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SOLUTIONS TO A PARTIAL DIFFERENTIAL EQUATION APPEARD IN MECHANICS

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A b s t r a c t. *The solutions, classical and generalized have been constructed and analyzed for a partial differential equation which appears as a mathematical model of many different phenomena.*

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1. *Introduction*

Equation

$$\frac{\partial^4}{\partial \xi^4}u(t,\xi) + \lambda \frac{\partial^2}{\partial \xi^2}u(t,\xi) + \frac{\partial^2}{\partial t^2}u(t,\xi) = 0, t > 0, 0 < \xi < 1, \tag{1.1}$$

appears in mathematical models for many different phenomena subject to different boundary or initial conditions (cf. for example [1],[3],[4],[5],[6],[8]). It is well-known that a solution to (1.1) is $u(t,\xi) = Y(\xi)T(t)$, where Y and T have the analytical from:

$$Y(\xi) = C_1 \cos hr_1\xi + C_2 \sin hr_1\xi + C_3 \cos r_2\xi + C_4 \sin r_2\xi \tag{1.2}$$

$$T(t) = C_5 \cos \Omega t + C_6 \sin \Omega t, \qquad \Omega^2 \in \mathbb{R}_+, \tag{1.3}$$

where

$$r_1 = \sqrt{\frac{\sqrt{\lambda^2 + 4\Omega^2} - \lambda}{2}};\ r_2 = \sqrt{\frac{\sqrt{\lambda^2 + 4\Omega^2} + \lambda}{2}} \tag{1.4}$$

(cf.[1],[2]). For Ω any compex number cf. [10]. Our aim is to analyse solutions, classical and generalized to equation (1.1).

2. *The corresponding equation to (1.1) in $\mathcal{D}'(\mathbb{R}^2)$ and its solutions*

2.1 *coresponding equation to (1.1) in $\mathcal{D}'(\mathbb{R}^2)$*

Suppose that there exists $u(t, \xi) \in \mathcal{C}_t^{(2)}(\mathbb{R}_+, \mathbb{R})$ such that:
1. $u(t, \xi)$ is a solution to (1.1),
2. there exist

$$\lim_{t \to 0^+} u(t, \xi) = u_1(\xi) \in \mathcal{C}(\mathbb{R}) \tag{2.1}$$

$$\lim_{t \to 0^+} u_t^{(1)}(t, \xi) = u_2(\xi) \in \mathcal{C}(\mathbb{R}). \tag{2.2}$$

Let $[Hu]$ denote the regular distribution defined by the function $H(t)u(t, \xi)$, where H is the Heaviside function ($H(t) = 0, t < 0; H(t) = 1, t \geq 0$).

We show by a simple manner the relation between the second partial derivative in the sense of distributions, $D_t^2[Hu]$, and the regular distribution, $\left[\frac{\partial^2}{\partial t^2} u(t, \xi)\right]$:

$$D_t^2[Hu] = \left[u_t^{(2)}(t, \xi)_0\right] - [u_2(\xi)] \otimes \delta(t) - [u_1(\xi)] \otimes \delta^{(1)}(t), \tag{2.3}$$

where $u_t^{(2)}(t, \xi)_0 = \dfrac{\partial^2}{\partial t^2} u(t, \xi), (t, \xi) \in \mathbb{R}_+ \times \mathbb{R}$; $u_t^{(2)}(t, \xi)_0 = 0, (t, \xi) \in \mathbb{R}_- \times \mathbb{R}$ and $u_t^{(2)}(t, \xi)_0$ is not defined for $(t, \xi) \in \{0\} \times \mathbb{R}$.

This is only a special case of a general theorem which gives the relation between partial derivatives in the sense of distributions and the classical ones.

Proof of (2.3). By definition of the derivative in $\mathcal{D}'(\mathbb{R}^2)$, for $\varphi \in \mathcal{D}(\mathbb{R}^2)$

$$\begin{aligned} &< D_t[Hu], \varphi(t, \xi) > = < [Hu], (-1)\varphi_t^{(1)}(t, \xi) > \\ &= - \int_{\mathbb{R}} \int_{\bar{\mathbb{R}}_+} H(t) u(t, \xi) \varphi_t^{(1)}(t, \xi) d\xi dt \\ &= \lim_{\epsilon \to 0^+} \int_{-\infty}^{\infty} d\xi \int_{\epsilon}^{\infty} u(t, \xi) \varphi_t^{(1)}(t, \xi) dt \end{aligned}$$

$$= \lim_{\epsilon\to 0^+} \int_{-\infty}^{\infty} d\xi \int_{\epsilon}^{\infty} u_t^{(1)}(t,\xi)\varphi(t,\xi)dt + \int_{-\infty}^{\infty} u(0,\xi)\varphi(0,\xi)d\xi$$

$$=< [u_t^{(1)}(t,\xi)_0, \varphi(t,\xi) > + < [u(0,\xi)] \otimes \delta(t), \varphi(t,\xi) > .$$

It follows that

$$D_t[Hu] = [u_t^{(1)}(t,\xi)_0] + [u_1(\xi)] \otimes \delta(t), \tag{2.4}$$

where

$$u_t^{(1)}(t,\xi)_0 = \frac{\partial}{\partial t} u(t,\xi), t > 0;\;\; u_t^{(1)}(t,\xi) = 0,\; t < 0 \text{ and } \;\; u_t^{(1)}(t,\xi)_0$$

is not defined in $t = 0,\ \xi \in \mathbb{R}$. If we repeat the mode of proceeding to (2.4), then it follows (2.3). Now, to (1.1) it correspouds in $\mathcal{D}'(\mathbb{R}^2)$

$$D_\xi^4 + \lambda D_\xi^2 + D_t^2)\tilde{u} = [u_1(\xi)] \otimes \delta^{(1)}(t) + [u_2(\xi)] \otimes \delta(t),$$

or

$$(D_t^2 + P(D_\xi))\tilde{u} = f, \tag{2.5}$$

where

$$P(D_\xi) = D_\xi^4 + \lambda D_\xi^2,\; f = [u_1(\xi)] \otimes \delta^{(1)}(t) + [u_2(\xi)] \otimes \delta(t) \;\text{ and }\; \tilde{u} \in \mathcal{D}'(\mathbb{R}^2).$$

We seek for solutions to (2.5) with the property supp $\tilde{u} \subset \overline{\mathbb{R}}_+ \times \mathbb{R}$.

2.2. *Solutions to (2.5)*

By the lemma in [7, p. 30] the operator $D_t^2 + P(D_\xi)$ is quasihyperbolic with respect to t if and only if the following condition is satisfied:

$$\exists c > 0, d \in \mathbb{R}, \forall \xi \in \mathbb{R} : ReP(i\xi) - c(ImP(i\xi))^2 \geq d.$$

In our case $P(i\xi) = \xi^4 - \lambda\xi^2$. For every $\xi \in \mathbb{R}, \xi^4 - \lambda\xi^2 \geq -\frac{\lambda^2}{4}$. Consequently the operator $D_t^2 + P(D_\xi)$ is quasihyperbolic.

By Proposition 5 in [7., p. 32] the unique fundamental solution E of $D_t^2 + P(D_\xi)$ with support in $\overline{\mathbb{R}}_+ \times \mathbb{R}$ and $E \in e^{\alpha t}\mathcal{S}'$ for an $\alpha \in \mathbb{R}$ is given by

$$E(t,\xi) = H(t)\mathcal{F}_x^{-1}\left(\frac{\sin(t\sqrt{P(2\pi i x)})}{\sqrt{P(2\pi i x)}}\right)(t,\xi), \tag{2.6}$$

where $\mathcal{F}^{-1}$ is the inverse Fourier transform.

Using Bochner's formula (cf. [9,(VII,7,22)] or [7,p 19])

$$E(t,|\xi|) = H(t)2\pi|\xi|^{1/2}\int_0^\infty \frac{\sin(t\sqrt{P(2\pi ix)})}{\sqrt{P(2\pi ix)}}x^{1/2}J_{-1/2}(2\pi|\xi|x)dx, \tag{2.7}$$

where J_v is the Bessel function.

Since

$$J_{-1/2}(2\pi|\xi|x) = \frac{1}{\pi}\frac{\cos 2\pi|\xi|x}{\sqrt{|\xi|x}},$$

we have

$$E(t,\xi) = 2H(t)\int_0^\infty \frac{\sin(t\sqrt{P(2\pi ix)})}{\sqrt{P(2\pi ix)}}\frac{\cos(2\pi|\xi|x)}{\sqrt{x}}dx. \tag{2.8}$$

Suppose now that $u_1(\xi)$ and $u_2(\xi)$ in (2.5) have the properties that:

$$([u_2(\xi)]\otimes\delta(t)) * [E(t,\xi)], ([u_1(\xi)]\otimes\delta^{(1)}(t)) * [E(t,\xi)] \tag{2.9}$$

exist, then there is a solution $\tilde{u}$ to (2.5) in $\mathcal{D}'(\mathbb{R}^2)$ with support in $\bar{\mathbb{R}}_+\times\mathbb{R}$

$$\begin{aligned}\tilde{u} &= (([u_1(\xi)]\otimes\delta^{(1)}(t))) + ([u_2(\xi)]\otimes\delta(t))) * [E(t,\xi)]\\ &= [u_2(\xi)] * [E(t,\xi)] + [u_1(\xi)] * D_t[E(t,\xi)].\end{aligned}$$

This solution is unique in the vector space $\mathcal{A}\subset\mathcal{D}'(\mathbb{R}^2)$. $\mathcal{A}$ consists of all $q\in\mathcal{D}'(\mathbb{R}^2)$ for which there exists $E*q$ (cf. [12 chapter III, §11.3]). We proved the following

Theorem 1. *Let E be qiven by (2.8) and let $\mathcal{A}$ be the vector space belonging to $\mathcal{D}'(\mathbb{R}^2)$ such that for every $g\in\mathcal{A}$ there is $[E]*g$.*

Suppose that $u_1(\xi)$ and $u_2(\xi)$ are in $\mathcal{C}(\mathbb{R})$ such that the convolutions (2.9) exist. Then

$$\tilde{u} = [u_2(\xi)] * [E(t,\xi)] + [u_1(\xi)] * D_t[E(t,\xi)] \tag{2.10}$$

is a solution to

$$(D_\xi^4 + \lambda D_\xi^2 + D_t^2)\tilde{u} = 0 \quad \textit{in } \mathcal{D}'(\mathbb{R}_+\times\mathbb{R}).$$

But it is also the unique solution in the space $\mathcal{A}\subset\mathcal{D}'(\mathbb{R}^2)$ satisfying the initial condition in t in the sense that

$$D_\xi^4 + \lambda D_\xi^2 + D_t^2)\tilde{u} = [u_2(\xi)]\otimes\delta(t) + [u_1(\xi)]\otimes\delta^{(1)}(t).$$

Remarks: 1. If $u_1(\xi)$ and $u_2(\xi)$ also belong to $\mathcal{C}^4(\mathbb{R})$, then by the property of convolution

$$D_\xi^i \tilde{u} = [u_2^{(i)}(\xi)] * [E(t,\xi)] + [u_1^{(i)}(\xi)] * D_t[E(t,\xi)],\ i = 1....,4.$$

2. If we have two solutions $u_1(t,\xi)$ and $u_2(t,\xi)$ to (1.1) with some initial condition $u_1(0,\xi) = u_2(0,\xi)$ and $\frac{d}{dt}u_1(t,\xi)|_{t=0} = \frac{d}{dt}u_2(t,\xi)|_{t=0}, \xi \in \mathbb{R}$, then

$$[u_2(t,\xi)] = [u_1(t,\xi)] + h, \tag{2.11}$$

where $h = 0$ or $h \notin \mathcal{A}$.

Let us prove it. The function $U(t,\xi) = u_2(t,\xi) - u_1(t,\xi)$ satisfies (1.1) with unitial condition $U_t^{(i)}(t,\xi)|_{t=0} = 0, i = 0,1, \xi \in \mathbb{R}$, consequently the regular distribution $[U(t,\xi)] \in \mathcal{D}'(\mathbb{R}^2)$ satisfies (2.5) with $f = 0$. Then $[U(t,\xi)] = h$, where $h = 0$ or $h \notin \mathcal{A}$. Hence $[U(t,\xi)] = [u_2(t,\xi)] - [u_1(t,\xi)] = h$.

3. The well-known solution to (1.1) $u(t,\xi) = Y(\xi)T(t)$, where Y and T have been given by (1.3) and (1.4), has not the convolution with $E(t,\xi)$ in the sense of distributions, i.e., $[u(t,\xi)] * [E(t,\xi)]$ does not exist. If were true that $[u(t,\xi)] * [E(t,\xi)]$ exists, then by 1. and the property of convolution:

$$[u(t,\xi)] = [u(t,\xi)] * \delta(t,\xi) = [u(t,\xi)] * (D_t^2 + P(D_\xi))[E(t,\xi)]$$

$$= ((D_t^2 + P(D_\xi))[u(t,\xi)]) * [E(t,\xi)]$$

$$= \left[\Big(\frac{\partial^2}{\partial t^2} + \frac{\partial^4}{\partial \xi^4} + \frac{\partial^2}{\partial \xi^2}\Big)u(t,\xi)\right] * \Big[E(t,\xi)\Big] = 0.$$

Thus $u(t,\xi) = 0,\ t > 0,\ \xi \in \mathbb{R}$.

4. If equation (2.5) with $f = 0$ has a solution belonging to $\mathcal{D}'(\mathbb{R}^2)$, it does not belong to $\mathcal{A}$.

P r o o f. A solution to (1.1) in $\mathcal{D}'(\mathbb{R}^2)$ is $u(t,\xi) \equiv 0,\ (t,\xi) \in \mathbb{R}^2$. By 2. if there is a solution to (1.1) belonging to $\mathcal{D}'(\mathbb{R}^2)$ which is not identical zero, then it does not belong to $\mathcal{A}$.

The solution $u(t,\xi) = Y(\xi)T(t)$, where Y and T have been given by (1.2) and (1.3), respectively, is in fact a solution to

$$\begin{aligned}(Y^{(4)}(\xi) \quad &+\lambda Y^{(2)}(\xi) + \omega^2 Y(\xi))T(t) \\ &+(T^{(2)}(t) - \omega^2 T(t))Y(\xi) = 0,\ \ t > 0,\ \xi \in \mathbb{R},\end{aligned} \tag{2.12}$$

for $\omega^2 \in \mathbb{R} \setminus \{0\}$. This equation can be written in the form

$$\left(P\Big(\frac{d}{d\xi}\Big) + \frac{d^2}{dt^2} - \omega^2\right) Y(\xi)T(t) = 0, \tag{2.13}$$

where

$$P\Big(\frac{d}{d\xi}\Big) = \frac{d^4}{d\xi^4} + \lambda\,\frac{d^2}{d\xi^2} + \omega^2.$$

In the sequel we suppose that $\omega^2 > 0$. Since

$$P(i\xi) = \xi^4 - \lambda\xi^2 + \omega^2 > 0, \;\; \xi \in \mathbb{R}, \quad \omega^2 - \frac{\lambda^2}{4} > 0,$$

by Proposition 6 in [7] there is the unique fundamental solution $E_\omega(t,\xi)$ of $P\Big(\dfrac{d}{d\xi}\Big) + \dfrac{d^2}{dt^2} - \omega^2$ with support in $\overline{\mathbb{R}}_+ \times \mathbb{R}$ and belonging to $e^{\alpha t}\mathcal{S}'$ for an $\alpha \in \mathbb{R}$. It has the following representation

$$E_\omega(t,\xi) = E(t,\xi) - \omega H(t) \int\limits_0^t \frac{\tau}{\sqrt{t^2-\tau^2}} J_1(\omega\sqrt{t^2-\tau^2}) E(\tau,\xi)\, d\tau, \tag{2.14}$$

where $E(t,\xi)$ is given by (2.8).

Theorem 2. *If in Theorem 1 instead of $E(t,\xi)$ we take $E_\omega(t,\xi)$, given by (2.14), then we obtain another form of solutions to*

$$\left(P\Big(\frac{d}{d\xi}\Big) + \frac{d^2}{dt^2} - \omega^2\right)[u(t,\xi)] = 0$$

with

$$P\Big(\frac{d}{d\xi}\Big) = \frac{d^4}{d\xi^4} + \lambda\,\frac{d^2}{d\xi^2} + \omega^2,$$

where $\omega^2 - \dfrac{\lambda^2}{4} > 0,\; \omega^2 > 0.$

2.3. *A convolutor to $E(t,\xi)$*

At the end of Part 2 we give a sufficient condition for a regular distribution to have convolution with $E(t,\xi)$, such that this convolution is also a regular distribution

Lemma 1. *If $f(\xi,t)$ has the property that*

$$|f(\xi,t)| \le H(t)\alpha(t)\beta(\xi),\; (\xi,t) \in \mathbb{R} \times \overline{\mathbb{R}}_+, \tag{2.15}$$

where $\alpha(t) \in \mathcal{L}_{loc}([0,\infty))$ *and* $\beta(\xi) \in \mathcal{L}^1(\mathbb{R})$, *then* $f(\xi,t)$ *defines a regular distribution* $[f(\xi,t)]$ *such that* $[f(\xi,t)] * [E(\xi,t)]$ *exists and is also a regular distribution defined by the function* $(f(\xi,t) * E(\xi,t))(\xi,t)$ *which is bounded in* $\xi \in \mathbb{R}$, *for every* $t \geq 0$.

P r o o f. It is enough to prove that there exists the convolution of two functions $f(\xi,t) * E(\xi,t)$ and that this convolution is a locally integrable function on $(\mathbb{R} \times \overline{\mathbb{R}}_+)$

$$\Big|f(\xi,t) * E(\xi,t)\Big| = \Big|\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi - x, t-\tau)E(x,\tau)dx\,d\tau\Big|$$

$$\leq \int_{-\infty}^{\infty} dx \int_0^t \alpha(t-\tau)\beta(\xi - x)|E(x,\tau)|d\tau \tag{2.16}$$

$$\leq H(t)\int_{-\infty}^{\infty} \beta(x)dx \int_0^t \alpha(t-\tau)B(\tau)d\tau,$$

where

$$B(\tau) = \sup_{\xi\in\mathbb{R}}|E(\xi,t)| = \sup_{\xi\in\mathbb{R}}\Big|\int_{-\infty}^{\infty} e^{2\pi i x\xi}\frac{\sin(2\pi\tau|x|\sqrt{4\pi^2x^2-\lambda})}{2\pi|x|\sqrt{4\pi^2x^2-\lambda}}dx\Big| \tag{2.17}$$

$$\leq \sup_{\xi\in\mathbb{R}}\Big|\int_{-\infty}^{\infty}\Big|\int_{-\infty}^{-\sqrt{\lambda}/2\pi} + \int_{\sqrt{\lambda}/2\pi}^{\infty} + \int_{-\sqrt{\lambda}/2\pi}^{\sqrt{\lambda}/2\pi} e^{2\pi i x\xi}\frac{\sin(2\pi\tau|x|\sqrt{4\pi^2x^2-\lambda})}{2\pi|x|\sqrt{4\pi^2x^2-\lambda}}dx\Big|.$$

In the first and the second integral we can use the inequality

$$\left|e^{2\pi i x\xi}\frac{\sin(2\pi\tau|x|\sqrt{4\pi^2x^2-\lambda})}{2\pi|x|\sqrt{4\pi^2x^2-\lambda}}\right| \leq \frac{1}{2\pi|x|\sqrt{4\pi^2x^2-\lambda}},\ \ t\geq 0,\ |x| \geq \frac{\sqrt{\lambda}}{2\pi}. \tag{2.18}$$

The third integral is:

$$\int_{-\frac{\sqrt{\lambda}}{2\pi}}^{\frac{\sqrt{\lambda}}{2\pi}} e^{2\pi i x\xi}\,\frac{1}{2\pi|x|\sqrt{\lambda-(2\pi x)^2}}\sin\, h(\tau 2\pi x\sqrt{\lambda-(2\pi x)^2})dx$$

$$= \int_{-\frac{\sqrt{\lambda}}{2\pi}}^{\frac{\sqrt{\lambda}}{2\pi}} e^{2\pi i x\xi} f(x,\tau)dx.$$

The function $f(x,\tau)$ is not defined for $x = 0$. But since there exists

$$\lim_{x\to 0} f(x,\tau) = \tau, \quad \tau \geq 0,$$

this function can be extended to $\Big(-\dfrac{\sqrt{\lambda}}{2\pi}, \dfrac{\sqrt{x}}{2\pi}\Big)$ as a continuous function.

Thus

$$\Bigg| \int_{-\frac{\sqrt{\lambda}}{2\pi}}^{\frac{\sqrt{\lambda}}{2\pi}} e^{2\pi i x\xi} f(x,\tau)dx \Bigg| \leq \int_{-\frac{\sqrt{\lambda}}{2\pi}}^{\frac{\sqrt{\lambda}}{2\pi}} |f(x,\tau)|dx.$$

Consequently, $B(\tau) \in \mathcal{L}_{loc}([0,\infty))$ and the Lemma is proved.

Remark. If $u_1(\xi) \equiv 0$ and $u_2(\xi) \in \mathcal{L}^1(\mathbb{R})$ then by Lemma we proved, it follows that the solution (2.10) is a regular distribution defined by the function $u_2(\xi) * E(t,\xi)$, with support in $\overline{\mathbb{R}}_+ \times \mathbb{R}$ and bounded in $\xi \in \mathbb{R}$, for every $t \geq 0$.

3. *Special case of equation (1.1)*

3.1. *Fourier's method*

In Part 2 the solutions to equation (2.5) have been limited by the space $\mathcal{A}$. Now we consider equation (1.1) in case $\lambda = 0$ without this limitation. A detailed discussion of the mentioned case by Fourier's method separation of variables one can find in [3]. Transverse vibrations of a homogeneous rod has been given by

$$\frac{\partial^4 u}{\partial x^4} + \frac{\partial^2 u}{\partial t^2} = 0. \tag{3.1}$$

Five various types of boundary conditions have been considered for a solution supposed in the form $u(x,t) = v(x)\ g(t)$

1. $v''(x) = v'''(x) = 0$, for $x = 0$ and $x = \pi$;
2. $v(x) = v''(x) = 0$, for $x = 0$ and $x = \pi$;
3. $v(x) = v'(x) = 0$, for $x = 0$ and $x = \pi$;
4. $v'(x) = v'''(x)$ for $x = 0$ and $x = \pi$;
5. $v^{(i)}(0) = v^{(i)}(\pi), \quad i = 0, 1, 2, 3.$

We would like to analyse the existence of other solutions (generalized or classical) to equation (3.1). Therefore we use a method explained in [11] and the Laplace Transform. We hope that not only our solution (3.28) but also the Comments at the end can be interesting for applications.

3.2 *Distributions and the Laplace transform*

We repeat some definitions and facts related to the space $\mathcal{S}'$ of tempered distributions and to the Laplace transform (in short LT) of them (cf. [12] and [13]).

Let Γ be a closed convex acute cone in $\mathbb{R}^n$, $\Gamma^* = \{t \in \mathbb{R}^n, tx \equiv t_1x_1 + ... + t_nx_n \geq 0, \ \forall x \in \Gamma\}$ and $C = int\Gamma^*$. Let K be a compact set in $\mathbb{R}^n$.

By $\mathcal{S}'(\Gamma + K)$ is denoted the space of tempered distributions defined on the close set $\Gamma + K \subset \mathbb{R}^n$. Then $\mathcal{S}'(\Gamma+)$ is defined by way of

$$\mathcal{S}'(\Gamma+) = \bigcup_{K \subset \mathbb{R}^n} \mathcal{S}'(\Gamma + K). \tag{3.2}$$

The set $\mathcal{S}'(\Gamma+)$ forms an algebra that is associative and commutative if for the operation of multiplication one takes the convolution, denoted by $*$.

If $\Gamma + K$ is convex, as it will be in our case, then the LT of $f \in \mathcal{S}'(\Gamma+)$ is defined by

$$\widehat{f}(z) = \mathcal{L}(f)(z) = \langle f(t), e^{-zt} \rangle, \ z \in C + i\mathbb{R}^n. \tag{3.3}$$

If $\sigma \geq 0, \ f \in \mathcal{S}'(\Gamma+)$, then

$$\mathcal{L}(e^{\sigma t} f)(z) = \langle f(t), e^{-(z-\sigma)t} \rangle, \ \ Rez > \sigma, \tag{3.4}$$

where $t = (t_1, ..., t_n), \ z = (z_1, ..., z_n)$ and $zt = z_1t_1 + ... + z_nt_n$. It is one to one operation.

For the properties of so defined LT one can consult [13]. We shall cite only some of them, we use in the sequel:

1) $\mathcal{L}\Big(\dfrac{\partial^m}{\partial t_i^m} f\Big)(z) = (z_i)^m \mathcal{L}(f)(z)$.

2) If $f \in \mathcal{S}'(\Gamma_1+)$ and $g \in \mathcal{S}'(\Gamma_2+)$, then $\mathcal{L}(f \times g)(z, s) = \mathcal{L}(f)(z)\mathcal{L}(g)(s)$, $z \in C_1 + i\mathbb{R}^n, \ s \in C_2 + i\mathbb{R}^n$.

3) If $f, g \in \mathcal{S}'(\Gamma+)$, then $f * g \in \mathcal{S}'(\Gamma+)$ and $\mathcal{L}(f * g)(z) = \mathcal{L}(f)(z)\mathcal{L}(g)(z), \ z \in C + i\mathbb{R}^n$.

4) $\mathcal{L}(\delta(t-t_0))(z) = e^{-zt_0}$.

5) $\mathcal{L}(f)(z+a) = \mathcal{L}(e^{-at}f)(z),\ \mathbb{R}ea > 0$.

6) If $f \in \mathcal{L}_{loc}(\mathbb{R}^n_+)$ and $|f(x)| \le Ce^{qx},\ x \ge x_0 > 0$, then $f(x)e^{-qx} \in \mathcal{S}'(\overline{\mathbb{R}}^n_+)$ and

$$\int\limits_{\mathbb{R}^n_+} e^{-(z+q)t} f(t)dt = \int\limits_{\mathbb{R}^n_+} e^{-zt}e^{-qt}f(t)dt = \mathcal{L}(e^{-qt}f)(z).$$

Let $\mathcal{H}_a^{(\alpha,\beta)}(C),\ \alpha \ge 0,\ \beta \ge 0,\ a \ge 0$, denote the sets of holomorphic functions on $C + i\mathbb{R}^n$ which satisfy the following growth condition

$$|f(z)| \le Me^{a|x|}(1+|z|^2)^{\alpha/2}(1+\Delta^{-\beta}(x,\partial C)),\ z = x+iy \in C + i\mathbb{R}^n, \tag{3.5}$$

where ∂C is the boundary of C and $\Delta(x,\partial C)$ is the distance between x and ∂C. We set

$$\mathcal{H}_a(C) = \bigcup_{\alpha\ge 0,\beta\ge 0} \mathcal{H}_a^{(\alpha,\beta)}(C) \ \text{ and } \mathcal{H}_+(C) = \bigcup_{a\ge 0} \mathcal{H}_a(C).$$

Proposition A. *([13] p.191). The algebras $\mathcal{H}_+(C)$ and $\mathcal{S}'(C^*+)$ and also their subalgebras $\mathcal{H}_0(C)$ and $\mathcal{S}'(C^*)$ are isomorphic. This isomorphism is accomplished via the LT.*

A property of the defined LT which can be used in a practical way is the following:

Let $f \in \mathcal{S}'(\overline{\mathbb{R}}^n_+ + \overline{P})$. The LT of f, $\mathcal{L}(f)$, can be obtained by one after the other applications of the LT-s $\mathcal{L}_1(f), ..., \mathcal{L}_n(f)$, $\quad \mathcal{L}(f) = \mathcal{L}_1(f) \circ ... \circ \mathcal{L}_n(f)$.

If $\sigma \ge 0$, $f \in \mathcal{S}'(C^*+)$ and $g = e^{\sigma t}f$ then by definition $\mathcal{L}(g)(s) = \langle f(t), e^{-(s-\sigma)t}\rangle$, $Res > \sigma$..

Let $F(s)$ be a function holomorphic for $Res > \sigma$. The function $F(\xi+\sigma)$ is holomorphic for $Re\xi > 0$. If $F(\xi+\sigma) \in \mathcal{H}(\mathbb{R}_+)$, then there exists $f \in \mathcal{S}'(\overline{\mathbb{R}}_+)$ such that $\mathcal{L}(e^{\sigma t}f)(s) = F(s),\ Re\, s > \sigma$.

We shall quote some auxiliary formulas for the classical Laplace Transform we need. Let H denotes the Heaviside function, $H(t) = 0,\ t < 0;\ H(t) = 1,\ t \ge 0$.

1. $\mathcal{L}_z^{-1}\Big(\dfrac{1}{z + a\sqrt{s}}\Big) = H(x)e^{-ax\sqrt{s}}$.

2. $\mathcal{L}_s^{-1}\Big(\dfrac{1}{\sqrt{s}}e^{-ax\sqrt{s}}\Big) = \dfrac{H(t)}{\sqrt{\pi t}}e^{-(ax)^2/(4t)},\ x > 0, ; Rea > 0$

$$= H(t)\chi(ax,t).$$

3. $\frac{1}{2i}\Big(\chi(e^{i\frac{\pi}{4}}x,t)-\chi(e^{-i\frac{\pi}{4}}x,t)\Big)=\frac{-1}{\sqrt{\pi t}}\frac{1}{2i}\Big(e^{i\frac{x^2}{4t}}-e^{-i\frac{x^2}{4t}}\Big)=-\frac{1}{\sqrt{\pi t}}\sin\frac{x^2}{4t}.$

4. $\frac{1}{ie^{i\frac{\pi}{4}}}\chi(e^{i\frac{\pi}{4}}x,t)=\frac{1}{\sqrt{\pi t}}\Big[-\frac{\sqrt{2}}{2}\cos\frac{x^2}{4t}-\frac{\sqrt{2}}{2}\sin\frac{x^2}{4t}-i\Big(-\frac{\sqrt{2}}{2}\cos\frac{x^2}{4t}+ +\frac{\sqrt{2}}{2}\sin\frac{x^2}{4t}\Big)\Big].$

5. $\frac{1}{ie^{-i\frac{\pi}{4}}}\chi(e^{-i\frac{\pi}{4}}x,t)=\frac{1}{\sqrt{\pi t}}\frac{\sqrt{2}}{2}\Big[\cos\frac{x^2}{4t}+\sin\frac{x^2}{4t}-i\Big(\cos\frac{x^2}{4t}-\sin\frac{x^2}{4t}\Big)\Big].$

3.3. *Solution to (3.1) in* $\mathcal{D}'(\overline{\mathbb{R}}^2_+)$

We consider the equation (3.1) with initial conditions

$$u(0,t)=\frac{\partial}{\partial x}(0,t)=0,\quad t\geq 0,$$

$$\frac{\partial^k}{\partial x^k}u(0,t)=A_k(t),\ \ k=2,3,\ \ t\geq 0, \tag{3.6}$$

$$u(x,0)=B_0(x),\quad \frac{\partial}{\partial t}u(x,0)=B_1(x),\ \ x\geq 0,$$

where $[H(t)A_k(t)]\in e^{\sigma t}\mathcal{S}'(\overline{\mathbb{R}}_+),\ k=2,3,\ p>0$ and $[H(x)B_i(x)]\in e^{\sigma x}\mathcal{S}'(\overline{\mathbb{R}}_+)$, $i=0,1,\ q>0,\ \sigma>0$.

To find an equation in $\mathcal{D}'(\overline{\mathbb{R}}^2_+)$ which corresponds to (3.1) for $x>0,\ t>0$ we need the following relations between derivatives in the sense of distributions and the classical ones.

Let $H^2(x_1,x_2)=H(x_1)H(x_2)$, where H is the Heaviside function. For a function f with continuous partial derivatives on $\mathbb{R}^2$, $[H^2f]$ is the distribution, defined by H^2f, belonging to $\mathcal{D}'(\mathbb{R}^2)$ and to $\mathcal{D}'(\overline{\mathbb{R}}^2_+)$, as well. Let $\Big(\frac{\partial^p}{\partial x_i^p}f\Big)_0$ denote the function equal to $\frac{\partial^p}{\partial x_i^p}f$ on the $\mathbb{R}^2_+$ and equal zero on $\mathbb{R}^2\setminus\mathbb{R}^2_+$, but is not defined for $(x_1,x_2)\in\{(0,x_2)\cup(x_1,0);\, x_1\geq 0,\, x_2\geq 0\}$.

With the notation as above we have (cf. [11])

$$D^p_{x_i}[H^2f]=\Big[H^2\Big(\frac{\partial^p}{\partial x_i^p}f\Big)_0\Big]+R_p(f),\ \ p\in\mathbb{N},$$

where

$$R_p(f)=\Big[H^2\frac{\partial^{p-1}}{\partial x_i^{p-1}}f(x)|_{x_i=0}\Big]\times\delta(x_i)+...+[H^2f(x)|_{x_i=0}]\times\delta^{(p-1)}(x_i).$$

To equation (3.1) with initial condition (3.6) it corresponds in $\mathcal{D}'(\overline{\mathbb{R}}^2_+)$

$$\frac{\partial^4}{\partial x^4}[u(x,t)] + \frac{\partial^2}{\partial t^2}[u(x,t)] = [H(t)A_2(t)] \times \delta^{(1)}(x) \tag{3.7}$$

$$+[H(t)A_3(t)] \times \delta(x) + [H(x)B_1(x)] \times \delta(t) + [H(x)B_0(x)] \times \delta^{(1)}(t).$$

Applying the LT we have

$$(z^4 + s^2)\mathcal{L}(u)(z,s) = \mathcal{L}(A_2)(s)z + \mathcal{L}(A_3)(s) + \mathcal{L}(B_1)(z) + \mathcal{L}(B_0)(z)s,$$

or

$$\mathcal{L}(u)(z,s) = \frac{Q(z,s)}{z^4 + s^2}, \tag{3.8}$$

with

$$Q(z,s) = \mathcal{L}(A_2)(s)z + \mathcal{L}(A_3)(s) + \mathcal{L}(B_1)(z) + \mathcal{L}(B_0)(z)s. \tag{3.9}$$

Since

$$\frac{1}{z^4 + s^2} = \frac{1}{2is}\Big(\frac{1}{z^2 - is} - \frac{1}{z^2 + is}\Big),$$

$$\frac{Q(z,s)}{z^4 + s^2} = \frac{Q(z,s)}{2is}\Big(\frac{1}{z^2 - is} - \frac{1}{z^2 + is}\Big). \tag{3.10}$$

By Proposition A in [11] $\dfrac{Q(z,s)}{z^4 + s^2}$ has to be holomorphic in $\{(z,s) \in \mathbb{C}^2;\ Rez > w_1 > 0,\ Res > w_2 > 0\}$. Since $z^4 + s^2 = (z - z_1)(z + z_1)(z - z_2)(z + z_2)$, where $z_1 = e^{i\frac{\pi}{4}}\sqrt{s},\ z_2 = e^{i\frac{3\pi}{4}}\sqrt{s}$, it is necessary to have

$$Q(e^{i\frac{\pi}{4}}\sqrt{s}, s) = 0 \quad \text{and} \quad Q(-e^{i\frac{3\pi}{4}}\sqrt{s}, s) = 0$$

or equivalently

$$Q(e^{i\frac{\pi}{4}}\sqrt{s}, s) = 0 \ \text{ and } \ Q(e^{-i\frac{\pi}{4}}\sqrt{s}, s) = 0. \tag{3.11}$$

First step

In the first step we consider the first addend in (3.10). Now we need (3.11) to be satisfied which gives:

$$\mathcal{L}(A_2)(s)e^{i\frac{\pi}{4}}\sqrt{s} + \mathcal{L}(A_3)(s) + \mathcal{L}(B_1)(e^{i\frac{\pi}{4}}\sqrt{s}) + s\mathcal{L}(B_0)(e^{i\frac{\pi}{4}}\sqrt{s}) = 0. \tag{3.12}$$

Now we can express $\mathcal{L}(A_3)(s)$,

$$\mathcal{L}(A_3)(s) = -\mathcal{L}(A_2)(s)e^{i\frac{\pi}{4}}\sqrt{s} - \mathcal{L}(B_1)(e^{i\frac{\pi}{4}}\sqrt{s}) - s\mathcal{L}(B_0)(e^{i\frac{\pi}{4}}\sqrt{s}).$$

With such expressed $\mathcal{L}(A_3)(s)$ the first addend in (3.10) is:

$$\frac{Q(z,s)}{2is(z^2-is)} = \frac{\mathcal{L}(A_2)(s)(z-e^{i\frac{\pi}{4}}\sqrt{s})}{2is(z^2-is)}+$$

$$+\frac{\mathcal{L}(B_1)(z)-\mathcal{L}(B_1)(e^{i\frac{\pi}{4}}\sqrt{s})+s(\mathcal{L}(B_0)(z)-\mathcal{L}(B_0(e^{i\frac{\pi}{4}}\sqrt{s}))}{2is(z^2-is)}$$

$$= \frac{\mathcal{L}(A_2)(s)}{2is(z+e^{i\frac{\pi}{4}}\sqrt{s})} + \left(\frac{\mathcal{L}(B_1)(z)-\mathcal{L}(B_1)(e^{i\frac{\pi}{4}}\sqrt{s})}{4ise^{i\frac{\pi}{4}}\sqrt{s}}+\right. \tag{3.13}$$

$$\left.+\frac{\mathcal{L}(B_0)(z)-\mathcal{L}(B_0)(e^{i\frac{\pi}{4}}\sqrt{s})}{4ise^{i\frac{\pi}{4}}\sqrt{s}}\right)\left(\frac{1}{z-e^{i\frac{\pi}{4}}\sqrt{s})} - \frac{1}{z+e^{i\frac{\pi}{4}}\sqrt{s})}\right).$$

By using the auxiliary formulas 1., 2. and 5. we quoted we find the LT of (3.13).

Let us consider the first addend in (3.13)

$$\mathcal{L}^{-1}\left(\frac{\mathcal{L}(\overline{A}_2)(s)}{2is(z+e^{i\frac{\pi}{4}}\sqrt{s})}\right) = \mathcal{L}_s^{-1}\circ\left(\mathcal{L}_z^{-1}\left(\frac{1}{z+e^{i\frac{\pi}{4}}\sqrt{s}}\right)\frac{\mathcal{L}(\overline{A}_2)(s)}{2is}\right)$$

$$= \frac{1}{2i}\mathcal{L}_s^{-1}\left(\frac{1}{\sqrt{s}}\,e^{-e^{i\frac{\pi}{4}}\sqrt{s}x}\right)\frac{1}{\sqrt{s}}\mathcal{L}(A_2)(s) \tag{3.14}$$

$$= \frac{H(x)H(t)}{2i\Gamma(1/2)}\,\chi(e^{i\frac{\pi}{4}}x,t) \overset{t}{*} \int_0^t (t-\tau)^{-1/2}A_2(\tau)d\tau.$$

The second addend in (3.13) is:

$$\frac{\mathcal{L}(B_1)(z)-\mathcal{L}(B_1)(e^{i\frac{\pi}{4}}\sqrt{s})}{4ise^{i\frac{\pi}{4}}\sqrt{s}}\left(\frac{1}{z-e^{i\frac{\pi}{4}}\sqrt{s}} - \frac{1}{z+e^{i\frac{\pi}{4}}\sqrt{s}}\right). \tag{3.15}$$

We shall start with

$$\mathcal{L}^{-1}\left(\frac{\mathcal{L}(B_1)(z)-\mathcal{L}(B_1)(e^{i\frac{\pi}{4}}\sqrt{s})}{4ise^{i\frac{\pi}{4}}\sqrt{s}(z+e^{i\frac{\pi}{4}}\sqrt{s})}\right)$$

$$= \mathcal{L}_z^{-1}\circ\mathcal{L}_s^{-1}\left(\frac{\mathcal{L}(B_1)(z)}{4ise^{i\frac{\pi}{4}}\sqrt{s}(z+e^{i\frac{\pi}{4}}\sqrt{s})}\right. \tag{3.16}$$

$$\left.-\mathcal{L}_s^{-1}\circ\mathcal{L}_z^{-1}\left(\frac{\mathcal{L}(B_1)(e^{i\frac{\pi}{4}}\sqrt{s})}{4ise^{i\frac{\pi}{4}}\sqrt{s}(z+e^{i\frac{\pi}{4}}\sqrt{s})}\right)..\right.$$

The first addend in (3.16) is

$$\begin{aligned}
&\mathcal{L}_z^{-1}\left(B_1(z)\mathcal{L}_s^{-1}\left(\frac{1}{4(e^{i\frac{\pi}{4}}\sqrt{s})^3(e^{i\frac{\pi}{4}}\sqrt{s}+z)}\right)\right)\\
&=\mathcal{L}_z^{-1}\left(B_1(z)\mathcal{L}_s^{-1}\frac{1}{4e^{i\frac{3\pi}{4}}s}\overset{t}{*}\mathcal{L}_s^{-1}\frac{1}{(z+e^{i\frac{\pi}{4}}\sqrt{s})\sqrt{s}}\right)\\
&=\frac{1}{4e^{i\frac{3\pi}{4}}}\int_0^t\chi(e^{i\frac{\pi}{4}}x,\tau)d\tau\overset{x}{*}B_1(x).
\end{aligned}\tag{3.17}$$

For the second addend in (3.16) we have

$$\begin{aligned}
&-\mathcal{L}_s^{-1}\circ\mathcal{L}_z^{-1}\left(\frac{\mathcal{L}(B_1)(e^{i\frac{\pi}{4}}\sqrt{s})}{4ise^{i\frac{\pi}{4}}\sqrt{s}(z+e^{i\frac{\pi}{4}}\sqrt{s})}\right)\\
&=-\mathcal{L}_s^{-1}\left(\mathcal{L}_s(B_1)(e^{i\frac{\pi}{4}}\sqrt{s})\cdot\frac{1}{4e^{i\frac{3\pi}{4}}s}\cdot\frac{1}{\sqrt{s}}\mathcal{L}_z^{-1}\left(\frac{1}{z+e^{i\frac{\pi}{4}}\sqrt{s}}\right)\right)\\
&=-\mathcal{L}_s^{-1}\left(\frac{1}{4e^{i\frac{3\pi}{4}}s}\frac{H(x)}{\sqrt{s}}\,e^{-e^{i\frac{\pi}{4}}x\sqrt{s}}\int_0^\infty e^{-e^{i\frac{\pi}{4}}\sqrt{s}\tau}B_1(\tau)d\tau\right)\\
&=d-\frac{1}{4e^{i\frac{3\pi}{4}}}\overset{t}{*}\mathcal{L}_s^{-1}\left(\frac{1}{\sqrt{s}}\int_0^\infty e^{-e^{i\frac{\pi}{4}}\sqrt{s}(x+\tau)}B_1(\tau)d\tau\right)\\
&=-\frac{1}{4e^{i\frac{3\pi}{4}}}\overset{t}{*}\int_0^\infty e^{-\frac{1}{4}i(x+\tau)^2/t}\frac{1}{\sqrt{\pi t}}B_1(\tau)d\tau\\
&=-\frac{1}{4ie^{i\frac{\pi}{4}}}\int_0^t du\int_0^\infty\chi(e^{i\frac{\pi}{4}}(x+\tau),u)B_1(\tau)d\tau.
\end{aligned}\tag{3.18}$$

The first addend in (3.15) gives

$$\begin{aligned}
&\mathcal{L}^{-1}\left(\frac{\mathcal{L}(B_1)(z)-\mathcal{L}(B_1)(e^{i\frac{\pi}{4}}\sqrt{s})}{4ise^{i\frac{\pi}{4}}s\sqrt{s}(z-e^{i\frac{\pi}{4}}\sqrt{s})}\right)\\
&=\mathcal{L}^{-1}\frac{\mathcal{L}(B_1)(z)}{4ie^{i\frac{\pi}{4}}s\sqrt{s}(z-e^{i\frac{\pi}{4}}\sqrt{s})}-\mathcal{L}^{-1}\frac{\mathcal{L}(B_1)(e^{i\frac{\pi}{4}}\sqrt{s})}{4ie^{i\frac{\pi}{4}}s\sqrt{s}(z-e^{i\frac{\pi}{4}}\sqrt{s})}
\end{aligned}$$

$$= \frac{1}{4ie^{i\frac{\pi}{4}}}\left(\mathcal{L}_s^{-1}\frac{1}{s\sqrt{s}}\, e^{e^{i\frac{\pi}{4}}x\sqrt{s}} \overset{x}{*} B_1(x) - \mathcal{L}_s^{-1}\frac{1}{s\sqrt{s}}\, e^{e^{i\frac{\pi}{4}}sx}\int_0^\infty e^{-e^{i\frac{\pi}{4}}\sqrt{s}u}B_1(u)du\right)$$

$$= \frac{1}{4ie^{i\frac{\pi}{4}}}\left(\mathcal{L}_s^{-1}\frac{1}{s\sqrt{s}}\int_0^x e^{i\frac{\pi}{4}(x-u)\sqrt{s}}B_1(u)du - \mathcal{L}_s^{-1}\frac{1}{s\sqrt{s}}\int_0^\infty e^{-e^{i\frac{\pi}{4}(u-x)\sqrt{s}}}B_1(u)du\right.$$

$$= \frac{-1}{4ie^{i\frac{\pi}{4}}}\mathcal{L}_s^{-1}\left(\frac{1}{s\sqrt{s}}\int_x^\infty e^{-e^{i\frac{\pi}{4}(u-x)\sqrt{s}}}B_1(u)du\right) \tag{3.19}$$

$$= \frac{-1}{4ie^{i\frac{\pi}{4}}}\int_0^t\int_x^\infty \chi(e^{i\frac{\pi}{4}}(u-x),\tau)B_1(u)du\,d\tau.$$

If we collect all the results obtained in (3.16)−(3.19), then the inverse LT of (3.15) is a function denoted by $F(B_1, x, t, \frac{\pi}{4})$,

$$F(B_1,x,t,\frac{\pi}{4}) = -\frac{1}{4ie^{i\frac{\pi}{4}}}\int_0^t\int_x^\infty \chi(e^{i\frac{\pi}{4}}(u-x),\tau)B_1(u)du\,d\tau$$

$$-\frac{1}{4ie^{i\frac{\pi}{4}}}\int_0^t \chi(e^{i\frac{\pi}{4}}x,\tau)d\tau \overset{x}{*} B_1(x) \tag{3.20}$$

$$+\frac{1}{4ie^{i\frac{\pi}{4}}}\int_0^t du\int_0^\infty \chi(e^{i\frac{\pi}{4}}(x+\tau),u)B_1(\tau)d\tau.$$

To find the inverse LT of (3.13), it is yet to be find the inverse LT of

$$\frac{s(\mathcal{L}(B_0)(z) - \mathcal{L}(B_0)(e^{i\frac{\pi}{4}}\sqrt{s}))}{4e^{i\frac{3\pi}{4}}s\sqrt{s}}\left(\frac{1}{z-e^{i\frac{\pi}{4}}\sqrt{s}} - \frac{1}{z+e^{i\frac{\pi}{4}}\sqrt{s}}\right). \tag{3.21}$$

If we compare (3.21) with (3.15), we can observe that in the structure of (3.21) we have additionally only a product by s. Since $F(B_0, x, 0, \frac{\pi}{4}) = 0$, the inverse LT of (3.21) is

$$\frac{\partial}{\partial t}F\Big(B_0,x,t,\frac{\pi}{4}\Big) = -\frac{1}{4ie^{\frac{i\pi}{4}}}\int_x^\infty \chi\Big(e^{i\frac{\pi}{4}}(u-x),t\Big)B_0(u)du$$

$$-\frac{1}{4ie^{i\frac{\pi}{4}}}\chi(e^{i\frac{\pi}{4}}x,t) \overset{x}{*} B_0(x) \tag{3.22}$$

$$+\frac{1}{4ie^{i\frac{\pi}{4}}}\int\limits_{x}^{\infty}\chi\Big(e^{i\frac{\pi}{4}}(x+\tau),t\Big)B_0(\tau)d\tau.$$

To finish the first step we collect the all obtained results which give

$$\mathcal{L}^{-1}\left(\frac{Q(z,s)}{2is(z^2-is)}\right)(x,t)=$$

$$=\frac{1}{2i\Gamma(1/2)}\chi(e^{i\frac{\pi}{4}}x,t)\overset{t}{*}\int\limits_0^t(t-\tau)^{-1/2}A_2(\tau)d\tau \tag{3.23}$$

$$+F\Big(B_1,x,t,\frac{\pi}{4}\Big)+\frac{\partial}{\partial t}F\Big(B_0,x,t,\frac{\pi}{4}\Big),$$

where F is given by (3.20).

Second step

In the second step we consider the second addend in (3.10). Now we need $(3.11)_2$ to be satisfied:

$$\mathcal{L}(A_2)(s)e^{-i\frac{\pi}{4}}\sqrt{s}+\mathcal{L}(A_3)(s)+\mathcal{L}(B_1)(e^{-i\frac{\pi}{4}}\sqrt{s})+s\mathcal{L}(B_0)(e^{-i\frac{\pi}{4}}\sqrt{s})=0. \tag{3.24}$$

The procedure to find the inverse LT of

$$\frac{Q(z,s)}{2is(z^2+is)}=\frac{Q(z,s)}{4ie^{-i\frac{\pi}{4}}s\sqrt{s}}\left(\frac{1}{z-e^{-i\frac{\pi}{4}}\sqrt{s}}-\frac{1}{z+e^{-i\frac{\pi}{4}}\sqrt{s}}\right) \tag{3.25}$$

is just the same as for the first addend in (3.10), which we applied in the first step. Consequently because the $Ree^{-i\frac{\pi}{4}}>0$, we have

$$\mathcal{L}^{-1}\left(\frac{Q(z,s)}{2is(z^2+is)}\right)(x,t)=\frac{1}{2i\Gamma(1/2)}\chi(e^{-i\frac{\pi}{4}}x,t)\overset{t}{*}\int\limits_0^t(t-\tau)^{-1/2}A_2(\tau)d\tau+$$

$$+F\Big(B_1,x,t,-\frac{\pi}{4}\Big)+\frac{\partial}{\partial t}F\Big(B_0,x,t,-\frac{\pi}{4}\Big). \tag{3.26}$$

Third step

It remains to find the solution $u(x,t)$ to equation (3.1). This can be done, now, by taking the inverse LT of (3.8) or in fact of (3.10).

By (3.23) and (3.26) we have

$$\begin{aligned} u(x,t) &= \frac{1}{2i\Gamma(1/2)}\chi(e^{i\frac{\pi}{4}}x,t) \overset{t}{*} \int_0^t (t-\tau)^{-1/2}A_2(\tau)d\tau \\ &+F(B_1,x,t,\frac{\pi}{4}) + \frac{\partial}{\partial t}F(B_0,x,t,\frac{\pi}{4}) \\ &-\frac{1}{2i\Gamma(1/2)}\chi(e^{-i\frac{\pi}{4}}x,t) \overset{t}{*} \int_0^t (t-\tau)^{-1/2}A_2(\tau)d\tau \\ &-F\Big(B_1,x,t,\frac{-\pi}{4}\Big) - \frac{\partial}{\partial t}F\Big(B_0,x,t,-\frac{\pi}{4}\Big). \end{aligned} \tag{3.27}$$

Now we can apply properties of χ 3.- 5. in Section 3.2 to (3.27):

$$\begin{aligned} u(x,t) &= \frac{1}{\Gamma(1/2)}\frac{1}{\sqrt{\pi t}}\sin\frac{x^2}{4t} \overset{t}{*} \int_0^t (t-\tau)^{-1/2}A_2(\tau)d\tau \\ &+\frac{\sqrt{2}}{4}\frac{1}{\sqrt{\pi t}}\int_0^t\int_x^\infty \left(\cos\frac{(u-x)^2}{4\tau}+\sin\frac{(u-x)^2}{4\tau}\right)B_1(u)du\,d\tau \\ &+\frac{\sqrt{2}}{4}\frac{1}{\sqrt{\pi t}}\int_0^t \left(\cos\frac{x^2}{4\tau}+\sin\frac{x^2}{4\tau}\right)d\tau \overset{x}{*} B_1(x) \\ &-\frac{\sqrt{2}}{4}\frac{1}{\sqrt{\pi t}}\int_0^t\int_0^\infty \left(\cos\frac{(x+\tau)^2}{4u}+\sin\frac{(x+\tau)^2}{4u}\right)B_1(\tau)d\tau\,du \end{aligned}$$

$$+\frac{\sqrt{2}}{4}\frac{1}{\sqrt{\pi t}}\int_x^\infty \left(\cos\frac{(u-x)^2}{4t}+\sin\frac{(u-x)^2}{4t}\right)B_0(u)du$$

$$
\begin{aligned}
&+\frac{\sqrt{2}}{4}\frac{1}{\sqrt{\pi t}}\left(\cos\frac{x^2}{4t}+\sin\frac{x^2}{4t}\right)\overset{x}{*}B_0(x)\\
&-\frac{\sqrt{2}}{4}\frac{1}{\sqrt{\pi t}}\int_0^\infty\left(\cos\frac{(x+\tau)^2}{4t}+\sin\frac{(x+\tau)^2}{4t}\right)B_0(\tau)d\tau.
\end{aligned}
\tag{3.28}
$$

Comments

1. Functions A_2 and A_3 we can express by B_0 and B_1 using (3.12) and (3.24).

2. If we have also some boundary conditions, then we try to settle B_0 and B_1 in such a way that they are satisfied, if it is possible.

3. If in the initial conditions (3.6)

$$
\begin{aligned}
&A_2(t)=-2\nu^2C_1g(t),\ A_3(t)=-2\nu^3C_2g(t)\\
&B_0(x)=K_1v(x)\quad\text{and}\quad B_1(x)=\mu K_2v(x),
\end{aligned}
\tag{3.29}
$$

then it follows by (3.8) that $u(x,t)=v(x)g(t)$ is a solution to (3.1), as well, with initial condition (3.6), where

$$
\begin{aligned}
&v(x)=C_1\cos\nu x+C_2\sin\nu x+C_3\cos h\nu x+C_4\sin h\nu x\\
&g(t)=K_1\cos\mu t+K_2\sin\mu t;\\
&\nu=\sqrt[4]{\omega^2},\ \mu=\sqrt{\omega^2},\ \omega^2>0;\ \ K_1,K_2,\,C_i,\ i=1,...,4,\ \ \text{are constants.}
\end{aligned}
\tag{3.30}
$$

P r o o f. To prove that $u(x,t)=v(x)g(t)$ is a solution to (3.1) with (3.6) and (3.29) which satisfies (3.8) we use the known properties of f and g (cf. [1]):

$$
\begin{aligned}
\widehat{v}(z)&=-\frac{2}{z^4-\omega^2}(\nu^2C_1z+\nu^3C_2)\\
\widehat{g}(s)&=\frac{1}{s^2+\omega^2}(K_1s+\mu K_2).
\end{aligned}
$$

Then

$$
\begin{aligned}
\mathcal{L}(u)(z,s)\ &=-\frac{2(\nu^2C_1z+\nu^3C_2)\widehat{g}(s)}{z^4+s^2}\\
&+\frac{(K_1s+\mu K_2)\widehat{v}(z)}{z^4+s^2}=\frac{z^4-\omega^2}{z^4+s^2}\widehat{v}(z)\widehat{g}(s)\\
&+\frac{s^2+\omega^2}{z^4+s^2}\widehat{v}(z)\widehat{g}(s)=\frac{(z^4+s^2)\widehat{v}(z)\widehat{g}(s)}{z^4+s^2}=\widehat{v}(z)\widehat{g}(s).
\end{aligned}
$$

4. There exists one and only one solution $u(x,t)$ to (3.1) for $x > 0,\ t > 0$ which satisfies $(3.6)_1$ and $(3.6)_3$ with fixed B_0, B_1, such that $[H_2(x,t)u(x,t)] \in e^{\sigma(x+t)}\mathcal{S}'(\overline{\mathbb{R}}^2_+)$.

P r o o f. Suppose to have two solutions to (3.1), u_1 and u_2. Let $U = u_1 - u_2$ and $a_i = A_i^1 - A_i^2,\ i = 2,3;\ A_i^1, A_i^2$ are given in $(3.6)_2$ for u_1 and u_2, respectively. Then U satisfies

$$\frac{\partial^4}{\partial x^4}[U(x,t)] + \frac{\partial^2}{\partial t^2}[U(x,t)] = [H(t)a_2(t)] \times \delta^{(1)}(x) + [H(t)a_3(t)] \times \delta(x).$$

Because of (3.11) $a_i(t) = 0,\ i = 2,3$. Therefore by (3.8), $U(x,t) = 0$, $x \geq 0, t \geq 0$.

REFERENCES

[1] T.M. A t a n a c k o v i ć, *Stability Theory of Elastic Rods,* World Scientific, Singapore, 1997.

[2] J. C a r r and Z.M. M a l h a r d e n, *Beck's problem,* SIAM J.Appl. Math. Vol. 37, No 2 (1979), 261-262.

[3] R.C o u r a n t, D.H i l b e r t, *Methods of Mathematical Physics.*

[4] N.D.F o w k e s and J.J.M a h o n y, *An Introduction to Mathematical Modeling, John Wiley and Sons,* New York, 1996.

[5] LI.G e n - g u o, ZHU Z h e n g - y o u, *CHENG Chang-jun Dynamical stability of viscoelastic column with fractional derivative constitutive relation,* Applied Mathematics and Mechanics, Shangai University, V. 22, No 3, (2001), 294-303.

[6] O.N.K i r i l o v, A.P.S e y r a n i a n, *Solution to the Hermann - Smith Problem,* Doklady Akademii Nauk, Vol. 386, No 6 (2002), 761-766. (Doklady Physics, Vol 47, No 10 (2002), 767-771).

[7] N.O r t n e r, *Construction of Fundamental Solutions,* (In Topics in Boundary element Research, ED. C. A. Brebbia, 1987).

[8] Yu.A.R o s s i k h i n, M.V.S h i t i k o v a, *Application of fractional calculus to dynamic problems of linear and nonlinear hereditary mechanics of solids,* Appl. Mech. Rev. 50 (1997), 15-67.

[9] L.S c h w a r t z, *Théory des distributions,* 3 éd., Hermann, Paris, 1966.

[10] A.P.S e y r a n i a n, *Bifurcations of Eigenvalues and Stability Problems in Mechanics. In Modern Pronlems of Structural Stability,* Springer, 2002.

[11] B.S t a n k o v i ć, *Solution of mathematical models by localization.* Bul. Acad. Serb. Sc. et Arts, Science Math. No 27, (2002), 1-18.

[12] V.S. V l a d i m i r o v, *Equations of Mathematical physics,* Nauka, Moscow, 1988 (in Russian).

[13] V.S.V l a d i m i r o v, *Generalized Functions in Mathematical Physics,* Mir Publishers, Moscow, 1979.

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