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ASYMPTOTICS OF SOME CLASSES OF NONOSCILLATORY SOLUTIONS OF SECOND-ORDER HALF-LINEAR DIFFERENTIAL EQUATIONS

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A b s t r a c t. The precise asymptotic behaviour at infinity of some classes of nonoscillatory solutions of the half-linear differential equations is determined.

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0. *Introduction*

Let $\alpha > 0$ be a constant and let $q : [0, \infty) \to \mathbb{R}$ be a continuous function which is conditionally integrable in the sense that

$$\int_0^\infty q(t)dt = \lim_{T\to\infty} \int_0^T q(s)ds \quad \text{exists and is finite.}$$

We consider the half-linear differential equation

$$(|y'|^{\alpha-1}y')' + q(t)|y|^{\alpha-1}y = 0, \quad t \geq 0, \tag{A}$$

and derive the precise asymptotic behaviour of some classes of its nonoscillatory solutions $y(t)$ meaning, as usual, that we construct a positive, continuous function $\varphi(t)$ defined on a positive half-axis such that $y(t)/\varphi(t) \to 1$ as $t \to \infty$, denoted as $y(t) \sim \varphi(t)$.

In particular, we treat in that respect the nonoscillatory solutuions of (A) which belong to the class of slowly varying functions in the sense of Karamata [1], which is of frequent occurrence in various branches of mathematical analysis.

For brevity, we use the canonical representation of these as the definition.

Definition 0.1. *A positive measurable function $L(t)$ defined on $(0, \infty)$ is slowly varying if and only if it can be written in the form*

$$L(t) = c(t) \exp\left\{\int_{t_0}^{t} \frac{\varepsilon(s)}{s} ds\right\}, \qquad t \geq t_0,$$

for some $t_0 > 0$, where $c(t)$ and $\varepsilon(t)$ are such that for $t \to \infty$

$$c(t) \to c \in (0, \infty) \quad \text{and} \quad \varepsilon(t) \to 0.$$

If $c(t)$ is identically a positive constant, then $L(t)$ is called normalized.

The present work is the first attempt at scrutinizing the asymptotic behaviour of slowly varying solutions of the half-linear differential eqautions. Note that the asymptotic analysis of slowly varying solutions for the linear equation $y'' + q(t)y = 0$, which is a special case of (A) with $\alpha = 1$, has been made by several authors; see e.g. [2, 3, 5, 6]

1. *Results*

The existence of nonoscillatory solutions of (A) is essentially proved (for $c(t) = c$) in [4, Lemma 2.2], but we present the proof here for the reader's benefit. We put

$$E(\alpha) = \frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}},$$

which is referred to as the generalized Euler constant with respect to (A), and make use of the asterisk notation:

$$\xi^{\gamma *} = |\xi|^{\gamma-1}\xi = |\xi|^{\gamma} \operatorname{sgn} \xi \quad \text{for} \quad \xi \in \mathbb{R} \quad \text{and} \quad \gamma > 0.$$

Theorem 1.1. *Put*
$$Q(t) = \int_t^\infty q(s)ds \tag{1.1}$$
and suppose that there exists a continuous function $P : [t_0, \infty) \to (0, \infty)$, $t_0 \geq 0$, *such that* $\lim_{t\to\infty} P(t) = 0$ *and*
$$|Q(t)| \leq P(t), \qquad t \geq t_0, \tag{1.2}$$
$$\int_t^\infty P(s)^{1+\frac{1}{\alpha}} ds \leq \frac{1}{\alpha} c(t)^{\frac{1}{\alpha}} P(t), \qquad t \geq t_0, \tag{1.3}$$
where $c(t)$ *is a continuous nonincreasing function satisfying*
$$0 < c(t) \leq c < E(\alpha), \qquad t \geq t_0, \tag{1.4}$$
for some constant c. *Then the equation* (A) *has a nonoscillatory solution of the form*
$$y(t) = \exp\left\{\int_{t_0}^t [v(s) + Q(s)]^{\frac{1}{\alpha}*} ds\right\}, \qquad t \geq t_0, \tag{1.5}$$
where $v(t)$ *is a solution of the integral equation*
$$v(t) = \alpha \int_t^\infty |v(s) + Q(s)|^{1+\frac{1}{\alpha}} ds, \qquad t \geq t_0, \tag{1.6}$$
satisfying
$$v(t) = O(P(t)) \quad \text{as } t \to \infty. \tag{1.7}$$

P r o o f. Consider the function $y(t)$ defined by (1.5). It is easy to see that $y(t)$ is a solution of (A) if $v(t)$ is chosen in such a way that $u(t) = v(t) + Q(t)$ satisfies the generalized Riccati equation
$$u' + \alpha |u|^{1+\frac{1}{\alpha}} + q(t) = 0, \qquad t \geq t_0. \tag{1.8}$$
This requirement yields the differential equation for $v(t)$:
$$v' + \alpha |v + Q(t)|^{1+\frac{1}{\alpha}} = 0, \qquad t \geq t_0, \tag{1.9}$$
from which the equation (1.6) follows via integration over $[t, \infty)$ under the additional condition $\lim_{t\to\infty} v(t) = 0$.

We shall show that a unique solution of (1.6) of the desired kind indeed exists by using the Banach contraction theorem. Let $C_P[t_0, \infty)$ denote the set of all continuous functions $v(t)$ on $[t_0, \infty)$ such that
$$\|v\|_P = \sup_{t\geq t_0} \frac{|v(t)|}{P(t)} < \infty. \tag{1.10}$$

It is clear that $C_P[t_0,\infty)$ is a Banach space with the norm $\|\cdot\|_P$.

Define the set $V \subset C_P[t_0,\infty)$ and the mapping $\mathcal{F} : V \to C_P[t_0,\infty)$ by

$$V = \{v \in C_P[t_0,\infty) : \quad \|v(t)\|_P \le \alpha, \quad t \ge t_0\} \tag{1.11}$$

and

$$\mathcal{F}v(t) = \alpha \int_t^\infty |v(s) + Q(s)|^{1+\frac{1}{\alpha}} ds, \quad t \ge t_0, \tag{1.12}$$

respectively. If $v \in V$, then

$$|\mathcal{F}v(t)| \le \alpha(1+\alpha)^{1+\frac{1}{\alpha}} \int_t^\infty P(s)^{1+\frac{1}{\alpha}} ds \le (1+\alpha)^{1+\frac{1}{\alpha}} c(t)^{\frac{1}{\alpha}} P(t), \quad t \ge t_0,$$

from which it follows, in view of (1.4), that

$$\|\mathcal{F}v\|_P \le (1+\alpha)^{1+\frac{1}{\alpha}} c^{\frac{1}{\alpha}} < (1+\alpha)^{1+\frac{1}{\alpha}} E(\alpha)^{\frac{1}{\alpha}} = \alpha. \tag{1.13}$$

This shows that $\mathcal{F}$ maps V into itself. If $v_1, v_2 \in V$, then, using the mean value theorem, we have

$$\begin{aligned} |\mathcal{F}v_1(t) - \mathcal{F}v_2(t)| &\le \alpha \int_t^\infty \left| |v_1(s) + Q(s)|^{1+\frac{1}{\alpha}} - |v_2(s) + Q(s)|^{1+\frac{1}{\alpha}} \right| ds \\ &\le \alpha \left(1 + \frac{1}{\alpha}\right) \int_t^\infty [(1+\alpha)P(s)]^{\frac{1}{\alpha}} |v_1(s) - v_2(s)| ds \\ &= (1+\alpha)^{1+\frac{1}{\alpha}} \int_t^\infty P(s)^{1+\frac{1}{\alpha}} \frac{|v_1(s) - v_2(s)|}{P(s)} ds \\ &\le (1+\alpha)^{1+\frac{1}{\alpha}} \frac{1}{\alpha} c(t)^{\frac{1}{\alpha}} P(t) \|v_1 - v_2\|_P, \quad t \ge t_0, \end{aligned}$$

which implies that

$$\|\mathcal{F}v_1 - \mathcal{F}v_2\|_P \le \frac{1}{\alpha}(1+\alpha)^{1+\frac{1}{\alpha}} c^{\frac{1}{\alpha}} \|v_1 - v_2\|_P. \tag{1.14}$$

Since $\frac{1}{\alpha}(1+\alpha)^{1+\frac{1}{\alpha}} c^{\alpha} < 1$ (cf. (1.13)), we conclude that $\mathcal{F}$ is a contraction mapping on V.

The contraction mapping principle then guarantees the existence of a unique element $v \in V$ such that $v = \mathcal{F}v$, which clearly is a solution of the integral equation (1.6). Then, the function $y(t)$ given by (1.5) with this $v(t)$

gives a solution of (A) on $[t_0, \infty)$. That $v(t)$ satisfies (1.7) is a consequence of the fact that $v \in V$. This completes the proof.

Corollary 1.1. *The equation* (A) *has a normalized slowly varying solution if*

$$\lim_{t\to\infty} t^\alpha \int_t^\infty q(s)ds = 0. \tag{1.15}$$

P r o o f. Here, one can take the function $c(t)$ from Theorem 1.1 to be

$$c(t) = \sup_{s\geq t} \left| s^\alpha \int_s^\infty q(r)dr \right|. \tag{1.16}$$

Then $c(t)$ is nonincreasing and tends to zero as $t \to \infty$. Choose $t_0 > 0$ so that

$$c(t) < E(\alpha) \quad \text{ and } \quad |Q(t)| \leq \frac{c(t)}{t^\alpha} \quad \text{ for } \ t \geq t_0.$$

The second inequality holds due to (1.15). Take in Theorem 1.1 $P(t) = c(t)/t^\alpha$. Then (1.2) holds and

$$\int_t^\infty P(s)^{1+\frac{1}{\alpha}}ds = \int_t^\infty \left[\frac{c(s)}{s^\alpha}\right]^{1+\frac{1}{\alpha}} ds \leq \frac{c(t)^{1+\frac{1}{\alpha}}}{\alpha t} = \frac{1}{\alpha} c(t)^{\frac{1}{\alpha}} P(t), \qquad t \geq t_0.$$

Consequently, by Theorem 1.1, (A) has a nonoscillatory solution $y(t)$ of the form (1.5) on $[t_0, \infty)$ with $v(t)$ satisfying (1.7). Since

$$t^\alpha v(t) = O(t^\alpha P(t)) = o(1) \quad \text{ and } \quad t^\alpha Q(t) = O(t^\alpha P(t)) = o(1)$$

as $t \to \infty$, $y(t)$ can be rewritten as

$$y(t) = \exp\left\{\int_{t_0}^t \frac{\varepsilon(s)}{s} ds\right\}, \qquad t \geq t_0,$$

with $\varepsilon(t) = [t^\alpha(v(t) + Q(t))]^{\frac{1}{\alpha}*} = o(1)$ as $t \to \infty$ due to Definition 0.1. This completes the proof.

Theorem 1.2. *Suppose that the hypotheses of Theorem* 1.1 *are satisfied. Suppose furthermore that there exists a positive integer* n *such that*

$$\int^\infty c(t)^{\frac{n}{\alpha}} P(t)^{\frac{1}{\alpha}} dt < \infty \qquad \textit{if} \quad 0 < \alpha \leq 1, \tag{1.17}$$

$$\int^\infty c(t)^{\frac{n}{\alpha^2}} P(t)^{\frac{1}{\alpha}} dt < \infty \qquad \textit{if} \quad \alpha > 1. \tag{1.18}$$

Then, for the solution (1.5) *of the equation* (A), *the following asymptotic formula holds for* $t \to \infty$

$$y(t) \ \sim \ A \exp\left\{\int_{t_0}^{t} [v_{n-1}(s) + Q(s)]^{\frac{1}{\alpha}*} ds\right\}, \tag{1.19}$$

where A *is a positive constant. Here the sequence* $\{v_n(t)\}$ *of successive approximations is defined by*

$$v_0(t) = 0, \qquad v_n(t) = \alpha \int_t^{\infty} |v_{n-1}(s) + Q(s)|^{1+\frac{1}{\alpha}} ds, \qquad n = 1,\ 2, \cdots. \tag{1.20}$$

P r o o f. Let $y(t)$ be the solution (1.5) of (A) obtained in Theorem 1.1. Recall that the function $v(t)$ used in (1.5) has been constructed as the fixed element in $C_P[t_0, \infty)$ of the contractive mapping $\mathcal{F}$ defined by (1.12). Th e standard proof of the contraction mapping principle shows that the sequence $\{v_n(t)\}$ defined by (1.20) converges to $v(t)$ uniformly on $[t_0, \infty)$. To see how fast $v_n(t)$ approaches $v(t)$ we proceed as follows. First, note that $|v_n(t)| \le \alpha P(t)$, $t \ge t_0$, $n = 1, 2, \cdots$. By definition, we have

$$|v_1(t)| = \alpha \int_t^{\infty} |Q(s)|^{1+\frac{1}{\alpha}} ds \le \alpha \int_t^{\infty} P(s)^{1+\frac{1}{\alpha}} ds \le c(t)^{\frac{1}{\alpha}} P(t),$$

and

$$\begin{aligned}
|v_2(t) - v_1(t)| &\le \alpha \int_t^{\infty} \left| |v_1(s) + Q(s)|^{1+\frac{1}{\alpha}} - |Q(s)|^{1+\frac{1}{\alpha}} \right| ds \\
&\le \alpha \left(1 + \frac{1}{\alpha}\right) \int_t^{\infty} [(1+\alpha)P(s)]^{1+\frac{1}{\alpha}} |v_1(s)| ds \\
&\le (\alpha+1)^{1+\frac{1}{\alpha}} \int_t^{\infty} c(s)^{\frac{1}{\alpha}} P(s)^{1+\frac{1}{\alpha}} ds \le (\alpha+1)^{1+\frac{1}{\alpha}} c(t)^{\frac{1}{\alpha}} \int_t^{\infty} P(s)^{1+\frac{1}{\alpha}} ds \\
&\le \frac{1}{\alpha} (\alpha+1)^{1+\frac{1}{\alpha}} c(t)^{\frac{2}{\alpha}} P(t) \le E(\alpha)^{\frac{1}{\alpha}} \left[\frac{c(t)}{E(\alpha)}\right]^{\frac{2}{\alpha}} P(t)
\end{aligned}$$

for $t \ge t_0$. Assuming that

$$|v_n(t) - v_{n-1}(t)| \le E(\alpha)^{\frac{1}{\alpha}} \left[\frac{c(t)}{E(\alpha)}\right]^{\frac{n}{\alpha}} P(t), \qquad t \ge t_0 \tag{1.21}$$

for some $n \in \mathbb{N}$, we compute

$$\begin{aligned}
|v_{n+1}(t) - v_n(t)| &\le \alpha \int_t^\infty \left| |Q(s) + v_n(s)|^{1+\frac{1}{\alpha}} - |Q(s) + v_{n-1}(s)|^{1+\frac{1}{\alpha}} \right| ds \\
&\le \alpha \left(1 + \frac{1}{\alpha}\right) \int_t^\infty [(1+\alpha)P(s)]^{\frac{1}{\alpha}} |v_n(s) - v_{n-1}(s)| ds \\
&= (\alpha+1)^{1+\frac{1}{\alpha}} \int_t^\infty E(\alpha)^{\frac{1}{\alpha}} \left[\frac{c(s)}{E(\alpha)}\right]^{\frac{n}{\alpha}} P(s)^{1+\frac{1}{\alpha}} ds \\
&= (\alpha+1)^{1+\frac{1}{\alpha}} E(\alpha)^{\frac{1}{\alpha}} \left[\frac{c(t)}{E(\alpha)}\right]^{\frac{n}{\alpha}} \int_t^\infty P(s)^{1+\frac{1}{\alpha}} ds \\
&\le (\alpha+1)^{1+\frac{1}{\alpha}} E(\alpha)^{\frac{1}{\alpha}} \left[\frac{c(t)}{E(\alpha)}\right]^{\frac{n}{\alpha}} \frac{1}{\alpha} c(t)^{\frac{1}{\alpha}} P(t) \\
&= E(\alpha)^{\frac{1}{\alpha}} \left[\frac{c(t)}{E(\alpha)}\right]^{\frac{n+1}{\alpha}} P(t), \qquad t \ge t_0,
\end{aligned}$$

which establishes the truth of (1.21) for all integers $n \in \mathbb{N}$.

Now we have

$$v(t) = v_{n-1}(t) + r_n(t)$$

with

$$r_n(t) = \sum_{k=n}^{\infty} [v_k(t) - v_{k-1}(t)],$$

from which, due to (1.21), it follows that

$$\begin{aligned}
|v(t) - v_{n-1}(t)| &\le \sum_{k=n}^{\infty} E(\alpha)^{\frac{1}{\alpha}} \left[\frac{c(t)}{E(\alpha)}\right]^{\frac{k}{\alpha}} P(t) \\
&\le E(\alpha)^{\frac{1}{\alpha}} \left[\frac{c(t)}{E(\alpha)}\right]^{\frac{n}{\alpha}} \sum_{k=0}^{\infty} \left(\frac{c}{E(\alpha)}\right)^k P(t) \qquad (1.22) \\
&= E(\alpha) \left[\frac{c(t)}{E(\alpha)}\right]^{\frac{n}{\alpha}} \frac{E(\alpha)}{E(\alpha) - c} P(t) = K c(t)^{\frac{n}{\alpha}} P(t)
\end{aligned}$$

for $t \ge t_0$, where K is a constant depending only on α and n.

Using (1.5) and (1.22), we obtain

$$\frac{y(t)}{\exp\left\{\int_{t_0}^{t}[Q(s)+v_{n-1}(s)]^{\frac{1}{\alpha}*}ds\right\}}$$
$$= \exp\left\{\int_{t_0}^{t}\left([Q(s)+v(s)]^{\frac{1}{\alpha}*}-[Q(s)+v_{n-1}(s)]^{\frac{1}{\alpha}*}\right)ds\right\}. \tag{1.23}$$

Let $0 < \alpha \leq 1$. Then, by the mean value theorem and (1.22),

$$\left|[Q(t)+v(t)]^{\frac{1}{\alpha}*}-[Q(t)+v_{n-1}(t)]^{\frac{1}{\alpha}*}\right| \leq \frac{1}{\alpha}[(1+\alpha)P(t)]^{\frac{1}{\alpha}-1}|v(t)-v_{n-1}(t)|$$
$$\leq Lc(t)^{\frac{n}{\alpha}}P(t)^{\frac{1}{\alpha}}, \quad t \geq t_0, \tag{1.24}$$

where L is a constant depending on α and n.

Let $\alpha > 1$. Then, using (1.22) and the inequality $|a^{\theta}-b^{\theta}| \leq 2|a-b|^{\theta}$ holding for $\theta \in (0,1)$ and $a,\ b \in \mathbb{R}$, we see that

$$\left|[Q(t)+v(t)]^{\frac{1}{\alpha}*}-[Q(t)+v_{n-1}(t)]^{\frac{1}{\alpha}*}\right| \leq 2|v(t)-v_{n-1}(t)|^{\frac{1}{\alpha}}$$
$$\leq Mc(t)^{\frac{n}{\alpha^2}}P(t)^{\frac{1}{\alpha}}, \quad t \geq t_0, \tag{1.25}$$

where M is a constant depending on α and n.

Combining (1.23) with (1.24) or (1.25) according as $0 < \alpha \leq 1$ or $\alpha > 1$, and using (1.17) or (1.18), we conculde that the right-hand side of (1.23) tends to a constant $A > 0$ as $t \to \infty$, which implies that $y(t)$ has the des ired asymptotic behaviour (1.19). This completes the proof.

Corollary 1.2. *Suppose that* (1.15) *holds and that the function* $c(t)$ *defined by* (1.16) *satisfies*

$$\int^{\infty}\frac{c(t)^{\frac{n+1}{\alpha}}}{t}dt < \infty \qquad \text{if} \quad 0 < \alpha \leq 1, \tag{1.26}$$

$$\int^{\infty}\frac{c(t)^{\frac{n+\alpha}{\alpha^2}}}{t}dt < \infty \qquad \text{if} \quad \alpha > 1. \tag{1.27}$$

Then the formula (1.19) *holds for the slowly varying solution* $y(t)$ *of* (A).

P r o o f. The conclusion follows from Theorem 1.2 combined with the observation that in this case $c(t)^{\frac{n}{\alpha}}P(t)^{\frac{1}{\alpha}} = c(t)^{\frac{n+1}{\alpha}}/t$ and $c(t)^{\frac{n}{\alpha^2}}P(t) = c(t)^{\frac{n+\alpha}{\alpha^2}}/t$ according to whether $0 < \alpha \leq 1$ and $\alpha > 1$.

2. *Examples*

Two examples illustrating our main results will be given below.

Example 2.1. Consider the equation

$$(|y'|^{\alpha-1}y')' + kt^{\beta}\sin(t^{\gamma})|y|^{\alpha-1}y = 0, \qquad t \geq 1, \tag{2.1}$$

where $k,\ \alpha,\ \beta$ and γ are positive constants satisfying

$$\gamma > 1 + \alpha + \beta. \tag{2.2}$$

Since

$$\int_t^\infty s^{\beta}\sin(s^{\gamma})ds = \frac{1}{\gamma}t^{1+\beta-\gamma}\cos(t^{\gamma}) + \frac{1+\beta-\gamma}{\gamma}\int_t^\infty s^{\beta-\gamma}\cos(s^{\gamma})ds,$$

there exists a positive constant K such that

$$\left|\int_t^\infty ks^{\beta}\sin(s^{\gamma})ds\right| \leq Kt^{1+\beta-\gamma}, \qquad t \geq 1, \tag{2.3}$$

which, in view of (2.2), implies that

$$\lim_{t\to\infty} t^{\alpha}\int_t^\infty ks^{\beta}\sin(s^{\alpha})ds = 0.$$

Therefore, the equation (2.1) has a slowly varying solution $y(t)$ by Corollary 1.1.

In this case the function $c(t)$ defined by (1.16) can be taken to be $c(t) = Kt^{1+\alpha+\beta-\gamma}$. Since $c(t)$ satisfies both (1.26) and (1.27) for any $n \in \mathbb{N}$ because of (2.2), from Corollary 1.2 for $n = 1$ we conclude that th e slowly varying solution $y(t)$ of (2.1) has the asymptotic behaviour

$$y(t) \ \sim\ A\exp\left\{\int_{t_0}^t \left(\int_s^\infty kr^{\beta}\sin(r^{\gamma})dr\right)^{\frac{1}{\alpha}*} ds\right\} \qquad \text{as } \ t \to \infty, \tag{2.4}$$

which is equivalent to $y(t) \ \sim\ A_0$ (constant), since the integral in the braces in (2.4) converges as $t \to \infty$ because of (2.3).

Example 2.2. Consider the equation

$$(|y'|^{\alpha-1}y')' + \frac{a + b\sin t}{t^{\beta}(\log t)^{\gamma}}|y|^{\alpha-1}y = 0, \qquad t \geq e, \tag{2.5}$$

where the constants appearing in (2.5) are positive except for a, and satisfy $\beta \geq \alpha + 1$ and $|a| < b$.

I) We first suppose that $a \neq 0$. Note that, for $\beta > 1$,

$$Q(t) = \int_t^\infty \frac{a + b\sin s}{s^\beta (\log s)^\gamma} ds = \frac{a}{\beta - 1} t^{1-\beta} (\log t)^{-\gamma} \left[1 + O\left(\frac{1}{t}\right)\right]. \tag{2.6}$$

Let $\beta > \alpha + 1$. Then, $(Q(t))^{\frac{1}{\alpha}*}$ is absolutely integrable on $[e, \infty)$ and $t^\alpha Q(t) \to 0$ as $t \to \infty$. Corollary 1.1 then implies that (2.5) possesses a slowly varying solution $y(t)$.

The function $c(t) = (2|a|/(\beta - 1)) t^{1+\alpha-\beta} (\log t)^{-\gamma}$ defined by (1.16) satisfies the conditions (1.26) and (1.27) for any $n \in \mathbb{N}$, so that, by Corollary 1.2 with $n = 1$, $y(t)$ enjoys the asymptotic property

$$y(t) \sim A \exp\left\{\int_{t_0}^t (Q(s))^{\frac{1}{\alpha}*} ds\right\} \sim A_0 \quad \text{as } t \to \infty.$$

Let $\beta = \alpha + 1$. We see that $t^\alpha Q(t) \to 0$ as $t \to \infty$ also in this case, so that (2.5) has a slowly varying solution $y(t)$. As easily verified, the function $c(t) = (2|a|/\alpha)(\log t)^{-\gamma}$ satisfies the conditi ons (1.26) and (1.27) become, respectively,

$$\int^\infty t^{-1} (\log t)^{-\frac{(n+1)\gamma}{\alpha}} dt < \infty \quad (0 < \alpha \leq 1) \tag{2.7}$$

and

$$\int^\infty t^{-1} (\log t)^{-\frac{(n+\alpha)\gamma}{\alpha^2}} dt < \infty \quad (\alpha > 1), \tag{2.8}$$

which are fulfilled if one determines n to satisfy

$$n > \frac{\alpha - \gamma}{\gamma} \quad (0 < \alpha \leq 1) \quad \text{or} \quad n > \frac{\alpha(\alpha - \gamma)}{\gamma} \quad (\alpha > 1). \tag{2.9}$$

For practical use write (2.9) as

$$\gamma > \frac{\alpha}{n+1} \quad (0 < \alpha \leq 1) \quad \text{or} \quad \gamma > \frac{\alpha^2}{n+\alpha} \quad (\alpha > 1),$$

which is equivalent to

$$\gamma > \alpha \max\left\{\frac{1}{n+1}, \frac{\alpha}{n+\alpha}\right\}. \tag{2.10}$$

Obviously, the range $\gamma > \alpha \max\left\{\frac{1}{2}, \frac{\alpha}{1+\alpha}\right\}$ is such that (2.10) i.e., (2.9) holds for $n = 1$, so that Corollary 1.2 can be applied with $n = 1$, leading to

$$
\begin{aligned}
y(t) &\sim A\exp\left\{\int_{t_0}^{t}(Q(s))^{\frac{1}{\alpha}*}ds\right\} \\
&\sim A'\exp\left\{\left(\frac{a}{\alpha}\right)^{\frac{1}{\alpha}*}\int_{t_0}^{t}s^{-1}(\log s)^{-\frac{\gamma}{\alpha}}ds\right\} \quad \text{as } t\to\infty,
\end{aligned}
\tag{2.11}
$$

from which it readily follows that

$$y(t) \sim A_0 \quad \text{if} \quad \gamma>\alpha$$

and

$$y(t) \sim A_0(\log t)^{\delta}, \quad \delta=\left(\frac{a}{\alpha}\right)^{\frac{1}{\alpha}*} \quad \text{if} \quad \gamma=\alpha.$$

Arguing in the same way, we conclude that (2.9) holds for $n=2$ in the range

$$\alpha\max\left\{\frac{1}{3},\frac{\alpha}{2+\alpha}\right\}<\gamma\le\alpha\max\left\{\frac{1}{2},\frac{\alpha}{1+\alpha}\right\}. \tag{2.12}$$

Then, the conclusion of Corollary 1.2 holds with $n=2$, that is,

$$y(t)\sim A\exp\left\{\int_{t_0}^{t}[v_1(s)+Q(s)]^{\frac{1}{\alpha}*}ds\right\} \quad \text{as } t\to\infty, \tag{2.13}$$

where $v_1(t)=\alpha\int_t^\infty|Q(s)|^{1+\frac{1}{\alpha}}ds$. Using (2.6), we have

$$
\begin{aligned}
v_1(t) &= \alpha\int_t^\infty\left|\frac{a}{\alpha}s^{-\alpha}(\log s)^{-\gamma}\left[1+O\left(\frac{1}{s}\right)\right]\right|^{1+\frac{1}{\alpha}}ds \\
&= \left|\frac{a}{\alpha}\right|^{1+\frac{1}{\alpha}}t^{-\alpha}(\log t)^{-\gamma\left(1+\frac{1}{\alpha}\right)}\left[1+O\left(\frac{1}{\log t}\right)\right].
\end{aligned}
\tag{2.14}
$$

Putting

$$w_1(t)=\left|\frac{a}{\alpha}\right|^{1+\frac{1}{\alpha}}t^{-\alpha}(\log t)^{-\gamma\left(1+\frac{1}{\alpha}\right)}, \tag{2.15}$$

we claim that

$$y(t)\sim A'\exp\left\{\int_{t_0}^{t}[w_1(s)+Q(s)]^{\frac{1}{\alpha}*}ds\right\} \quad \text{as } t\to\infty. \tag{2.16}$$

In fact, if $\alpha>1$, then

$$\int_{t_0}^{t} \left| [v_1(s)+Q(s)]^{\frac{1}{\alpha}*} - [w_1(s)+Q(s)]^{\frac{1}{\alpha}*} \right| ds$$
$$\le 2\int_{t_0}^{t} |v_1(s)-w_1(s)|^{\frac{1}{\alpha}*} ds \le K \int_{t_0}^{t} s^{-1}(\log s)^{-\frac{\gamma}{\alpha}(1+\frac{1}{\alpha})-\frac{1}{\alpha}} ds, \tag{2.17}$$

where K is a constant depending on α and a. Since $\gamma > \alpha/(\alpha+2)$ by (2.12),

$$\frac{\gamma}{\alpha}\left(1+\frac{1}{\alpha}\right)+\frac{1}{\alpha} > 1+\frac{2}{\alpha(\alpha+2)} > 1,$$

which implies that the last integral in (2.17) converges as $t \to \infty$. If $0 < \alpha \le 1$, then, using the inequality $|v_1(t)|,\ |w_1(t)| \le \alpha P(t) = 2|a|t^{-\alpha}(\log t)^{-\gamma}$ already known, we obtain

$$\int_{t_0}^{t} \left| [v_1(s)+Q(s)]^{\frac{1}{\alpha}*} - [w_1(s)+Q(s)]^{\frac{1}{\alpha}*} \right| ds$$
$$\le M_1 \int_{t_0}^{t} [s^{-\alpha}(\log s)^{-\gamma}]^{\frac{1}{\alpha}-1} |v_1(s)-w_1(s)| ds$$
$$\le M_2 \int_{t_0}^{t} [s^{-\alpha}(\log s)^{-\gamma}]^{\frac{1}{\alpha}-1} s^{-\alpha}(\log s)^{-\gamma(1+\frac{1}{\alpha})-1} ds \tag{2.18}$$
$$= M_3 \int_{t_0}^{t} s^{-1}(\log s)^{-\frac{2\gamma}{\alpha}-1} ds,$$

the last integral of which clearly converges as $t \to \infty$. Here M_i, $i = 1,\ 2,\ 3$, are constants depending only on α and a. Combining (2.16) with (2.15) establishes the asymptotic formula for $t \to \infty$

$$y(t) \sim A'' \exp\left\{ \int_{t_0}^{t} \left[\left|\frac{a}{\alpha}\right|^{1+\frac{1}{\alpha}} s^{-\alpha}(\log s)^{-\gamma(1+\frac{1}{\alpha})} + \frac{a}{\alpha} s^{-\alpha}(\log s)^{-\gamma} \right]^{\frac{1}{\alpha}*} ds \right\}. \tag{2.19}$$

Observe that when specialized to the case $\alpha = 1$, (2.19) reduces to the following formulas obtained in [3], cf. [5, p.67],

$$y(t) \sim A_1 (\log t)^{a^2} \exp\left\{2a(\log t)^{\frac{1}{2}}\right\} \qquad \text{if} \quad \gamma = \frac{1}{2},$$

$$y(t) \sim A_1 \exp\left\{\frac{a}{1-\gamma}(\log t)^{1-\gamma}\right\} \exp\left\{\frac{a^2}{1-2\gamma}(\log t)^{1-2\gamma}\right\} \qquad \text{if} \quad \frac{1}{3} < \gamma < \frac{1}{2}.$$

Let $\alpha = \frac{1}{2}$, for example. Then, (2.19) implies

$$y(t) \sim A_2 (\log t)^{32|a|^3 a} \exp\left\{8a^2 (\log t)^{\frac{1}{2}}\right\} \qquad \text{if} \quad \gamma = \frac{1}{4},$$

$$y(t) \sim A_2 \exp\left\{\frac{32|a|^3 a}{1-4\gamma}(\log t)^{1-4\gamma}\right\} \exp\left\{\frac{4a^2}{1-2\gamma}(\log t)^{1-2\gamma}\right\} \qquad \text{if} \quad \frac{1}{6} < \gamma < \frac{1}{4}.$$

II) Next we consider the equation (2.5) with $a = 0$, that is,

$$(|y'|^{\alpha-1}y')' + \frac{b\sin t}{t^\beta (\log t)^\gamma}|y|^{\alpha-1}y = 0, \qquad t \geq e, \tag{2.20}$$

where $b > 0$ is a constant. We suppose that $\beta \geq \alpha$. In this case we have

$$Q(t) = bt^{-\beta}(\log t)^{-\gamma}\cos t + O(t^{-\beta-1}(\log t)^{-\gamma}) \quad \text{as} \ \ t \to \infty,$$

and $t^\alpha Q(t) \to 0$ as $t \to \infty$, which implies that (2.20) possesses a slowly varying solution $y(t)$.

If $\beta > \alpha$, then, by taking $c(t) = 2bt^{\alpha-\beta}(\log t)^{-\gamma}$, we see that (1.26) and (1.27) are satisfied for all $n \in \mathbb{N}$, and so from Corollary 1.2 with $n = 1$ it follows that $y(t) \sim A_0$ as $t \to \infty$ since $[Q(t)]^{\frac{1}{\alpha}*}$ is integrable on $[e, \infty)$.

If $\beta = \alpha$, then $c(t) = 2b(\log t)^{-\gamma}$ satisfies (1.26) and (1.27) if and only if (2.10) holds. Consequently, if $\gamma > \alpha \max\left\{\frac{1}{2}, \frac{\alpha}{1+\alpha}\right\}$, then Corollary 1.2 is applicable to the c ase $n = 1$ and, using the conditional integrability of $[Q(t)]^{\frac{1}{\alpha}*}$ which is implied by that of $t^{-1}(\log t)^{-\frac{\alpha}{\gamma}}\cos t$, we conclude that $y(t) \sim A_0$ as $t \to \infty$. Furthermore, if γ satisfies (2.12), then from Corollary 1.2 with $n = 2$ we obtain (2.13), which, with the use of the fact

$$v_1(t) = \alpha \int_t^\infty |Q(s)|^{1+\frac{1}{\alpha}} ds = |b|^{1+\frac{1}{\alpha}} t^{-\alpha} (\log t)^{-\gamma\left(1+\frac{1}{\alpha}\right)} |\cos t|^{1+\frac{1}{\alpha}} \left(1 + O\left(\frac{1}{\log t}\right)\right)$$

as $t \to \infty$, yields the following asymptotic formula for $y(t)$:

$$y(t) \sim A' \exp\left\{\int_{t_0}^t \left[|b|^{1+\frac{1}{\alpha}} s^{-\alpha} (\log s)^{-\gamma\left(1+\frac{1}{\alpha}\right)} |\cos s|^{1+\frac{1}{\alpha}} + bs^{-\alpha}(\log s)^{-\gamma}\cos s\right]^{\frac{1}{\alpha}*} ds\right\}. \tag{2.21}$$

When specialized to the case $\alpha = 1$, (2.21) reduces to

$$y(t) \sim A_1(\log t)^{\frac{b^2}{2}} \qquad \text{if} \quad \gamma = \frac{1}{2}$$

$$y(t) \sim A_1 \exp\left\{\frac{b^2}{2(1-2\gamma)}(\log t)^{1-2\gamma}\right\} \qquad \text{if} \quad \frac{1}{3} < \gamma < \frac{1}{2},$$

which have been obtained in [5, p. 68]. Letting $\alpha = \frac{1}{3}$ in (2.21), an elementary calculation shows that

$$y(t) \sim A_2(\log t)^{\frac{3}{8}b^6} \qquad \text{if} \quad \gamma = \frac{1}{6}$$

$$y(t) \sim A_2 \exp\left\{\frac{3b^2}{8(1-6\gamma)}(\log t)^{1-6\gamma}\right\} \qquad \text{if} \quad \frac{1}{9} < \gamma < \frac{1}{6}.$$

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