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# CONVOLUTED $C$-COSINE FUNCTIONS AND CONVOLUTED $C$-SEMIGROUPS

M. KOSTIĆ

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*A b s t r a c t.* *Convoluted $C$-cosine functions and convoluted $C$-semigroups are introduced and investigated.Then their relations to the well-posedness of second order abstract Cauchy problem and the $\Theta$-convoluted $C$-regularized Cauchy problem are studied. Generation results for convoluted $C$-semigroups are given.*

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## 1. *Introduction*

The classical theory of $C_0$-semigroups of operators is generalized in many different directions. In 1987, Arendt obtained an extension of Widder's representation theorem for the Laplace transform in a Banach space. This version of Widder's theorem stimulated the development of the theory of integrated semigroups. Many new types of operator families in Banach spaces are defined on the basis of Arendt's ideas. Convoluted semigroups were introduced by Cioranescu and Lumer in 1994. In contrast to the case of integrated semigroups, whose resolvents have polynomial growth in some logarithmic regions, in the case of convoluted semigroup the resolvent exists in some smaller region and it may grow not faster than $Ce^{M(\lambda)}$, where $M(\lambda)$ is the associated function of a sequence $(M_n)$ which defines a space of ultradifferentiable functions. We also note that convoluted semigroups are closely related to ultradistribution semigroups of Komatsu.

On the other hand, $C$-semigroups were studied by many authors. In 1990, Tanaka and Okazawa ([18]) defined local $C$-semigroups and local integrated semigroups. The use of $C$-semigroups is a powerful method in studying ill-posed abstract Cauchy problems. Local $C$-cosine functions as well as integrated $C$-semigroups and integrated $C$-cosine functions were introduced later ; see [14] , [20] and [21].

Here, we deal with some new types of operator families in Banach spaces. They unify the classes of integrated $C$-cosine functions and integrated $C$-semigroups. The corresponding Cauchy problems

$$(ACP_2):\begin{cases} u\in C([0,\tau):D(A))\cap C^2([0,\tau):E),\\ u''(t)=Au(t),\\ u(0)=x, u'(0)=y,\end{cases}$$

and

$$(\Theta C):\begin{cases} u\in C([0,\tau):D(A))\cap C^1([0,\tau):E),\\ u'(t)=Au(t)+\Theta(t)Cx,\\ u(0)=0,\end{cases}$$

are considered. Convoluted $C$-semigroups are characterized by their asymptotic $\Theta C$-resolvents.

Throughout this paper $E$ denotes a complex Banach space and $L(E)$ denotes the space of bounded linear operators from $E$ into $E$. For a closed linear operator $A$, its domain, range and null space are denoted by $D(A)$, $R(A)$ and $N(A)$, respectively. We will assume in the sequel that $C\in L(E)$ is an injective operator with $CA\subset AC$. The $C$ resolvent set of $A$, $\rho_C(A)$, is the set of complex numbers $\lambda$ such that $\lambda-A$ is injective and $R(C)\subset R(\lambda-A)$.

## 2. *Definition and elementary properties of global convoluted $C$-cosine functions*

Throughout this section we will always assume that $K \in C([0,\infty) : \mathbb{C})$ is an exponentially bounded function such that $\tilde{K}(\lambda) \neq 0$ $(Re\lambda \geq \beta)$, where $\tilde{K}(\lambda)$ is the Laplace transform of $K(t)$.

**Notation:** $\Theta(t) := \int\limits_0^t K(s)ds$, $t \geq 0$.

The next definition is the convoluted version of [21], Chapter 1., Definition 4.1.

**Definition 2.1.** *Let $w \in \mathbb{R}$. If $(w^2, \infty) \subset \rho_C(A)$ and there exists a strongly continuous operator family $(C_K(t))_{t\geq 0}$, $(C_K(t) \in L(E), t \geq 0)$, such that $\|C_K(t)\| = O(e^{wt})$, $t \geq 0$, and*

$$\lambda(\lambda^2 - A)^{-1}Cx = \frac{1}{\tilde{K}(\lambda)} \int\limits_0^\infty e^{-\lambda t} C_K(t)x dt, \ \lambda > \max(w, \beta), x \in E,$$

*then it is said that $A$ is a subgenerator of a $K$-convoluted $C$-cosine function $(C_K(t))_{t\geq 0}$. The operator $\bar{A} := C^{-1}AC$ is called the generator of $(C_K(t))_{t\geq 0}$.*

**Remark.** Since $CA \subset AC$, one has that $\bar{A}$ is an extension of $A$. If $C = I$, a $K$-convoluted $C$-cosine function has a unique subgenerator which coincides with its generator. With $K(t) = \frac{t^{k-1}}{(k-1)!}$, $k \in \mathbb{N}$, one obtains the class of $k$-times integrated $C$-cosine functions. The generator of a $K$-convoluted $C$-cosine function is uniquely determined. From the previous definition it is also clear that $(C_K(t))_{t\geq 0}$ is a nondegenerate family, i.e.,

$$\text{if } C_K(t)x = 0 \text{ for all } t \geq 0 \text{ , then } x = 0.$$

**Theorem 2.2.** *Let $(C_K(t))_{t\geq 0}$ be a strongly continuous, exponentially bounded operator family and let $A$ be a closed operator. Then the statements (i) and (ii) are equivalent, where:*

*(i) $A$ is the generator of a $K$-convoluted $C$-cosine function $(C_K(t))_{t\geq 0}$,*

*(ii) a) $C_K(t)C = CC_K(t)$, $t \geq 0$,*

*b) $C_K(t)A \subset AC_K(t)$, $t \geq 0$,* *c)* $A\int\limits_0^t (t-s)C_K(s)x\,ds = C_K(t)x - \Theta(t)Cx,\ t \geq 0,\ x \in E.$

P r o o f. (i) $\Rightarrow$ (ii) Let $x \in E$. Clearly, $(\lambda^2 - A)^{-1}C^2x = C(\lambda^2 - A)^{-1}Cx$, for all sufficiently large $\lambda$. By Definition 2.1, the previous equality implies

$$\frac{1}{\tilde{K}(\lambda)}\int\limits_0^\infty e^{-\lambda t}C_K(t)Cx\,dt = \frac{1}{\tilde{K}(\lambda)}\int\limits_0^\infty e^{-\lambda t}CC_K(t)x\,dt.$$

Thus, a) follows from the uniqueness theorem for the Laplace transforms. Assume now $x \in D(A)$. One has

$$\lambda(\lambda^2 - A)^{-1}CAx = \frac{1}{\tilde{K}(\lambda)}\int\limits_0^\infty e^{-\lambda t}C_K(t)Ax\,dt,\ i.e.,$$

$$\lambda A(\lambda^2 - A)^{-1}Cx = \frac{1}{\tilde{K}(\lambda)}\int\limits_0^\infty e^{-\lambda t}C_K(t)Ax\,dt.$$

This easily implies

$$A\int\limits_0^\infty e^{-\lambda t}C_K(t)x\,dt = \int\limits_0^\infty e^{-\lambda t}C_K(t)Ax\,dt,$$

and b) follows from [21], Theorem 1.10. In order to prove c), we note that

$$\begin{aligned}
\mathcal{L}\left(\int\limits_0^t (t-s)C_K(s)x\,ds\right)(\lambda) &= \mathcal{L}(t)(\lambda)\mathcal{L}(C_K(t)x)(\lambda) \\
&= \frac{1}{\lambda^2}\tilde{K}(\lambda)\lambda(\lambda^2 - A)^{-1}Cx \\
&= \frac{\tilde{K}(\lambda)}{\lambda}(\lambda^2 - A)^{-1}Cx,
\end{aligned}$$

which implies

$$\begin{aligned}
A\left(\mathcal{L}\left(\int\limits_0^t (t-s)C_K(s)x\,ds\right)(\lambda)\right) &= \tilde{K}(\lambda)\lambda(\lambda^2 - A)^{-1}Cx - \frac{\tilde{K}(\lambda)}{\lambda}Cx \\
&= \mathcal{L}(C_K(t)x - \Theta(t)Cx)(\lambda).
\end{aligned}$$

Hence, 3) is a consequence of [21], Theorem 1.10.

(ii) $\Rightarrow$ (i) Suppose $\|C_K(t)\| \leq Me^{wt}$, $t \geq 0$, for some real number $w$ with $w \geq \beta$. Using b) and c), we have

$$\mathcal{L}(C_K(t)x)(\lambda) = \frac{\tilde{K}(\lambda)}{\lambda}Cx + \frac{1}{\lambda^2}A\mathcal{L}(C_K(t)x)(\lambda), \ i.e.,$$

$$(\lambda^2 - A)\mathcal{L}(C_K(t)x)(\lambda) = \lambda\tilde{K}(\lambda)Cx, \ \lambda > w, \ x \in E. \tag{1}$$

Hence, $R(C) \subset R(\lambda^2 - A)$, for $\lambda > w$. Let us show that $\lambda^2 - A$ is injective if $\lambda$ satisfies $\lambda > w$. Suppose $(\lambda^2 - A)x = 0$. It implies

$$C_K(t)x - \Theta(t)x = \int_0^t (t-s)C_K(s)Axds = \lambda^2 \int_0^t (t-s)C_K(s)xds, \ t \geq 0,$$

and consequently,

$$\mathcal{L}(C_K(t)x)(\lambda) = \frac{\tilde{K}(\lambda)}{\lambda}Cx + \lambda^2\mathcal{L}(t)\mathcal{L}(C_K(t)x)(\lambda) = \frac{\tilde{K}(\lambda)}{\lambda}Cx + \mathcal{L}(C_K(t)x)(\lambda),$$

for all sufficiently large $\lambda$. Thus, $Cx = 0$ and $x = 0$. By (1), it follows $(w^2, \infty) \subset \rho_C(A)$ and

$$\lambda(\lambda^2 - A)^{-1}Cx = \frac{1}{\tilde{K}(\lambda)} \int_0^\infty e^{-\lambda t}C_K(t)xdt, \ \lambda > w, x \in E.$$

Using the well-known Arendt-Widder theorem (see [3]), we obtain the following Hille-Yoshida type theorem:

**Theorem 2.3.** *(a) The following statements are equivalent:*

*(ai) $A$ is a subgenerator of an exponentially bounded $\Theta$-convoluted $C$-cosine function $(C_\Theta(t))_{t\geq 0}$ satisfying condition*

$$\lim_{\sigma\to 0} \sup_{h\leq\sigma} \frac{\|C_\Theta(t+h) - C_\Theta(t)\|}{h} \leq Me^{wt}, \ t \geq 0, \ \textit{for some } M > 0.$$

*(aii) There exists $a \geq w$ such that $(a^2, \infty) \subset \rho_C(A)$ and*

$$\left\| \frac{d^k}{d\lambda^k}[\lambda\tilde{K}(\lambda)(\lambda^2 - A)^{-1}C] \right\| \leq \frac{Mk!}{(\lambda - w)^{k+1}}, \ k \in \mathbb{N}_0, \lambda > a.$$

*(b) Assume that $A$ is densely defined. Then the following statements are equivalent:*

*(bi) $A$ is a subgenerator of an exponentially bounded $K$-convoluted $C$-cosine function $(C_K(t))_{t\geq 0}$ satisfying $\|C_K(t)\| \leq Me^{wt}$, $t \geq 0$, for some $M > 0$.*

*(bii) There exists $a \geq w$ such that $(a^2, \infty) \subset \rho_C(A)$ and*

$$\left\| \frac{d^k}{d\lambda^k}[\lambda \tilde{K}(\lambda)(\lambda^2 - A)^{-1}C] \right\| \leq \frac{Mk!}{(\lambda - w)^{k+1}},\ k \in \mathbb{N}_0, \lambda > a.$$

## 3. *Global convoluted $C$-semigroups*

**Definition 3.1.** *Let $\beta < w < \infty$. If $(w, \infty) \subset \rho_C(A)$ and there exists a strongly continuous operator family $(S_K(t))_{t\geq 0}$ such that $\|S_K(t)\| = O(e^{wt})$, $t \geq 0$, and*

$$(\lambda - A)^{-1}Cx = \frac{1}{\tilde{K}(\lambda)} \int_0^\infty e^{-\lambda t} S_K(t)x dt,\ \lambda > w, x \in E,$$

*then it is said that $A$ is a subgenerator of a $K$-convoluted $C$-semigroup $(S_K(t))_{t\geq 0}$.*

Note, Definition 3.1 implies that $(S_K(t))_{t\geq 0}$ is a nondegenerate family. If $C = I$ then Definition 3.1 means that $A$ is the generator of a global exponentially bounded convoluted semigroup $(S_K(t))_{t\geq 0}$.

Using the similar arguments as in the proofs of Theorem 2.2 and Theorem 2.3, one has the following results:

**Theorem 3.2.** *Let $(S_K(t))_{t\geq 0}$ be a strongly continuous, exponentially bounded operator family and let $A$ be a closed operator. Then the assertions (i) and (ii) are equivalent, where:*

*(i) $A$ is a subgenerator of $K$-convoluted $C$-semigroup $(S_K(t))_{t\geq 0}$,*

*(ii) a) $S_K(t)C = CS_K(t)$, $t \geq 0$,*

*b) $S_K(t)A \subset AS_K(t)$, $t \geq 0$,*

*c) $A\int_0^t C_K(s)xds = C_K(t)x - \Theta(t)Cx$, $t \geq 0$, $x \in E$.*

**Theorem 3.3.** *a) The following assertions are equivalent:*

*(ai) $A$ is a subgenerator of an exponentially bounded $\Theta$-convoluted $C$-semigroup $(S_\Theta(t))_{t\geq 0}$ satisfying condition*

$$\lim_{\sigma\to 0}\sup_{h\leq\sigma}\frac{\|S_\Theta(t+h)-S_\Theta(t)\|}{h}\leq Me^{wt},\ t\geq 0,\ \textit{for some } M>0.$$

*(aii) There exists $a\geq w$ such that $(a,\infty)\subset\rho_C(A)$ and*

$$\left\|\frac{d^k}{d\lambda^k}\left[\frac{1}{\tilde{K}(\lambda)}(\lambda-A)^{-1}C\right]\right\|\leq\frac{Mk!}{(\lambda-w)^{k+1}},\ k\in\mathbb{N}_0,\lambda>a.$$

*b) Assume that $A$ is densely defined. Then the following statements are equivalent:*

*(bi) $A$ is a subgenerator of an exponentially bounded $K$-convoluted $C$-semigroup $(S_K(t))_{t\geq 0}$ satisfying $\|S_K(t)\|\leq Me^{wt}$, $t\geq 0$, for some $M>0$.*

*(bii) There exists $a\geq w$ such that $(a,\infty)\subset\rho_C(A)$ and*

$$\left\|\frac{d^k}{d\lambda^k}\left[\frac{1}{\tilde{K}(\lambda)}(\lambda-A)^{-1}C\right]\right\|\leq\frac{Mk!}{(\lambda-w)^{k+1}},\ k\in\mathbb{N}_0,\lambda>a.$$

The next proposition relates a $K$-convoluted $C$-cosine function and a $K$-convoluted $C$-semigroup. Its proof can be obtained as in the case of integrated semigroups. We give it here for the sake of completness.

**Proposition 3.4.** *Suppose that $A$ and $-A$ are subgenerators of a $K$-convoluted $C$-semigroups,* $S_K$ and $V_K$, *respectively. Then $A^2$ is a subgenerator of a $K$-convoluted $C^2$-cosine function.*

P r o o f. It is not hard to verify that $\lambda^2-A^2$ is injective for all sufficiently large $\lambda\in\mathbb{R}$. Moreover, $R(C^2)\subset R(\lambda^2-A^2)$ and

$$\begin{aligned}(\lambda^2-A^2)^{-1}C^2x &= [(\lambda-A)^{-1}C][(\lambda+A)^{-1}Cx]\\ &= \frac{1}{2\lambda}[(\lambda+A)^{-1}Cx+(\lambda-A)^{-1}Cx]\\ &= (2\tilde{K}(\lambda))^{-1}\mathcal{L}(S_K(t)x+V_K(t)x)(\lambda),\\ &\quad x\in E,\lambda \text{ sufficiently large.}\end{aligned}$$

Hence,

$$\begin{aligned}\lambda(\lambda^2-A^2)^{-1}C^2x &= \frac{1}{\tilde{K}(\lambda)}\mathcal{L}\left(\frac{1}{2}\left(S_K(t)x+V_K(t)x\right)\right)(\lambda),\\ &\quad x\in E,\lambda \text{ sufficiently large.}\end{aligned}$$

In general, the converse of Proposition 3.4 is not true. For some special integrated semigroups ($E = L^p(\Omega)$, $\Omega \subset \mathbb{R}^n$ open) , the converse has been proved by Hieber.

## 4. *Local $K$-convoluted $C$-cosine functions*

Using Theorem 2.2, we define the class of local $K$-convoluted $C$-cosine functions as follows.

**Definition 4.1.** *Let $A$ be a closed operator and $K(\cdot)$ be a continuous function on $[0,\tau)$, $0<\tau\leq\infty$. If there exists a strongly continuous operator family $(C_K(t))_{t\in[0,\tau)}$ such that:*
*(i) $C_K(t)C = CC_K(t)$, $t\in[0,\tau)$,*
*(ii) $C_K(t)A \subset AC_K(t)$, $t\in[0,\tau)$,*
*(iii) for all $x\in E$ and $t\in[0,\tau)$:*

$$\int_0^t (t-s)C_K(s)x\,ds \in D(A), \text{ and}$$

$$A\int_0^t (t-s)C_K(s)x\,ds = C_K(t)x - \Theta(t)Cx,$$

*then $C_K$ is called a (local) $K$-convoluted $C$-cosine function and $A$ is called a subgenerator of $C_K$.*

Properties and examples of integrated $C$-cosine functions, which are not exponentially bounded, can be found in [20]. Clearly, if $A$ is a subgenerator of $C_K$, then for all $x\in D(A)$ function $t\mapsto C_K(t)x$ is continuously differentiable on $[0,\tau)$ and

$$\frac{d}{dt}C_K(t)x = \int_0^t C_K(s)Ax\,ds + K(t)Cx,\ t\in[0,\tau).$$

We say that a function $t\mapsto v(t)$ belonging to $C([0,\tau):E)$ is a $K$-convoluted mild solution of $(ACP_2)$ for $(x,y)\in E^2$ if for all $t\in[0,\tau)$, $\int_0^t (t-s)v(s)\,ds \in D(A)$ and

$$A\int_0^t (t-s)v(s)\,ds = v(t) - \Theta(t)x - \int_0^t \Theta(s)y\,ds,\ t\in[0,\tau).$$

Using the same arguments as in [20, Theorem 1.5], one has the following proposition.

**Proposition 4.2.** *Let $A$ be a subgenerator of a $K$-convoluted $C$-cosine function $(C_K(t))_{t\in[0,\tau)}$ and $x, y \in E$. If $K$ is a kernel, i.e.*

$$(\forall u \in C([0,\tau) : E)) \left( \int_0^t K(t-s)u(s)ds \equiv 0 \Rightarrow u \equiv 0 \right),$$

*then all $K$-convoluted mild solutions of $(ACP_2)$ for $(x, y)$ are unique.*

**Remark.** By Titchmarsh's theorem ([3],p.[106]), the condition $0 \in$supp$K$ implies that $K$ is a kernel.

**Proposition 4.3.** *Suppose that for all $x \in R(C)$ there exists a unique $K$-convoluted mild solution of $(ACP_2)$ for $(x, 0)$. Then $A$ is a subgenerator of a $K$-convoluted $C$-cosine function.*

P r o o f. Define $C_K(t)x := v(t)$, where $v(t)$ is the $K$-convoluted mild solution of $(ACP_2)$ for $(Cx, 0)$. The uniqueness of mild solutions implies that $(C_K(t))_{t\in[0,\tau)}$ is a strongly continuous family of linear operators satisfying $(iii)$ of Definition 4.1. It is easy to see that $t \mapsto CC_K(t)x$ is a $K$-convoluted mild solution of $(ACP_2)$ for $(C^2x, 0)$. As a consequence, we have $(i)$ of Definition 4.1. The proof of $(ii)$ of Definition 4.1 can be obtained as in [20, Theorem 1.5]. So, we have only to prove that $C_K(t)$, $t \in [0,\tau)$, is a bounded operator. We will follow the proof of [2, Proposition 2.3] with appropriate changes. Consider the mapping $\Phi : E \to C([0,\tau) : D(A))$, given by

$$\Phi(x)(t) = \int_0^t (t-s)C_K(s)xds, t \in [0,\tau), x \in E,$$

where $D(A)$ is equipped with the graph norm $\|x\|_A = \|x\| + \|Ax\|$, and $C([0,\tau) : D(A))$ is a Frechet space for the seminorms

$$p_n(v) := \sup_{t\in\left[0,\tau-\frac{1}{n}\right]} \|v(t)\|_{D(A)}, v \in C([0,\tau) : D(A)).$$

Clearly, $\Phi$ is a linear mapping. Let us show that $\Phi$ has a closed graph. Suppose $x_n \to x$, and $\sup\limits_{t\in\left[0,\tau-\frac{1}{n}\right]} \int_0^t (t-s)C_K(s)x_nds \to f(t)$, $n \to \infty$. It implies

$$Af(t) = \lim_{n\to\infty} A\int_0^t (t-s)C_K(s)x_n ds = \lim_{n\to\infty}[C_K(t)x_n - \Theta(t)Cx_n],\ t\in[0,\tau),$$

and

$$\lim_{n\to\infty} C_K(t)x_n = Af(t) + \Theta(t)Cx,\ t\in[0,\tau).$$

Using the dominated convergence theorem, we have

$$f(t) = \lim_{n\to\infty}\int_0^t (t-s)C_K(s)x_n ds = \int_0^t (t-s)[Af(s)+\Theta(s)Cx]ds,\ t\in[0,\tau).$$

So, $f(0)=f'(0)=0$, $f\in C^2([0,\tau):E)$ and

$$Af(t) = f''(t) - \Theta(t)Cx,\ t\in[0,\tau).$$

Hence, $A\int_0^t (t-s)v(s)ds = v(t) - \Theta(t)Cx$, $t\in[0,\tau)$, where $v = f''$. This implies $v(t) = C_K(t)x$, $t\in[0,\tau)$, and $f=\Phi(x)$. Hence, for all sufficiently large $n\in\mathbb{N}$ there exists $c_n$ such that

$$\left\| A\int_0^t (t-s)C_K(s)x ds \right\| \leq c_n\|x\|,\ x\in E,\ t\in[0,\tau-\frac{1}{n}).$$

Since $A\int_0^t (t-s)C_K(s)x ds = C_K(t)x - \Theta(t)Cx$, $x\in E$, $t\in[0,\tau)$, one can easily conclude that $C_K(t)\in L(E)$, $t\in[0,\tau)$.

**Lemma 4.4.** *Let $A$ be a subgenerator of a $K$-convoluted $C$-cosine function $(C_K(t))_{t\in[0,\tau)}$, $0<\tau\leq\infty$, and $H\in L^1_{loc}((0,\tau):\mathbb{C})$. Then $A$ is a subgenerator of $(H*_0K)$-convoluted $C$-cosine function $((H*_0C_K)(t))_{t\in[0,\tau)}$, where $*_0$ is the convolution like mapping given by*

$$f*_0 g(t) := \int_0^t f(t-s)g(s)ds.$$

P r o o f. Clearly, $((H *_0 C_K)(t))_{t\in[0,\tau)}$ is a strongly continuous operator family satisfying $(i)$ and $(ii)$ of Definition 4.1. Moreover,

$$A\int_0^t (t-s)\int_0^s H(s-r)C_K(r)xdrds$$
$$= A((id *_0 H *_0 C_K)(t))x = A((H *_0 id *_0 C_K)(t))x$$
$$= A\int_0^t H(t-s)\int_0^s (s-r)C_K(r)xdrds$$
$$= \int_0^t H(t-s)[C_K(s)x - \Theta(s)Cx]ds = (H *_0 C_K)(t)x - (H *_0 \Theta)(t)Cx,$$
$$x \in E, \quad t \in [0,\tau).$$

This completes the proof.

**Example 4.5.** *Let $E := L^p(\mathbb{R}^n)$, $1 < p < \infty$. A remarkable result of Hieber says that the Laplacian $\Delta$ with domain $H^{p,2}(\mathbb{R}^n)$ is the generator of an $\alpha$-times integrated cosine function if and only if $\alpha \geq (n-1)\left|\frac{1}{p} - \frac{1}{2}\right|$ (cf. [3], [8]). Hence, the Laplacian $\Delta$ is the generator of a $\left(H *_0 \frac{t^{\alpha-1}}{\Gamma(\alpha)}\right)$-convoluted cosine function on $E$ for any $H \in L^1_{loc}((0,\infty) : \mathbb{C})$ and $\alpha \geq \max\left\{(n-1)\left|\frac{1}{p} - \frac{1}{2}\right|, 1\right\}$.*

**Lemma 4.6.** *Suppose that $A$ is a subgenerator of a $K$-convoluted $C$-cosine function $(C_K(t))_{t\in[0,\tau)}$. Then for all $(x,y) \in R(C)\times R(C)$ there exists a $K$-convoluted mild solution of $(ACP_2)$ .*

P r o o f. Let $x = Cx_1$ and $y = Cy_1$. Using Lemma 4.4, it is easy to see that the function $v(t) := C_K(t)x_1 + \int_0^t C_K(s)y_1ds$ is a $K$-convoluted mild solution of $(ACP_2)$ for $(x,y)$.

By the foregoing we have the next proposition (see also [20]).

**Proposition 4.7.** *If $K$ is a kernel then the following statements are equivalent:*

*a) $A$ is a subgenerator of a $K$-convoluted $C$-cosine function.*

*b) for all $(x,y) \in R(C) \times R(C)$ there exists a unique $K$-convoluted mild solution of $(ACP_2)$ .*

5. *Basic properties of $K$-convoluted $C$-semigroups*

Let $0 < \tau \leq \infty$ and let $K$ be a continuous function on $[0, \tau)$ with $K \neq 0$.

**Definition 5.1.** *A strongly continuous operator family $(S_t)_{t \in [0,\tau)}$ is called a (local) $K$-convoluted $C$-semigroup on $E$ if it satisfies:*
*1) $S(0) = 0$, $S(t)C = CS(t)$, $\forall t \in [0, \tau)$.*
*2)* $$S(t)S(s)x = \left[\int_0^{t+s} - \int_0^{t} - \int_0^{s}\right] K(t+s-r)S(r)Cx\,dr,$$
$$x \in E,\ 0 \leq s, t,\ s+t < \tau.$$

$S(\cdot)$ is said to be nondegenerate if $S(t)x = 0$ for all $t \in [0, \tau)$ implies $x = 0$. For a nondegenerate $K$-convoluted $C$-semigroup we may define its generator $A$ via

$$A := \left\{ (x, y) \in E^2 : S(t)x - \Theta(t)Cx = \int_0^t S(s)y\,ds,\ \forall t \in [0, \tau) \right\}.$$

It is a closed linear operator in $E$.

Let us consider now the abstract $(\Theta C)$-problem.

**Definition 5.2.** *The $\Theta$-convoluted $C$-regularized Cauchy problem, $(\Theta C)$ in short, is well-posed if for all $x \in E$ there exists a unique solution of $(\Theta C)$. Here $D(A)$ is supplied with the graph norm $\|x\|_A = \|x\| + \|Ax\|$.*

We give now the result analogous to [3, Proposition 2.3]. The proof is similar, so it is omitted.

**Proposition 5.3.** *Let $\tau < \infty$. Suppose $(\Theta C)$ is well-posed. Then there exists a unique nondegenerate strongly continuous function $S : [0, \tau) \to L(E)$ such that for every $x \in E$, $\int_0^t S(s)x\,ds \in D(A)$ and*

$$A \int_0^t S(s)x\,ds = S(t)x - \Theta(t)Cx,\ t \in [0, \tau). \qquad (\bullet)$$

If $0 \leq t < \tau$ and $0 \leq s < t$ we have the following equality:

$$\Theta(s)\Theta(t-s) - \int_{t-s}^{t} K(t-r)\Theta(r)\,dr + \int_0^s K(t-r)\Theta(r)\,dr = 0.$$

It can be proved using the standard arguments. Now we can obtain the results analogous to [13, Proposition 2.4, Theorem 2.5]. The consideration is similar, so it is omitted. We have only to add that the last equality gives that the coeficient of $C^2x$ in the proof of [13, Proposition 2.4] equals zero.

**Proposition 5.4.** *Let $(S_t)_{t\in[0,\tau)}$ be a nondegenerate strongly continuous operator family such that $S(t)A \subset AS(t)$, $CS(t) = S(t)C$, $t \in [0,\tau)$. Suppose that for every $x \in E$, $\int_0^t S(s)xds \in D(A)$ and satisfies $(\bullet)$. Then:*

*1) $S(t)$ is a local $K$-convoluted $C$-semigroup whose generator is an extension of $A$.*

*2) If $K$ is a kernel then $(\Theta C)$ is well-posed.*

**Proposition 5.5.** *Suppose that $(\Theta C)$ is well-posed. Let $S$ be as in Proposition 5.3. Then:*

*1) $SC = CS$, $SA \subset AS$.*

*2) $S(t)S(s) = S(s)S(t)$, $0 \le s, t < \tau$.*

*3) $S(\cdot)$ is a local $K$-convoluted $C$-semigroup generated by $C^{-1}AC$.*

*4) For all $\lambda \in \rho_C(A)$ we have*

$$(\lambda - A)^{-1}CS(t) = S(t)(\lambda - A)^{-1}C,\ t \in [0,\tau).$$

**Remarks.** Putting $C = I$ in Proposition 5.4. we have [15, Theorem 1.3.4], which gives the semigroup property for local convoluted semigroups. Even if $K(t) = \frac{t^{k-1}}{(k-1)!}$, $k \in \mathbb{N}$, there exist examples of local integrated $C$-semigroups whose generators have empty $C$-resolvents ([13]). Combining Proposition 5.4 and Proposition 5.5, one has the following: if $A$ is a subgenerator of an exponentially bounded $K$-convoluted $C$-semigroup $(S_K(t))_{t\ge 0}$ in the sense of Definition 3.1, then $(S_K(t))_{t\ge 0}$ is a $K$-convoluted $C$-semigroup in the sense of Definition 5.1, and if additionally $K$ is a kernel, then the generator of $(S_K(t))_{t\ge 0}$ is $C^{-1}AC$.

The next proposition can be proved as in the case of integrated semigroups.

**Proposition 5.6.** *Let $(S_t)_{t\in[0,\tau)}$ be a (local) $K$-convoluted $C$-semigroup generated by $A$. If $x \in D(A^k)$ and $K \in C^{k-1}([0,\tau) : \mathbb{C})$ for some $k \in \mathbb{N}$ then we have*

$$\frac{d^k}{dt^k}S(t)x = S(t)A^kx + \sum_{i=0}^{k-1} K^{(i)}(t)CA^{k-1-i}x,\ t \in [0,\tau).$$

### 6. *The corresponding* $(\Theta C)$ *problem and generation results*

We start with

**Proposition 6.1.** *Suppose* $K \in C^k([0,\tau) : \mathbb{C})$, $k \in \mathbb{N}$. *Assume also that the problem* $(\Theta C)$ *is well-posed for* $A$. *Then for all* $x \in D(A^{k+1})$ *there exists a unique solution of*

$$(\Theta C_k): \begin{cases} u \in C^1([0,\tau) : E) \cap C([0,\tau) : D(A)) \\ u'(t) = Au(t) + \frac{d^k}{dt^k} K(t)Cx \\ u(0) = \sum\limits_{i=0}^{k-1} K^{(i)}(0)A^{k-1-i}Cx. \end{cases}$$

P r o o f. Let $(S_t)_{t\in[0,\tau)}$ be as in Proposition 5.3. One can easily verify that

$$u(t) := \int\limits_0^t S(s)A^{k+1}x ds + \sum_{i=0}^{k} \Theta^{(i)}(t)A^{k-i}Cx;\ t \in [0,\tau),\ x \in D(A^{k+1}),$$

is the solution of $(\Theta C_k)$. The uniqueness follows from the well-posedness of $(\Theta C)$ at $x = 0$.

Of course, if $a \in L^1_{(loc)}([0,\tau) : \mathbb{C})$ and if the problem $(\Theta C)$ is well-posed then the problem $(a *_0 \Theta, C)$ is well-posed. The semigroups $S_K$ and $S_{a*_0K}$ satisfy $S_{a*_0K} = a *_0 S_K$.

**Theorem 6.2.** *Let* $k \in \mathbb{N}\backslash\{1\}$ *and let* $K$ *satisfies* $K \in C^{k+1}([0,\tau) : \mathbb{C})$ *and* $K^{(i)}(0) = 0$, $0 \leq i \leq k-2$. *If* $A$ *is a closed linear operator with* $\lambda_0 \in \rho(A)$ *and if for all* $x \in D(A^{k+1})$ *there exists a unique solution of* $(\Theta C_k)$ *then:*

*1)* $(\Theta C)$ *is well-posed for* $A$.

*2)* $C^{-1}AC$ *generates a* $K$*-convoluted* $C$*-semigroup on* $[0,\tau)$*; here* $\tau < \infty$.

P r o o f. Let $z = Cy$, $y \in D(A^{k+1})$. Define $u_1(t) := (\lambda_0 - A)\int\limits_0^t u_y(s)ds$, $t \in [0,\tau)$, where $u_y$ is the solution of $(\Theta C_k)$ for $y$. One can simply verify that $u_1$ is a solution of

$$(\Theta C_{k-1}) : \begin{cases} u \in C^1([0,\tau) : E) \cap C([0,\tau) : D(A)) \\ u'(t) = Au(t) + \frac{d^{k-1}}{dt^{k-1}} K(t)x \\ u(0) = 0, \end{cases}$$

for $x = (\lambda_0 - A)z$. Moreover, $(\Theta C_{k-1})$ has a solution for all $x \in (\lambda_0 - A)CD(A^{k+1})$.

Similarly, the problem

$$(\Theta C_{k-2}) : \begin{cases} u \in C^1([0,\tau) : E) \cap C([0,\tau) : D(A)) \\ u'(t) = Au(t) + \frac{d^{k-2}}{dt^{k-2}} K(t)x \\ u(0) = 0 \end{cases}$$

has a solution for all $x \in (\lambda_0 - A)^2 CD(A^{k+1})$ (we can take $u_2(t) := (\lambda_0 - A) \int_0^t u_1(s)ds$).

By induction we obtain a solution $u_{k+1}$ of

$$(\Theta C_{-1}) : \begin{cases} u \in C^1([0,\tau) : E) \cap C([0,\tau) : D(A)) \\ u'(t) = Au(t) + \Theta(t)x \\ u(0) = 0, \end{cases}$$

for all $x \in (\lambda_0 - A)^{k+1} CD(A^{k+1})$. Since $R(C) \subset (\lambda_0 - A)^{k+1} CD(A^{k+1})$ we have proved 1), because the uniqueness follows from the well-posedness of $(\Theta C_k)$ at $x = 0$. The rest of proof is clear by Proposition 5.5.

**Remarks:** 1) We have proved the converse of [13, Remark 2.6, (d)]. Our results in this section also generalize [3, Proposition 3.3]. See also ([21], p. [36]).

Propositions 6.1, 5.4, 5.5 and Theorem 6.2 yield immediately

**Theorem 6.3.** *Let $\tau$, $K$ and $k$ be as in Theorem 6.2. If $A$ is a closed linear operator with $\lambda_0 \in \rho(A)$, then the following statements are equivalent:*

*1) $(\Theta, I)$ is well-posed for $A$ on $[0,\tau)$.*
*2) $(\Theta^{(k)}, R(\lambda_0 : A)^k)$ is well-posed for $A$ on $[0,\tau)$.*

If $K$ is a kernel then *1*) and *2*) are equivalent to:

*1)' A* generates a $K$-convoluted semigroup on $[0,\tau)$.

*2)' A* generates a $K^{(k)}$-convoluted $R(\lambda_0 : A)^k$-semigroup on $[0,\tau)$, respectively.

For the similar results see [15, Theorem 1.4.8] or [21, Theorem 6.7].

Let $\tau < \infty$. Assume that $(\Theta C)$ is well-posed. Define

$$L_\gamma(\lambda) := \int_0^\gamma e^{-\lambda s} S(s)ds;\ \gamma \in [0,\tau),\ \lambda \in [0,\infty),$$

where $S(\cdot)$ is given by Proposition 5.3. We collect some properties of the operators $L_\gamma(\lambda)$ in the next proposition. The proof is analogous to [13, Proposition 5.1] and so it can be omitted.

**Proposition 6.4.** *Let $x \in E$. Then*

*1) The function $\lambda \to L_\gamma(\lambda)x$ belongs to $C^\infty([0,\infty) : E)$ and there exists $M_\gamma$ such that*

$$\left\| \frac{\lambda^n}{(n-1)!} \frac{d^{n-1}}{d\lambda^{n-1}} L_\gamma(\lambda) \right\| \le M_\gamma,\ \forall \lambda \ge 0,\ n \in \mathbb{N}.$$

*2) $L_\gamma(\lambda)$ commutes with $C$ and $A$.*

*3) $(\lambda - A)L_\gamma(\lambda) = -e^{-\lambda\gamma} S(\gamma)x + \int_0^\gamma e^{-\lambda s} K(s)Cxds$.*

*4) $L_\gamma(\lambda)L_\gamma(\eta) = L_\gamma(\eta)L_\gamma(\lambda)$.*

A family $\{L_\gamma(\lambda) : \gamma \in [0,\tau), \lambda \ge 0\} \subset L(E)$ is called **asymptotic $\Theta C$-resolvent** for $A$ if there exists a strongly continuous operator family $(V_t)_{t\in[0,\tau)}$ such that 1), 2) and 4) hold and 3) holds with $S(\gamma)$ replaced by $V(\gamma)$.

Using the same arguments as in [13, Theorem 5.2, Corollary 5.3], we have the following results.

**Theorem 6.5.** *Let $A$ be a closed operator. Assume that $A$ has an asymptotic $\Theta C$-resolvent $\{L_\gamma(\lambda) : \gamma \in [0,\tau), \lambda \ge 0\}$. If $K$ is a kernel then $\left(\int_0^t \Theta(s)ds, C\right)$ is well-posed.*

**Proposition 6.6.** *Suppose $D(A)$ is dense in $E$ and $K$ is a kernel. Then $(\Theta C)$ is well-posed for $A$ on $[0,\tau)$ if and only if $A$ has an asymptotic*

$\Theta C$*-resolvent* $\{L_\gamma(\lambda) : \gamma \in [0, \tau), \lambda \geq 0\}$.

It is clear that the characterizations for local $K$-convoluted $C$-semigroups are generalizations of the characterizations for local integrated $C$-semigroups, and hence, for local $C$-cosine functions ([13]).

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Institute of Mathematics
University of Novi Sad
Trg Dositeja Obradovića 4
21000 Novi Sad
Serbia and Montenegro
e-mail: makimare@neobee.net