L^p \times L^q \times A$ to L^r whenever $1 < p, q \leq \infty$, $\frac{1}{p} + \frac{1}{q} = \frac{1}{p}$, and $\frac{2}{3} < r < \infty$. That is the case when $n = 4, k = 2$ and $s = 1$.

3. *Mappings $T_{f,g}$, $T_{f,h}$ and $T_{g,h}$*

Theorem 3.1*Let $f \in \mathcal{D}_{L^p}$, $g \in \mathcal{D}_{L^q}$ and $h \in \mathcal{D}_A$. Then for $1 < p, q < \infty$ mappings $T_{f,g}$, $T_{f,h}$ and $T_{g,h}$ from $\mathcal{D}_A$ to $\mathcal{D}_{L^r}$, from $\mathcal{D}_{L^q}$ to $\mathcal{D}_{L^r}$ and from $\mathcal{D}_{L^p}$ to $\mathcal{D}_{L^r}$ respectively are linear and continuous.*

P r o o f. Let $f \in \mathcal{D}_{L^p}$, $g \in \mathcal{D}_{L^q}$ and $h \in \mathcal{D}_A$.

$$T(f,g,h)(x) = \lim_{\varepsilon\to 0^+}\int_{|t|>\varepsilon} f(x-t)g(x+t)h(x+2t)\frac{dt}{t}$$

$$= \int_{|t|>N} f(x-t)g(x+t)h(x+2t)\frac{dt}{t}$$

$$+ \lim_{\varepsilon\to 0^+}\int_{N\geq|t|>\varepsilon} f(x-t)g(x+t)h(x+2t)\frac{dt}{t}$$

$$= \int_{|t|>N} f(x-t)g(x+t)h(x+2t)\frac{dt}{t}$$

$$+ \lim_{\varepsilon\to 0^+}\int_{N\geq|t|>\varepsilon} f(x-t)\frac{g(x+t)-g(x)}{t}h(x+2t)dt$$

$$+ \lim_{\varepsilon\to 0^+}\int_{N\geq|t|>\varepsilon} f(x-t)g(x)h(x+2t)\frac{dt}{t}$$

$$= \int_{|t|>N} f(x-t)g(x+t)h(x+2t)\frac{dt}{t}$$

$$+ \lim_{\varepsilon\to 0^+}\int_{N\geq|t|>\varepsilon} f(x-t)\frac{g(x+t)-g(x)}{t}$$

$$\cdot h(x+2t)dt + \lim_{\varepsilon\to 0^+}\int_{2N\geq|y|>2\varepsilon} f(x-\frac{y}{2})g(x)h(x+y)\frac{dy}{y}$$

$$= \int_{|t|>N} f(x-t)g(x+t)h(x+2t)\frac{dt}{t}$$

$$+ \lim_{\varepsilon\to 0^+}\int_{N\geq|t|>\varepsilon} f(x-t)\frac{g(x+t)-g(x)}{t}h(x+2t)dt$$

$$+ \lim_{\varepsilon\to 0^+}\int_{2N\geq|y|>2\varepsilon} f(x-\frac{y}{2})g(x)\frac{h(x+y)-h(x)}{y}dy$$

$$+ \lim_{\varepsilon\to 0^+}\int_{2N\geq|y|>2\varepsilon} f(x-\frac{y}{2})g(x)h(x)\frac{dy}{y}$$

$$= \int_{|t|>N} f(x-t)g(x+t)h(x+2t)\frac{dt}{t}$$

$$+ \lim_{\varepsilon\to 0^+}\int_{N\geq|t|>\varepsilon} f(x-t)\frac{g(x+t)-g(x)}{t}h(x+2t)dt$$

$$+ \lim_{\varepsilon\to 0^+}\int_{2N\geq|y|>2\varepsilon} f(x-\frac{y}{2})g(x)\frac{h(x+y)-h(x)}{y}dy$$

$$
\begin{aligned}
&- \lim_{\varepsilon\to 0+}\int_{N\geq|z|>\varepsilon} f(x+z)g(x)h(x)\frac{dz}{z}\\
=&\int_{|t|>N} f(x-t)g(x+t)h(x+2t)\frac{dt}{t}\\
&+\lim_{\varepsilon\to 0+}\int_{N\geq|t|>\varepsilon} f(x-t)\frac{g(x+t)-g(x)}{t}h(x+2t)dt\\
&+\lim_{\varepsilon\to 0+}\int_{2N\geq|y|>2\varepsilon} f(x-\frac{y}{2})g(x)\frac{h(x+y)-h(x)}{y}dy\\
&-\lim_{\varepsilon\to 0+}\int_{N\geq|z|>\varepsilon} \frac{f(x+z)-f(x)}{z}\cdot g(x)h(x)dz\\
&-\lim_{\varepsilon\to 0+}\int_{N\geq|z|>\varepsilon} f(x)g(x)h(x)\frac{dz}{z}\\
=&\int_{|t|>N} f(x-t)g(x+t)h(x+2t)\frac{dt}{t}\\
&+\lim_{\varepsilon\to 0+}\int_{N\geq|t|>\varepsilon} f(x-t)\frac{g(x+t)-g(x)}{t}h(x+2t)dt\\
&+\lim_{\varepsilon\to 0+}\int_{2N\geq|y|>2\varepsilon} f(x-\frac{y}{2})g(x)\frac{h(x+y)-h(x)}{y}dy\\
&-\lim_{\varepsilon\to 0+}\int_{N\geq|z|>\varepsilon} \frac{f(x+z)-f(x)}{z}g(x)h(x)dz\\
=&\int_{|t|>N} f(x-t)g(x+t)h(x+2t)\frac{dt}{t}\\
&+\lim_{\varepsilon\to 0+}\int_{N\geq|t|>\varepsilon} f(x-t)G(x,t)h(x+2t)dt\\
&+\lim_{\varepsilon\to 0+}\int_{2N\geq|y|>2\varepsilon} f(x-\frac{y}{2})g(x)H(x,y)dy\\
&-\lim_{\varepsilon\to 0+}\int_{N\geq|z|>\varepsilon} F(x,z)g(x)h(x)dz
\end{aligned}
$$

where

$$
G(x,t)=\begin{cases} \frac{g(x+t)-g(x)}{t}, & t\neq 0\\ \frac{d}{dx}g(x), & t=0,\ x\in\mathbf{R},\end{cases}
$$

$$
H(x,y)=\begin{cases} \frac{h(x+y)-h(x)}{y}, & y\neq 0\\ \frac{d}{dx}h(x), & y=0,\ x\in\mathbf{R}\end{cases}.
$$

$$F(x,z) = \begin{cases} \frac{f(x+z)-f(x)}{z}, & z \neq 0 \\ \frac{d}{dx}f(x), & z = 0,\ x \in \mathbf{R}. \end{cases}$$

It is obvious that $F(x,z)$, $G(x,t)$ and $H(x,y)$, along with all of their partial derivatives are continuous functions of x, y, z, t, respectively.

Let κ_N be characteristic function of $(-\infty, -N] \cup [N, \infty)$. Then, according to Muscalu's proof in [7] we have

$$||\kappa_N(t)\frac{\partial}{\partial x}f(x-t)g(x+t)h(x+2t)\frac{1}{t}||_r \leq C||\kappa_N\frac{\partial}{\partial x}f(x-\cdot)||_p||g||_q||h||_A,$$

$$||\kappa_N(t)f(x-t)\frac{\partial}{\partial x}g(x+t)h(x+2t)\frac{1}{t}||_r \leq C||\kappa_N f(x-\cdot)||_p||\frac{\partial}{\partial x}g||_q||h||_A,$$

$$||\kappa_N(t)f(x-t)g(x+t)\frac{\partial}{\partial x}h(x+2t)\frac{1}{t}||_r \leq C||\kappa_N f(x-\cdot)||_p||g||_q||\frac{\partial}{\partial x}h||_A.$$

So we get

$$\begin{aligned}
\frac{d}{dx}T(f,g,h)(x) &= \int_{|t|>N}\frac{\partial}{\partial x}f(x-t)g(x+t)h(x+2t)\frac{dt}{t} \\
&+ \int_{|t|>N}f(x-t)\frac{\partial}{\partial x}g(x+t)h(x+2t)\frac{dt}{t} \\
&+ \int_{|t|>N}f(x-t)g(x+t)\frac{\partial}{\partial x}h(x+2t)\frac{dt}{t} \\
&+ \lim_{\varepsilon\to 0^+}\int_{N\geq|t|>\varepsilon}\frac{\partial}{\partial x}[f(x-t)G(x,t)h(x+2t)]dt \\
&+ \lim_{\varepsilon\to 0^+}\int_{2N\geq|y|>2\varepsilon}\frac{\partial}{\partial x}[f(x-y/2)g(x)H(x,y)]dy \\
&- \lim_{\varepsilon\to 0^+}\int_{N\geq|z|>\varepsilon}\frac{\partial}{\partial x}[F(x,z)g(x)h(x)]dz \\
&= \int_{|t|>N}\frac{\partial}{\partial x}f(x-t)g(x+t)h(x+2t)\frac{dt}{t} \\
&+ \int_{|t|>N}f(x-t)\frac{\partial}{\partial x}g(x+t)h(x+2t)\frac{dt}{t} \\
&+ \int_{|t|>N}f(x-t)g(x+t)\frac{\partial}{\partial x}h(x+2t)\frac{dt}{t} \\
&+ \lim_{\varepsilon\to 0^+}\int_{N\geq|t|>\varepsilon}\frac{\partial}{\partial x}[f(x-t)G(x,t)h(x+2t)]dt
\end{aligned}$$

$$
\begin{aligned}
&+\lim_{\varepsilon\to 0+}\int_{N\geq|t|>\varepsilon}\frac{\partial}{\partial x}f(x-t)G(x,t)h(x+2t)dt\\
&+\lim_{\varepsilon\to 0+}\int_{N\geq|t|>\varepsilon}f(x-t)\frac{\frac{\partial}{\partial x}g(x+t)-\frac{\partial}{\partial x}g(x)}{t}h(x+2t)dt\\
&+\lim_{\varepsilon\to 0+}\int_{N\geq|t|>\varepsilon}f(x-t)G(x,t)\frac{\partial}{\partial x}h(x+2t)dt\\
&+\lim_{\varepsilon\to 0+}\int_{2N\geq|y|>2\varepsilon}\frac{\partial}{\partial x}f(x-\frac{y}{2})g(x)H(x,y)dy\\
&+\lim_{\varepsilon\to 0+}\int_{2N\geq|y|>2\varepsilon}f(x-\frac{y}{2})\frac{\partial}{\partial x}g(x)H(x,y)dy\\
&+\lim_{\varepsilon\to 0+}\int_{2N\geq|y|>2\varepsilon}f(x-\frac{y}{2})g(x)\frac{\frac{\partial}{\partial x}h(x+y)-\frac{\partial}{\partial x}h(x)}{y}dy\\
&-\lim_{\varepsilon\to 0+}\int_{N\geq|z|>\varepsilon}\frac{\frac{\partial}{\partial x}f(x+z)-\frac{\partial}{\partial x}f(x)}{z}g(x)h(x)dz\\
&-\lim_{\varepsilon\to 0+}\int_{N\geq|z|>\varepsilon}F(x,z)\frac{\partial}{\partial x}g(x)h(x)dz\\
&-\lim_{\varepsilon\to 0+}\int_{N\geq|z|>\varepsilon}F(x,z)g(x)\frac{\partial}{\partial x}h(x)dz\\
=&\int_{|t|>N}\frac{\partial}{\partial x}f(x-t)g(x+t)h(x+2t)\frac{dt}{t}\\
&+\int_{|t|>N}f(x-t)\frac{\partial}{\partial x}g(x+t)h(x+2t)\frac{dt}{t}\\
&+\int_{|t|>N}f(x-t)g(x+t)\frac{\partial}{\partial x}h(x+2t)\frac{dt}{t}\\
&+\lim_{\varepsilon\to 0+}\int_{N\geq|t|>\varepsilon}\frac{\partial}{\partial x}f(x-t)\frac{g(x+t)-g(x)}{t}h(x+2t)dt\\
&+\lim_{\varepsilon\to 0+}\int_{N\geq|t|>\varepsilon}f(x-t)\frac{\partial}{\partial x}g(x+t)h(x+2t)\frac{dt}{t}\\
&-\frac{\partial}{\partial x}g(x)\lim_{\varepsilon\to 0+}\int_{N\geq|t|>\varepsilon}f(x-t)h(x+2t)\frac{dt}{t}\\
&+\lim_{\varepsilon\to 0+}\int_{N\geq|t|>\varepsilon}f(x-t)\frac{g(x+t)-g(x)}{t}\cdot\frac{\partial}{\partial x}h(x+2t)dt\\
&+\lim_{\varepsilon\to 0+}\int_{2N\geq|y|>2\varepsilon}\frac{\partial}{\partial x}f(x-\frac{y}{2})g(x)\frac{h(x+y)-h(x)}{y}dy
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\partial}{\partial x}g(x)\lim_{\varepsilon\to 0^+}\int_{2N\geq|y|>2\varepsilon} f(x-\frac{y}{2})\frac{h(x+y)-h(x)}{y}dy \\
&+g(x)\lim_{\varepsilon\to 0^+}\int_{2N\geq|y|>2\varepsilon} f(x-\frac{y}{2})\frac{\partial}{\partial x}h(x+y)\frac{dy}{y} \\
&-g(x)\frac{\partial}{\partial x}h(x)\lim_{\varepsilon\to 0^+}\int_{2N\geq|y|>2\varepsilon} f(x-\frac{y}{2})\frac{dy}{y} \\
&-g(x)h(x)\lim_{\varepsilon\to 0^+}\int_{N\geq|z|>\varepsilon}\frac{\partial}{\partial x}f(x+z)\frac{dz}{z} \\
&+\frac{\partial}{\partial x}f(x)g(x)h(x)\lim_{\varepsilon\to 0^+}\int_{N\geq|z|>\varepsilon}\frac{dz}{z} \\
&-\frac{\partial}{\partial x}g(x)h(x)\lim_{\varepsilon\to 0^+}\int_{N\geq|z|>\varepsilon}\frac{f(x+z)-f(x)}{z}dz \\
&-g(x)\frac{\partial}{\partial x}h(x)\lim_{\varepsilon\to 0^+}\int_{N\geq|z|>\varepsilon}\frac{f(x+z)-f(x)}{z}dz \\
=&\int_{|t|>N}\frac{\partial}{\partial x}f(x-t)g(x+t)h(x+2t)\frac{dt}{t} \\
&+\int_{|t|>N}f(x-t)\frac{\partial}{\partial x}g(x+t)h(x+2t)\frac{dt}{t} \\
&+\int_{|t|>N}f(x-t)g(x+t)\frac{\partial}{\partial x}h(x+2t)\frac{dt}{t} \\
&+\lim_{\varepsilon\to 0^+}\int_{N\geq|t|>\varepsilon}\frac{\partial}{\partial x}f(x-t)g(x+t)h(x+2t)\frac{dt}{t} \\
&+\lim_{\varepsilon\to 0^+}\int_{N\geq|t|>\varepsilon}f(x-t)\frac{\partial}{\partial x}g(x+t)h(x+2t)\frac{dt}{t} \\
&+\lim_{\varepsilon\to 0^+}\in t_{N\geq|t|>\varepsilon}f(x-t)g(x+t)\frac{\partial}{\partial x}h(x+2t)\frac{dt}{t} \\
=&\lim_{\varepsilon\to 0^+}\int_{|t|>\varepsilon}\frac{\partial}{\partial x}f(x-t)g(x+t)h(x+2t)\frac{dt}{t} \\
&+\lim_{\varepsilon\to 0^+}\int_{|t|>\varepsilon}f(x-t)\frac{\partial}{\partial x}g(x+t)h(x+2t)\frac{dt}{t} \\
&+\lim_{\varepsilon\to 0^+}\int_{|t|>\varepsilon}f(x-t)g(x+t)\frac{\partial}{\partial x}h(x+2t)\frac{dt}{t} \\
=&\,T(f',g,h)(x)+T(f,g',h)(x)+T(f,g,h')(x)
\end{aligned}
$$

Using a similar technique, we can show by induction that

$$[T(f,g,h)]^{(n)}(x) = \sum_{k=0}^{n}\sum_{m=0}^{k}\binom{n}{k}\binom{k}{m}T(f^{(n-k)},g^{(k-m)},h^{(m)})(x)$$

for all $n \in N$.
Now we have

$$||[T(f,g,h)]^{(n)}(x)||_r \leq \sum_{k=0}^{n}\sum_{m=0}^{k}\binom{n}{k}\binom{k}{m}||T(f^{(n-k)},g^{(k-m)},h^{(m)})(x)||_r$$
$$\leq \sum_{k=0}^{n}\sum_{m=0}^{k}\binom{n}{k}\binom{k}{m}||f^{(n-k)}||_p \cdot ||g^{(k-m)}||_q \cdot ||h^{(m)}||_A$$

Theorem 3.2*Bilinear mappings*

$$(f,g) \mapsto T_h(f,g) from \mathcal{D}_{L^p} \times \mathcal{D}_{L^q} \to \mathcal{D}_{L^r},$$
$$(f,h) \mapsto T_g(f,h) from \mathcal{D}_{L^p} \times \mathcal{D}_{A} \to \mathcal{D}_{L^r},$$
$$(g,h) \mapsto T_f(g,h) from \mathcal{D}_{L^q} \times \mathcal{D}_{A} \to \mathcal{D}_{L^r},$$

are continuous.

P r o o f. According to Theorem 1 we have that these mappings are separately continuous. Then Corollary of Theorem 34.1 from [11] implies that these mappings are continuous. □

Corollary 3.3*Mapping*

$$T : \mathcal{D}_{L^p} \times \mathcal{D}_{L^q} \times \mathcal{D}_{A} \to \mathcal{D}_{L^r},$$

is continuous.

REFERENCES

[1] A. B u č k o v s k a, S. P i l i p o v ić, *An Extension of Bilinear Hilbert Transform to Distributions*, Integral Transforms and Special Functions, Taylor and Francis, accepted for publication.

[2] A. B u č k o v s k a, S. P i l i p o v i ć, *Bilinear Hilbert Transform of Ultradistributions*, Integral Transforms and Special Functions, Taylor and Francis, accepted for publication.

[3] M. T. L a c e y and C. M. T h i e l e, *L^p estimates on the bilinear Hilbert transform for $2 < p < \infty$*, Annals of Math.,146(1997), 693-724.

[4] M. T. L a c e y and C. M. T h i e l e, *On Calderon's Conjecture for the Bilinear Hilbert Transform*, Proc.Natl.Acad.Sci USA, 95(1998), 4828-4830.

[5] M. T. L a c e y and C. M. T h i e l e, *On Calderon's Conjecture*, Annals of Math., 149(1999), 475-496.

[6] J. H o r v a t, *Topological Vector Spaces and Distributions*, Vol. 1, Addison Wesley, Massachusetts, 1966.

[7] C. M u s c a l u, T. T a o, C. T h i e l e, *Multi-linear Operators Given by Singular Multipliers*, Journal of AMS, 2001.

[8] W. R u d i n, *Real and Complex Analysis*, McGraw-Hill, 1987

[9] L. S c h w a r t z, *Théorie des distributions*, I.2nd ed., Hermann, Paris, 1957.

[10] E. M. S t e i n and G. W e i s s, *Introduction to Fourier Analysis on Euclidean Spaces*, Princeton University Press, 1971

[11] F. T r e v e s, *Topological Vector Spaces, Distributions and Kernels*, Acad. Press, New York, 1967

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