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ON SOME MEAN VALUE RESULTS INVOLVING $|\zeta(\frac{1}{2}+it)|$

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A b s t r a c t. *Several problems involving $E(T)$ and $E_2(T)$, the error terms in the mean square and mean fourth moment formula for $|\zeta(\frac{1}{2}+it)|$, are discussed. In particular it is proved that*

$$\int_0^T E(t)E_2(T)\mathrm{d}t \ll T^{7/4}(\log T)^{7/2}\log\log T.$$

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1. *Introduction and statement of results*

Let, as usual ($\gamma = 0.5772157...$ is Euler's constant),

$$E(T) \;=\; \int_0^T |\zeta(\tfrac{1}{2}+it)|^2\mathrm{d}t - T\left(\log\left(\frac{T}{2\pi}\right)+2\gamma-1\right)$$

denote the error term in the mean square formula for $|\zeta(\frac12+it)|$, and let

$$E_2(T) \;=\; \int_0^T |\zeta(\tfrac12+it)|^4 \mathrm{d}t - TP_4(\log T)$$

denote the error term in the asymptotic formula for the fourth moment of $|\zeta(\frac12+it)|$. Here $P_4(x)$ is a polynomial of degree four in x with leading coefficient $1/(2\pi^2)$ (see [5] for the explicit evaluation of all the coefficients). Both of these functions play an important rôle in the theory of the Riemann zeta-function $\zeta(s)$, and the aim of this note is to discuss several problems involving their mean values. Especially interesting seems the evaluation of the integral

$$\int_0^T E(t)E_2(t)\mathrm{d}t,$$

or (which is technically more convenient)

$$I(T) \;:=\; \int_T^{2T} E(t)E_2(t)\mathrm{d}t, \tag{1.1}$$

since this integral exhibits the superpositions of oscillations of the functions $E(t)$ and $E_2(t)$. Namely both functions are oscillating, and we have $E(t) = \Omega_\pm(t^{1/4})$ (see [1], [4]) and $E_2(t) = \Omega_\pm(t^{1/2})$ (see [4], [6] and [12]). As usual, $f = \Omega_\pm(g)$ means that $\limsup f/g > 0$ and $\liminf f/g < 0$. We also have (see [2] and [4])

$$\int_0^T E^2(t)\mathrm{d}t = DT^{3/2} + O(T\log^4 T) \tag{1.2}$$

with $D = 2(2\pi)^{-1/2}\zeta^4(3/2)/(3\zeta(3))$, and (see [7], [12])

$$\int_T^{2T} E_2^2(t)\mathrm{d}t \ll T^2\log^{22} T. \tag{1.3}$$

As usual, $f \ll g$ (same as $f = O(g)$) means that $|f(x)| \le Cg(x)$ for some $C > 0$ and $x \ge x_0$). Hence by (1.2), (1.3), and the Cauchy-Schwarz inequality for integrals one obtains

$$I(T) \;\ll\; T^{7/4}\log^{11} T, \tag{1.4}$$

and one naturally asks whether (1.4) can be improved. This is indeed so, as shown by the following

Theorem 1. *We have*

$$I(T) \;=\; \int_T^{2T} E(t)E_2(t)\mathrm{d}t \;\ll\; T^{7/4}(\log T)^{7/2}\log\log T. \tag{1.5}$$

We remark here that an analogous formula to (1.5) holds if $E(t)$ is replaced ($d(n)$ is the number of divisors of n) by

$$\Delta(x) = \sum_{n\le x} d(n) - x(\log x + 2\gamma - 1),$$

the error term in the classical divisor problem. Namely the analogues of (2.4) and (2.5) will hold for $\Delta(x)$ by the Voronoi explicit formula for $\Delta(x)$ (see [2, Chapter 3]). Thus, following the proof of Theorem 1, we shall obtain

$$\int_T^{2T} \Delta(t)E_2(t)\mathrm{d}t \ll T^{7/4}(\log T)^{7/2}\log\log T.$$

In the course of the proof of Theorem 1 we shall encounter the function

$$g(t) := \tfrac{1}{2}\left(\frac{2}{\pi}\right)^{3/4}\sum_{n=1}^{\infty}(-1)^n d(n)n^{-5/4}\sin(\sqrt{8\pi nt} - \tfrac{1}{4}\pi), \tag{1.6}$$

which appears in the relation

$$\int_T^{2T} E(t)E_2(t)\mathrm{d}t = -\int_T^{2T} t^{3/4}g(t)|\zeta(\tfrac{1}{2}+it)|^4\mathrm{d}t + O(T^{3/2}\log^{10}T). \tag{1.7}$$

It is thus seen that the natural question of the true order of magnitude of $I(T)$ involves the evaluation of the integral on the right-hand side of (1.7), which contains the oscillatory function $g(t)$ (it is both $O(1)$ and $\Omega_\pm(1)$; see [1] and [4]). Although it appears to the author that the true order of $I(T)$ is $T^{3/2+o(1)}$, this is certainly hard to prove. However if we set

$$g_+(t) = \max(g(t),\, 0), \qquad g_-(t) = \min(g(t),\, 0), \tag{1.8}$$

then we have

Theorem 2. *If $g_\pm(t)$ is given by* (1.8), *then for $k = 1, 2$ we have*

$$\begin{aligned}\int_T^{2T} t^{3/4}g_+(t)|\zeta(\tfrac{1}{2}+it)|^{2k}\mathrm{d}t &\ll T^{7/4}(\log T)^{k^2},\\ \int_T^{2T} t^{3/4}g_+(t)|\zeta(\tfrac{1}{2}+it)|^{2k}\mathrm{d}t &\gg T^{7/4}(\log T)^{k^2},\end{aligned} \tag{1.9}$$

and likewise

$$\begin{aligned} \int_T^{2T} t^{3/4} g_-(t)|\zeta(\tfrac12+it)|^{2k}\mathrm{d}t \;&\ll\; T^{7/4}(\log T)^{k^2},\\ \int_T^{2T} t^{3/4} g_-(t)|\zeta(\tfrac12+it)|^{2k}\mathrm{d}t \;&\gg\; T^{7/4}(\log T)^{k^2}. \end{aligned}\tag{1.10}$$

Comparing Theorem 1 and Theorem 2 we see that there must be some cancellation when we deal with $g(t) = g_+(t) + g_-(t)$ instead of only $g_+(t)$ or $g_-(t)$. We note that, similarly to (1.7), we obtain (with suitable $B > 0$, see the remark after (2.2))

$$\int_T^{2T} t^{3/4} g(t)|\zeta(\tfrac12+it)|^2\mathrm{d}t = -BT^{3/2} + O(T^{5/4}\log T). \tag{1.11}$$

The integrals in (1.7) and (1.11) containing the function $g(t)$ are similar, which is why there is reason to think that $I(T)$ is also of the the order $T^{3/2+o(1)}$. The original motivation for the study of $I(T)$ was to try to obtain a lower bound for the integral on the right-hand side of (1.7). This would in turn, by the Cauchy-Schwarz inequality, provide a lower bound for the mean square integral of $E_2(T)$. The author proved in [7] the lower bound

$$\int_T^{2T} E_2^2(t)\mathrm{d}t \;\gg\; T^2, \tag{1.12}$$

which complements (1.3). However, in view of Theorem 1, it does not appear likely that this procedure can shed some new light on the behaviour of the integral in (1.12).

Since the function $g(t)$ is $\Omega_\pm(1)$, it means that it takes positive and negative values for some arbitrarily large values of t. Thus it seems of interest to characterize the sets where $g(t) > 0$ and $g(t) < 0$. In this direction we have ($\mu(\cdot)$ denotes measure) the following result, which will be used in proving Theorem 2.

Theorem 3. *There is a number* $\eta > 0$ *such that* $[T, 2T]$ *contains a subset* $\mathcal{A}(T)$ *in which* $g(t) > \eta$ *and*

$$\mu(\mathcal{A}(T)) \;\gg_\eta\; T, \tag{1.13}$$

and a subset $\mathcal{B}(T)$ *in which* $g(t) < -\eta$ *and*

$$\mu(\mathcal{B}(T)) \;\gg_\eta\; T. \tag{1.14}$$

2. *Proof of Theorem 1*

Let us define

$$G(T) := \int_0^T (E(t)-\pi)\mathrm{d}t. \tag{2.1}$$

If $P_4(x)$ is the polynomial appearing in the definition of $E_2(T)$ and $Q_4(x) := P_4(x)+P_4'(x)$, then integrating by parts we have

$$\begin{aligned}
&\int_T^{2T} E(t)E_2(t)\mathrm{d}t = \int_T^{2T} (E(t)-\pi+\pi)E_2(t)\mathrm{d}t\\
&= O(T^{3/2}) - \int_T^{2T} G(t)E_2'(t)\mathrm{d}t\\
&= -\int_T^{2T} G(t)(|\zeta(\tfrac12+it)|^4 - Q_4(\log t))\mathrm{d}t + O(T^{3/2}) \qquad (2.2)\\
&= -\int_T^{2T} G(t)|\zeta(\tfrac12+it)|^4\mathrm{d}t + O(T^{3/2}).
\end{aligned}$$

Proceeding with $E(t)$ in place of $E_2(t)$, we obtain (1.11). Here we used the facts that

$$E_2(T) = O(T^{2/3}\log^8 T), \qquad \int_0^T E_2(t)\mathrm{d}t = O(T^{3/2}), \tag{2.3}$$

and

$$G(T) = O(T^{3/4}), \qquad \int_0^T G(t)\mathrm{d}t = O(T^{5/4}). \tag{2.4}$$

For a proof of the bounds in (2.3), see [4] or [12]. The bounds in (2.4) follow from the explicit formula of Hafner–Ivić [1], namely

$$\begin{aligned}
G(t) &= 2^{-3/2}\sum_{n\le t}(-1)^n d(n)n^{-1/2}\left(\frac{t}{2\pi n}+\frac14\right)^{-1/4}\left(\operatorname{ar\,sinh}\sqrt{\frac{\pi n}{2t}}\right)^{-2}\sin f(t,n)\\
&\quad -2\sum_{n\le c_0 t} d(n)n^{-1/2}\left(\log\frac{t}{2\pi n}\right)^{-2}\sin\left(t\log\frac{t}{2\pi n}-t-\frac{\pi}{4}\right)+O(t^{1/4}),
\end{aligned} \tag{2.5}$$

where $c_0 = 1/(2\pi)+1/2-\sqrt{1/4+1/(2\pi)} = 0.019502\ldots$ and

$$f(t,k) = 2t\operatorname{ar\,sinh}\sqrt{\frac{\pi k}{2t}}+\sqrt{2\pi kt+\pi^2k^2}-\tfrac14\pi, \ \operatorname{ar\,sinh} x = \log(x+\sqrt{1+x^2}).$$

By using the Cauchy–Schwarz inequality for integrals, the bound (e.g., see [2] for a proof)

$$\int_0^T |\zeta(\tfrac{1}{2}+it)|^8 \mathrm{d}t \ll T^{3/2}\log^{21/2} T \tag{2.6}$$

and the mean theorem for Dirichlet polynomials (e.g., see [2, Chapter 5]), it is seen that the contribution of $\sum_{n\le c_0 t}$ in (2.5) to the right-hand side of (1.7) is $\ll T^{3/2}\log^{10} T$ (the exponent of the logarithm is not optimal, but it is unimportant). Simplifying the first sum in (2.5) by Taylor's formula (truncating it at $n = \sqrt{T}$), we obtain from (2.2) the asympottic formula (1.7), namely

$$I(T) = -\int_T^{2T} t^{3/4} g(t)|\zeta(\tfrac{1}{2}+it)|^4 \mathrm{d}t + O(T^{3/2}\log^{10} T) \tag{2.7}$$

with $g(t)$ given by (1.6). Since clearly

$$|g(t)| \le \tfrac{1}{2}\left(\frac{2}{\pi}\right)^{3/4}\sum_{n=1}^\infty d(n)n^{-5/4} = \tfrac{1}{2}\left(\frac{2}{\pi}\right)^{3/4}\zeta^2(\tfrac{5}{4}) = O(1),$$

one obtains easily from (2.7) and the weak bound

$$\int_0^T |\zeta(\tfrac{1}{2}+it)|^4 \mathrm{d}t \ll T\log^4 T, \tag{2.8}$$

the upper bound

$$I(T) \;=\; O(T^{7/4}\log^4 T). \tag{2.9}$$

Although (2.9) improves (1.4), it is poorer than (1.5) of Theorem 1, so that we must use different tools to obtain the assertion of Theorem 1. To this end we appeal to the following explicit formula of Ivić–Motohashi (see [4] and [12]): For $V^{1/2}\log^{-A} V \le \Delta \le V^{3/4}$ ($A > 0$ is an arbitrary, but fixed constant)

$$\begin{aligned}\int_0^V I(T,\Delta)\mathrm{d}t \; &= VP_4(\log V) + O(\Delta\log^5 V)\\ &+\pi\sqrt{\tfrac{1}{2}V}\sum_{j=1}^\infty \alpha_j H_j^3(\tfrac{1}{2})c_j\cos(\kappa_j\log\frac{\kappa_j}{4\mathrm{e}V})\mathrm{e}^{-(\kappa_j\Delta/2V)^2},\end{aligned} \tag{2.10}$$

where $c_j \sim \kappa_j^{-3/2}$ as $\kappa_j \to \infty$ and

$$I(T,\Delta) \;:=\; \frac{1}{\Delta\sqrt{\pi}}\int_{-\infty}^{\infty}|\zeta(\tfrac{1}{2}+iT+iu)|^4 \mathrm{e}^{-(u/\Delta)^2}\mathrm{d}u. \tag{2.11}$$

For the definitions and properties of the spectral quantities α_j, κ_j and $H_j(\frac{1}{2})$, see Y. Motohashi's monograph [13]. What will be needed here, besides (2.10) and (2.11), is essentially the bound (cf. [13])

$$\sum_{\kappa_j\le K}\alpha_j H_j^3(\tfrac{1}{2}) \ll K^2\log^3 K. \tag{2.12}$$

Although (2.12) is not stated explicitly in [13], it follows when one integrates the last formula on p. 130 (with $G = T^{3/4}$, say) from K to $2K$ with the help of the estimate for the sum in (2.12) in short intervals, obtained recently by the author in [9]. An asymptotic formula for the sum in (2.12) has been obtained recently by the author in [10]. This is

$$\sum_{\kappa_j\le K}\alpha_j H_j^3(\tfrac{1}{2}) = K^2P_3(\log K) + R(K), \quad R(K) = O(K^{5/4}\log^{25/2}K), \tag{2.13}$$

where $P_3(x)$ is a cubic polynomial in x with leading coefficient equal to $4/(3\pi^2)$.

The proof of (1.5) consists of three steps: the first is to show that (2.2) can be simplified to give (1.7). Then we show that $|\zeta(\frac{1}{2}+it)|^4$ can be replaced by $I(t,\Delta)$ (with suitable Δ) and permissible error. The last step is to use the spectral decomposition (2.10) and obtain (1.5). The bound in (1.5) is actually the limit of the method, set by the condition $V^{1/2}\log^{-A}V \le \Delta \le V^{3/4}$ in (2.10). Namely we wish Δ to be as small as possible, so any further improvements of (1.5) will necessitate the widening of this range, or obtaining another type of the asymptotic formula for the integral in (2.10).

We proceed now with the proof. By using (2.6) and the mean theorem for Dirichlet polynomials (e.g., see [2, Chapter 5]), it is seen that the contribution of $\sum_{n\le c_0 t}$ in (2.5) to (2.2) is $\ll T^{3/2}\log^{10}T$ (the exponent of the logarithm is not optimal, but it is unimportant). Then we simplify the first sum in (2.5) by Taylor's formula (truncating it at $n = \sqrt{T}$) to obtain (1.7), as claimed.

Before we go to the second step, let $(1 \ll N \ll \sqrt{T})$

$$g_N(t) := \tfrac{1}{2}\left(\frac{2}{\pi}\right)^{3/4}\sum_{n\le N}(-1)^n d(n)n^{-5/4}\sin(\sqrt{8\pi nt}-\tfrac{1}{4}\pi). \tag{2.14}$$

By using trivial estimation and (2.8) we have

$$\begin{aligned}\int_T^{2T} &\sum_{n\ge N}(-1)^n d(n)n^{-5/4}\sin(\sqrt{8\pi nt}-\tfrac14\pi)|\zeta(\tfrac12+it)|^4\mathrm{d}t\\ &\ll TN^{-1/4}\log N\log^4 T \ll T(\log T)^{7/2}\log\log T \qquad (2.15)\end{aligned}$$

by choosing

$$N \;=\; (\log T)^2. \tag{2.16}$$

Hence, in view of (1.6) and (2.7), it remains to prove that

$$\int_T^{2T} g_N(t)|\zeta(\tfrac12+it)|^4\mathrm{d}t \;\ll\; T(\log T)^{7/2}\log\log T. \tag{2.17}$$

Now we have, with $I(T,\Delta)$ given by (2.11),

$$\begin{aligned}H(T,\Delta) &:= \int_T^{2T} g_N(t)(|\zeta(\tfrac12+it)|^4 - I(t,\Delta))\mathrm{d}t\\
&= \frac{1}{\Delta\sqrt{\pi}}\int_{-\infty}^{\infty}\mathrm{e}^{-(u/\Delta)^2}\int_T^{2T} g_N(t)(|\zeta(\tfrac12+it)|^4-|\zeta(\tfrac12+it+iu)|^4)\mathrm{d}t\,\mathrm{d}u\\
&= \frac{1}{\Delta\sqrt{\pi}}\int_{-\infty}^{\infty}\mathrm{e}^{-(u/\Delta)^2}\int_T^{2T} g_N(t)\left(\int_u^0 \frac{\partial}{\partial x}|\zeta(\tfrac12+it+ix)|^4\mathrm{d}x\right)\mathrm{d}t\,\mathrm{d}u\\
&= \frac{1}{\Delta\sqrt{\pi}}\int_{-\infty}^{\infty}\mathrm{e}^{-(u/\Delta)^2}\int_u^0\left(\int_T^{2T} g_N(t)\frac{\partial}{\partial t}|\zeta(\tfrac12+it+ix)|^4\mathrm{d}t\right)\mathrm{d}x\,\mathrm{d}u\\
&= \frac{1}{\Delta\sqrt{\pi}}\int_{-\infty}^{\infty}\mathrm{e}^{-(u/\Delta)^2}\int_{-|u|}^{|u|} O(|\zeta(\tfrac12+iT+ix)|^4)\mathrm{d}x\,\mathrm{d}u\\
&+\frac{1}{\Delta\sqrt{\pi}}\int_{-\infty}^{\infty}\mathrm{e}^{-(u/\Delta)^2}\int_{-|u|}^{|u|} O(|\zeta(\tfrac12+2iT+ix)|^4)\mathrm{d}x\,\mathrm{d}u\\
&-\frac{1}{\Delta\sqrt{\pi}}\int_{-\infty}^{\infty}\mathrm{e}^{-(u/\Delta)^2}\int_u^0\int_T^{2T} g_N'(t)|\zeta(\tfrac12+it+ix)|^4\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}u.\end{aligned}$$

The integrals over u can be truncated at $|u| = \Delta\log T$ with a negligible error. Since we have (see [4] and [12])

$$\int_{T-G}^{T+G}|\zeta(\tfrac12+it)|^4\mathrm{d}t \ll G\log^4 T + TG^{-1/2}\log^C T \qquad (C>0,\, T^\varepsilon\le G\le T),$$

and

$$g_N'(t) \;\ll\; T^{-1/2}N^{1/4}\log N \qquad (T\le t\le 2T), \tag{2.18}$$

it follows that

$$
\begin{aligned}
H(T,\Delta) &\ll \frac{1}{\Delta}\int\limits_{-\Delta\log T}^{\Delta\log T} \mathrm{e}^{-(u/\Delta)^2}\mathrm{d}u\Big(\int\limits_{T-\Delta\log T}^{T+\Delta\log T}|\zeta(\tfrac{1}{2}+ix)|^4\mathrm{d}x \\
&+\int\limits_{2T-\Delta\log T}^{2T+\Delta\log T}|\zeta(\tfrac{1}{2}+ix)|^4\mathrm{d}x\Big) \qquad (2.19)\\
&+\Delta^{-1}T^{-1/2}N^{1/4}\log N\int_{-\Delta\log T}^{\Delta\log T}\mathrm{e}^{-(u/\Delta)^2}|u|\mathrm{d}u\,T\log^4 T \\
&\ll \Delta\log^5 T + T\Delta^{-1/2}\log^C T + T^{1/2}\Delta N^{1/4}\log N\log^4 T \\
&\ll T\log^{6-A}T \ll T(\log T)^{7/2}\log\log T
\end{aligned}
$$

in view of (2.16), where we choose with $A > 0$ sufficiently large (this is the lower bound in the permissible range for which (2.10) holds)

$$\Delta \;=\; T^{1/2}\log^{-A}T.$$

We are now at the final step of the proof of Theorem 1. From (2.10) and (2.11) we obtain, on integrating by parts (again $Q_4(x) = P_4(x) + P_4'(x)$),

$$
\begin{aligned}
&\int_T^{2T} g_N(t)I(t,\Delta)\mathrm{d}t = \int_T^{2T} g_N(t)Q_4(\log t)\mathrm{d}t \\
&+O\left(\Delta\log^5 T\int_T^{2T}|g_N'(t)|\mathrm{d}t\right) + O(\Delta\log^5 T) + O(T\Delta^{-1/2}\log^4 T) \\
&-\pi\int_T^{2T} g_N'(t)\left(\sqrt{\tfrac{1}{2}t}\sum_{\kappa_j\le T\Delta^{-1}\log T}\alpha_j H_j^3(\tfrac{1}{2})c_j\cos(\kappa_j\log\frac{\kappa_j}{4\mathrm{e}t})\mathrm{e}^{-(\kappa_j\Delta/(2t))^2}\right)\mathrm{d}t,
\end{aligned}
$$

where we used (2.12). We have

$$\int_T^{2T} g_N(t)Q_4(\log t)\mathrm{d}t \ll T^{1/2}\log^4 T$$

by the first derivative test, and by using (2.18) we obtain

$$
\begin{aligned}
&\int_T^{2T} g_N(t)I(t,\Delta)\mathrm{d}t \qquad (2.20)\\
&\ll T^{1/2}\log^4 T + T^{1/2}\Delta N^{1/4}\log N\log^5 T + T\Delta^{-1/2}\log^4 T + |K(T,\Delta)|,
\end{aligned}
$$

where we have set

$$\begin{aligned} K(T,\Delta) \;&= \sum_{n\le N}(-1)^n d(n)n^{-3/4}\sum_{\kappa_j\le T\Delta^{-1}\log T}\alpha_j H_j^3(\tfrac12)c_j\,\times\\ &\times\int_T^{2T}\cos(\sqrt{8\pi nt}-\tfrac14)\cos(\kappa_j\log\frac{\kappa_j}{4\mathrm{e}t})\mathrm{e}^{-(\kappa_j\Delta/(2t))^2}\mathrm{d}t. \end{aligned}$$

We write the cosines as exponentials and note that the saddle point of the ensuing integral is at $t_0 = \kappa_j^2/(2\pi n) \in [T, 2T]$ for $\kappa_j \asymp \sqrt{Tn}$. By the saddle point method (see e.g., [2]) the main contribution will be a multiple of ($F(t) = \sqrt{8\pi nt} - \kappa_j \log t$)

$$\begin{aligned} &\sum_{n\le N}(-1)^n d(n)n^{-3/4}\sum_{\kappa_j\asymp\sqrt{Tn}}\alpha_j H_j^3(\tfrac12)c_j\mathrm{e}^{iF(t_0)}(F''(t_0))^{-1/2}\mathrm{e}^{-(\kappa_j\Delta/(2t_0))^2}\\ &\ll \sum_{n\le N} d(n)n^{-7/4}\sum_{\kappa_j\asymp\sqrt{Tn}}\alpha_j H_j^3(\tfrac12)\\ &\ll TN^{1/4}\log^3 T\log N \ll T(\log T)^{7/2}\log\log T, \end{aligned}\tag{2.21}$$

where we used again (2.12). Thus from (2.20) and (2.21) we obtain the bound in (2.17), as asserted. This completes the proof of Theorem 1. In concluding, note that the inner sum in (2.21) was estimated trivially. However, one hopes that there is a lot of cancellation in such type of exponential sum with $\alpha_j H_j^3(\frac12)$. Indeed, it was conjectured by the author in [8] that such a cancellation occurs, and it was heuristically justified why one does expect this fact. Also there is hope to use the explicit expression which stands for the function $R(K)$ in the proof of the asympotic formula (2.13). The small improvement of the bound in (1.5) of Theorem 1 over the bound in (2.9), which is relatively not difficult to obtain, is precisely significant for this reason: it does show that cancellation in a sum with $\alpha_j H_j^3(\frac12)$ does occur.

3. *Proof of Theorem 2 and Theorem 3*

We shall first deal with Theorem 3, which is needed for the proof of Theorem 2. The proof is based on the method used by the author in [3]. Suppose $1 \ll H \ll T$. We note that, by the first derivative test,

$$\int_T^{T+H} g(t)\mathrm{d}t \;\ll\; \sqrt{T} \tag{3.1}$$

holds uniformly in H, and proceed as follows. Let

$$E = \frac{1}{8}\left(\frac{2}{\pi}\right)^{3/2}\frac{\zeta^4(\frac{5}{2})}{\zeta(5)}.$$

Then we have (EH comes from the terms $m = n$), by the first derivative test,

$$\begin{aligned}
&\int_T^{T+H} g^2(t)\mathrm{d}t = EH + O(\sqrt{T})\\
&+O\Big(\sum_{m,n=1;m\ne n}^{\infty} d(m)d(n)(mn)^{-5/4}\left|\int_T^{T+H}\exp(i\sqrt{8\pi mt}\pm i\sqrt{8\pi nt})\mathrm{d}t\right|\Big)\\
&= EH + O\Big(\sqrt{T}\sum_{m,n=1;m\ne n}^{\infty} d(m)d(n)(mn)^{-5/4}|\sqrt{m}-\sqrt{n}|^{-1}\Big) + O(\sqrt{T})\\
&= EH + O\Big(\sqrt{T}\sum_{r=1}^{\infty}\sum_{n=1}^{\infty}(n(n+r))^{\varepsilon-5/4}r^{-1}(n+r)^{1/2}\Big) + O(\sqrt{T})\\
&= EH + O(\sqrt{T}).
\end{aligned}\tag{3.2}$$

Thus (3.2) implies that

$$\int_T^{T+H} g^2(t)\mathrm{d}t \;\gg\; H, \tag{3.3}$$

provided that $H = D\sqrt{T}$ with a sufficiently large constant $D > 0$. We shall show now that there exists $\tau \in [T, T+H]$ such that $g(\tau) > 2\eta$ for some constant $\eta > 0$ (and also a point τ_1 such that $g(\tau_1) < -2\eta$). Suppose on the contrary that $g(t) < \varepsilon$ if $g(t) > 0$ for any given $\varepsilon > 0$ (the case when $g(t) > -\varepsilon$ if $g(t) < 0$ is treated analogously). Let (cf. (1.8))

$$g_+(t) = \max(g(t),\,0), \qquad g_-(t) = \min(g(t),\,0). \tag{3.4}$$

Since $g(t)$ is bounded we obtain, for some constants $C_1, C_2 > 0$, on using (3.1),

$$\begin{aligned}
\textstyle\int_T^{T+H} g^2(t)\mathrm{d}t \;&= \int_T^{T+H} g_+^2(t)\mathrm{d}t + \int_T^{T+H}(-g_-(t))^2\mathrm{d}t\\
&\le H\varepsilon^2 + C_1\int_T^{T+H}(-g_-(t))\mathrm{d}t\\
&= H\varepsilon^2 - C_1\int_T^{T+H} g(t)\mathrm{d}t + C_1\int_T^{T+H} g_+(t)\mathrm{d}t\\
&\le H\varepsilon^2 + C_2\sqrt{T} + C_1\varepsilon H.
\end{aligned}$$

However the above bound contradicts (3.3) if ε is small enough and D is large enough, since $H = D\sqrt{T}$. Hence there exists $\tau \in [T, T+H]$ such that $g(\tau) > 2\eta$ for some constant $\eta > 0$. Setting for brevity $c = \frac{1}{2}(2/\pi)^{3/4}$, using $|\sin x - \sin y| \le |x-y|$ $(x, y \in \mathbb{R})$, we have $(N \ge 3, x \gg 1)$

$$
\begin{aligned}
&|g(\tau+x) - g(\tau)| \\
&= \Big|c\sum_{n\le N}(-1)^n d(n) n^{-5/4}\left(\sin(\sqrt{8\pi n(\tau+x)} - \pi/4) - \sin(\sqrt{8\pi n\tau} - \pi/4)\right)\Big| \\
&+O\left(\sum_{n>N} d(n)n^{-5/4}\right) \\
&\ll \sum_{n\le N} d(n)n^{-5/4}n^{1/2}T^{-1/2}|x| + \sum_{n>N} d(n)n^{-5/4} \\
&\ll T^{-1/2}|x|N^{1/4}\log N + N^{-1/4}\log N < \eta
\end{aligned}
$$

if $|x| \le L(T) := C\eta T^{1/2}N^{-1/4}\log N$ and $N = [\eta^{-8}]$, provided that η is sufficiently small (which may be assumed), and $C > 0$ is a suitable absolute constant. One obtains then

$$
|g(\tau+x)| \ge g(\tau) - |g(\tau+x) - g(\tau)| > \eta, \quad |x| \le L(T).
$$

This means that every interval $[T_1, T_1 + D\sqrt{T}]$ $(T \le T_1 \le 2T, D$ sufficiently large) contains a subinterval of length $\gg_\eta \sqrt{T}$ in which $g(t) > \eta$. Consequently we divide $[T, 2T]$ into $\gg \sqrt{T}$ subintervals of the form $[T + (j-1)D\sqrt{T}, T + jD\sqrt{T}]$, $(j = 1, 2, \ldots)$, and we obtain that $g(t) > \eta$ on a set $\mathcal{A}(T)$ satisfying $\mu(\mathcal{A}(T)) \gg_\eta T$, as claimed.

We pass now to the proof of Theorem 2. The upper bounds in (1.9) and (1.10) follow easily from the fact that $g(t)$ is bounded and that one has

$$
\int_T^{2T} |\zeta(\tfrac{1}{2}+it)|^{2k}\mathrm{d}t \ \ll\ T(\log T)^{k^2} \qquad (k = 1, 2).
$$

For the lower bounds in (1.9) and (1.10) (they actually hold for any fixed $k \in \mathbb{N}$) we use the well-known bound (see e.g., [2] or [4])

$$
\int_T^{T+C\sqrt{T}} |\zeta(\tfrac{1}{2}+it)|^{2k}\mathrm{d}t \ \gg\ \sqrt{T}(\log T)^{k^2} \qquad (C > 0,\ k \in \mathbb{N}). \tag{3.5}
$$

In each interval of length $\ge C\sqrt{T}$ in the proof of Theorem 2 (where we had $g(t) > \eta$) we use (3.5) and gather the resulting lower bounds to obtain the lower bound in (1.9). The proof of the lower bound in (1.10) is analogous.

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