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QUASIASYMPTOTIC EXPANSION OF DISTRIBUTIONS WITH APPLICATIONS TO THE STIELTJES TRANSFORM

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A b s t r a c t. *Following the approach of Drozinov, Vladimirov and Zavialov we investigate the quasiasymptotic expansion of distributions and give Abelian type results for the ordinary asymptotic behaviour of the distributional modified Stieltjes transform of a distribution with appropriate quasiasymptotic expansion*

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1. *Introduction*

In the papers [9, 10] authors followed the definition of the distributional Stieltjes transform given in [7] which enabled them to use the strong theory of the space of tempered distributions $\mathcal{S}'$. In fact, they generalized slightly the definition of Lavoine and Misra. Using the notion of the quasiasymptotic behaviour of distributions from $\mathcal{S}'_+ = \{f \in \mathcal{S}', \text{supp} f \subset [0, \infty)\}$, introduced by Zavialov in [15], they obtained more general results than in [6, 7, 2] for the asymptotic behaviour of the distributional Stieltjes transform at ∞ and 0^+.

McClure and Wong [3, 14] studied the asymptotic expansion of the generalized Stieltjes transform of some classes of locally integrable functions characterized by their expansions at ∞ and 0^+.

Our approach to the asymptotic expansion of the distributional modified Stieltjes transform which we study in this paper is quite different from the approach given in [3, 14].

In the first part of the paper we give the definition of the quasiasymptotic expansion at ∞ of a distribution from $\mathcal{S}'_+$ given in [4, p.385]. Also in [4] is given the definition of the qiasi-asymptotic expansion at 0^+ of an element from $\mathcal{S}'_+$. In this paper we give the definition of space $\mathcal{M}'(r)$, Stieltjes transformation, Modified Stieltjes transformation T_{r+1}, and Generalized Modified Stieltjes transformation $\overline{T_{r+1}}$. This enables us to obtain, in the second part of the paper, the asymptotic expansion at ∞ and at 0^+ of the modified Stieltjes transforms of appropriate distributions from $\mathcal{S}'_+$.

Domains in [14] and in this paper on which the modified Stieltjes transform is defined do not contain each other.

Notation: As usually R, C, N are the spaces of real, complex and natural numbers; $N_0 = N \cup \{0\}$, $\mathcal{S}'_+$ is the space of tempered distributions with supports in the $[0, \infty)$. The space of rapidly decreasing functions is denoted by $\mathcal{S}$ and by $\mathcal{S}_m$, $m \in N_0$ its closure with the norm

$$\gamma_m(\varphi) = \sup\{(1 + |t|^2)^{m/2}|\varphi^{(i)}(t)|, \ \ t \in \mathbb{R}^n, \ |i| \le m\},$$

$\mathcal{S}'$ denotes the space of all distributions of slow growth.

A positive continuous function L defined on $(0, \infty)$ is called slowly varying at ∞ (0^+) if for every $k > 0$

$$\lim_{t\to\infty} \frac{L(kt)}{L(t)} = 1, \qquad \left(\lim_{t\to 0^+} \frac{L(kt)}{L(t)} = 1\right).$$

We denote by $\sum\limits_{\infty}(\sum\limits_{0^+})$ the set of all slowly varying functions at $\infty(0^+)$. For the properties of slowly varying functions we refer the reader to [11].

If L is a slowly varying function at $\infty(0^+)$, then for every $\varepsilon > 0$ there is $A_\varepsilon > 0$ such that $x^{-\varepsilon} < L(x) < x^{\varepsilon}$ $(x^{-\varepsilon} > L(x) > x^{\varepsilon})$ if $x > A_\varepsilon$ $(0 < x < A_\varepsilon)$.

This property of L and corresponding properties of $\mathcal{S}_m$ ([12, p. 93]) imply the following assertion: Let $G \in L^1_{loc}$, supp$G \subset [0, \infty)$, $\alpha > -1$, and $G(x) \sim x^\alpha L(x)$ as $x \to \infty$ $(x \to 0^+)$. Then $G(kx)/(k^\alpha L(k)) \to x^\alpha_+$, $k \to \infty$, in $\mathcal{S}'_+$ for $\alpha+1>0$. Recall, for $\alpha>-1$, $x_+^\alpha = H(x)x^\alpha$; H is Heaviside's function. (The symbol $\sim$ is related to the ordinary asymptotic behaviour).

The following scale of distributions from $\mathcal{S}'_+$ has been used in investigations of the quasiasymptotic behaviour of distributions.

$$f_{\alpha+1} = \begin{cases} \dfrac{Ht^\alpha}{\Gamma(\alpha+1)}, & \alpha > -1 \\ \\ D^n f_{\alpha+n+1}, & \alpha \le -1, \;\; \alpha+n > -1, \end{cases} \tag{1.1}$$

where D is the distributional derivative.

2. *Definitions*

2.1. *Definition of q.a.e.*

Definition 1. *The quasiasymptotic behaviour of distribution (q.a.b) at infinity. If T be a distribution in $\mathcal{S}'_+$ such that the distributional limit*

$$\lim_{k\to\infty} \frac{T(kx)}{\rho(k)} = \gamma(x)$$

exists in $\mathcal{S}'$ $(\gamma \neq 0)$, then T is called to have the quasiasymptotic behaviour at infinity related to the regularly varying function $\rho(k) = k^\alpha L(k)$ with the limit γ; we write this as $T \overset{q}{\sim} \gamma$ in $\mathcal{S}'$ as $x \to \infty$.

Here ρ is regularly varying at infinity and the limit $\gamma \in \mathcal{S}'_+$, is of the form $\gamma(x) = Cf_{\alpha+1}(x)$.

We shall repeat in this section some well known facts about the quasi-asymptotic behaviour from [13].

Let $f \in \mathcal{S}'_+$. It is said that f has the q.a.b. at $\infty(0^+)$ with the limit $g \neq 0$ with respect $k^\alpha L(k), L \in \Sigma_\infty((1/k)^\alpha L(1/k),\ L \in \Sigma_0),\ \alpha \in R$, if

$$\lim_{k\to\infty} \Big\langle \frac{f(kt)}{k^\alpha L(k)}, \phi(t) \Big\rangle = \langle g(t), \phi(t) \rangle, \ \phi \in \mathcal{S}$$

$$\Big(\lim_{k\to\infty} \Big\langle \frac{f(t/k)}{(1/k^\alpha)L(1/k)}, \phi(t) \Big\rangle = \langle g(t), \phi(t) \rangle, \ \phi \in \mathcal{S} \Big).$$

Let us notice that in [10] is reformulated the definition of the q.a.b. at 0^+ from [13].

We need in the paper the following Structural theorem (for the q.a.b. at ∞, see [13] and for the q.a.b. at 0^+, see [10]).

Structural theorem Let $f \overset{q}{\sim} g$ at ∞ (0^+) with respect to $k^\alpha L(k)$ $((1/k)^\alpha L(1/k))$. Then there exist $F \in L^1_{loc}$, $\text{supp}F \subset [0,\infty)$, $C \neq 0$ and $m \in N_0$, $m+\alpha > -1$, such that

$$f = D^m F,\ F(k) \sim Ck^{m+\alpha}L(k),\ \ k \to \infty$$
$$(F(1/k) \sim C(1/k)^{m+\alpha}L(1/k),\ \ k \to \infty). \tag{1.2}$$

We remark that the q.a.b. at 0^+ is a local property while the q.a.b. at ∞ is a global property of an $f \in \mathcal{S}'_+$.

For the quasiasymptotic Expansion of Distributions we extend slightly the definitions of the closed and open quasiasymptotic expansion, in short the q.a.e. at ∞, given in [4] and using the same idea we give the definition of the q.a.e. at 0^+.

Definition 2. *Let $\alpha \in R$ and $L \in \sum\limits_{\infty}$ $(L \in \sum\limits_{0})$. We put*

$$(f_L)_{\alpha+1} = \begin{cases} H(t)L(t)\cdot t^\alpha/\Gamma(\alpha+1), & \alpha > -1 \\ D^n(f_L)_{\alpha+n+1}, \alpha < -1, & \alpha+n > -1, \end{cases} \tag{1.3}$$

where n is the smallest natural number such that $\alpha + n > -1$. Obviously, $(f_L)_{\alpha+1} \overset{q}{\sim} f_{\alpha+1}$ at ∞ (0^+) with respect to $k^\alpha L(k)$ $((1/k)^\alpha L(1/k))$.

Definition 3. *An $f \in \mathcal{S}'_+$ has the closed q.a.e. at $\infty\,(0^+)$ of order $(\alpha, L) \in R \times \sum\limits_{\infty}$ $((\alpha, L) \in R \times \sum\limits_{0})$ and of length ℓ, $0 \le \ell < \infty$, with respect to $k^{\alpha-\ell}L_0(k)$ $((1/k)^{\alpha+\ell} \times L_0(1/k))$ if f has the q.a.b. it $\infty\,(0^+)$ with respect to $k^\alpha L(k)$ and if there exist $\alpha_i \in R$, $L_i \in \sum\limits_{\infty}$ $(L_i \in \sum\limits_{0^+})$, $c_i \in C$, $i = 1,2,...,N$, $\alpha_1 \ge \alpha_2 \ge \alpha_N$ $(\alpha_1 \le \alpha_2 \le ... \le \alpha_N)$ and that f is of the form*

$$f(t) = \sum_{i=1}^{N} c_i (f_{L_i})_{\alpha_i+1}(t) + h(t) \tag{1.4}$$

such that

$$\begin{cases} \lim\limits_{x\to\infty} \left\langle \dfrac{h(kt)}{k^{\alpha-\ell}L_0(k)}, \phi(t) \right\rangle = 0, \ \ \phi \in \mathcal{S} \\ \left(\lim\limits_{x\to\infty} \left\langle \dfrac{h(k/t)}{(1/k)^{\alpha+\ell}L_0(1/k)}, \phi(t) \right\rangle = 0, \ \ \phi \in \mathcal{S} \right). \end{cases} \tag{1.5}$$

Obviously, we can (and we shall) assume that $c_i \neq 0,\ i = 1, ..., N$, and that $\alpha_N \geq \alpha - \ell$ ($\alpha_N \leq \alpha + \ell$). Since the sum of the two slowly varying functions is the slowly varying one, we can and we shall always assume that in the representation (1.4) $\alpha_1 > \alpha_2 > ... > \alpha_N (\alpha_1 < \alpha_2 < ... < \alpha_N)$. Namely $(f_{L_j})_{\beta+1} + (f_{L_k})_{\beta+1} = (f_{L_j+L_k})_{\beta+1}$. $(f_{L_j})_{\beta_1+1}$ and $(f_{L_k})_{\beta_2+1}$ have the same q.a.b. at $\infty(0^+)$ iff $\beta_1 = \beta_2$ and $L_j \sim L_k$. So, we have

Proposition 1. *Let $f \in \mathcal{S}'_+$ satisfy conditions of above definition and assume that there are two representation of f*

$$f(t) = \sum_{i=1}^{N} c_i (f_{L_i})_{\alpha_i+1} + h(t)$$

$$f(t) = \sum_{i=1}^{M} \tilde{c}_i (f_{L_i})_{\tilde{\alpha}_i+1} + \tilde{h}(t)$$

for which all the assumptions given above hold. Then $M = N$,
$\alpha_1 = \tilde{\alpha}_1 ... \alpha_N = \tilde{\alpha}_N,\ L_1 \sim \tilde{L}_1 ... L_N \sim \tilde{L}_N,\ \ \alpha_1 = \alpha,\ \ L_1 \sim L.$

We shall use the following notation for in $f \in \mathcal{S}'_+$ in Definition 3.

$$f \overset{q.e}{\sim} \sum_{i=1}^{N} c_i (f_{L_i})_{\alpha_i+1} \text{ at } \infty(0^+) \text{ of order } (\alpha, L) \text{ with respect to}$$

$$k^{\alpha-\ell} L_0(k)((1/k)^{\alpha+\ell} L_0(1/k)).$$

2.2. *Space $\mathcal{M}'(r)$*

We extend the definition of the space $\mathcal{J}'(r)$ given in [10] and using the same idea we give the definition of space $\mathcal{M}'(r)$.

$\mathcal{M}'(r),\ r \in R \setminus (-N)$ denotes the space of all generalized functions $f \in \mathcal{S}'_+(R)$ such that there exist $k \in N_0$ and a locally integrable function F, $\mathrm{supp} F \subset [0, \infty)$. so that f is of the form

$$f = t^{-r} D^k F, \tag{2.1.}$$

where F is continuous on $[0, \infty)$ and

$$\int_R |F(t)| (t + \beta)^{r-1-k} dt < \infty, \quad \text{for } \beta > 0, \tag{2.2}$$

More precisely $\mathcal{M}'(r)$ is the dual space for i.e.

$$\mathcal{M}'(r) = \{\psi \in C^\infty \setminus (0,\infty);\ \psi = t^r \varphi;\ \ \psi \in \mathcal{S}([0,\infty))\}$$

with the multinorm
$\|\psi\|_\alpha = \sup\{t^r \varphi^{(i)}(t)(1+t)^k;\ i \leq k,\ t \in [0,\infty),\ k \in N_0\}$.

The Stieltjes transformation $S_r(f)(s),\ r \in R \setminus (-N)$ is complex valued function, defined by

$$S_r(f(t))(s) = \int_0^\infty \frac{f(t)}{(s+t)^{r+1}}\,dt. \tag{2.3}$$

$$s \in C \setminus (-\infty, 0],\ \ 0 < t < \infty,\ \ r \in R \setminus (-N)$$

Modified Stieltjes transformation introduced by Marichev is defined as

$$(T_\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_0^\infty \left(1 + \frac{x}{y}\right)^{-\alpha} \frac{1}{y}\, f(y)dy,\ \ x \in C \setminus (-\infty, 0], \tag{2.4}$$

$0 < y < \infty,\ \ \alpha \in R \setminus (-N)$. It can be written as

$$T_\alpha(f)(x) = \frac{1}{\Gamma(\alpha)} \int_0^\infty \frac{y^{\alpha-1} f(y)}{(x+y)^\alpha}\,dy,\ \ x \in C \setminus (-\infty, 0]. \tag{2.5}$$

Now, we shall find relation between (2.3) and (2.5). Putting $r = \alpha - 1,\ f(t) = y^{\alpha-1} f(y)$ in (2.3) we get,

$$S_{\alpha-1}(y^{\alpha-1} f(y))(x) = \int_0^\infty \frac{y^{\alpha-1} f(y)}{(x+y)^\alpha}\,dy,\ \ x \in C \setminus (-\infty, 0]. \tag{2.6}$$

By (2.5) and (2.6), we have

$$T_\alpha(f)(x) = \frac{1}{\Gamma(\alpha)}\, S_{\alpha-1}(y^{\alpha-1} f(y))(x),\ \ x \in C \setminus (-\infty, 0].$$

Interchanging x by z and α by $r+1$, it follows that

$$\Gamma(r+1) T_{r+1}(f)(z) = S_r(y^r f)(z),\ \ z \in C \setminus (-\infty, 0]. \tag{2.7}$$

2.3. *Modified Stieltjes transformation* T_{r+1}

Let us define the T_{r+1}-transformation $r \in R \setminus (-N)$. Assume that $f \in \mathcal{M}'(r)$ that is $f = t^{-r} D^k F$ and $\int_R |F(t)|(t+\beta)^{-r-1-k} dt < \infty$, for $\beta > 0$.

Modified Stieltjes transformation is defined.

$T_{r+1}(f),\ r \in R \setminus (-N)$ is complex valued function defined by

$$\Gamma(r+1) T_{r+1}(f)(s) = (r+1)_k \int_0^\infty \frac{F(t)}{(s+t)^{r+1+k}}\, dt,$$

$$r \in R \setminus (-N), \quad s \in C \setminus (-\infty, 0], \quad 0 < t < \infty.$$

2.4. *Generalized Modified Stieltjes Transformation* $\tilde{T}_{r+1}$

The $\tilde{T}_{r+1}$-transformation of a distribution $f \in \mathcal{S}'_+(R)$ is complex valued function $\tilde{T}_{r+1}(f)$ defined by

$$\Gamma(r+1)\tilde{T}_{r+1}(f)(s) = \lim_{w \to \infty} \langle f(t),\ \eta(t)(s+t)^{-r-1} \exp(-wt) \rangle.$$

$$w \in R, \quad s \in \Lambda \subset (C \setminus (-\infty,]), \quad \eta \in A(s).$$

Here Λ is the set of complex numbers for which this limit exists; $A(s)$ is the family of all smooth functions, defined on R for which there exists $\varepsilon = \varepsilon_{\eta,s} > 0$ such that $0 \le \eta(t) \le 1,\ t \in R,\ \eta(t) = 1$ if t belongs to the ε-neighbourhood of $\overline{R}_+,\ \eta(t) = 0$ if it belongs to the complement of the 2ε-neighbourhood of $\overline{R}_+$, where $\varepsilon > 0$ is arbitrary if $Im\, s \ne 0$ and $0 < 2\varepsilon < \max \mathrm{Re} s$, if $Im\, s = 0$. Moreover, assume $|D^p \eta(t)| \le Cp,\ t \in R$. If $\eta(t) \in A(s),\ s \in (C \setminus \overline{R}_-)$, then $\eta(t)(s+t)^{-r-1} \exp(-wt) \in \mathcal{S}(R)$ for $w \in R_+,\ r \in R$.

3. *Main results*

For the main results of this section we need the following assertion from [10].

Theorem 1. *Let $f \in \mathcal{S}'_+$, and $f = D^m F$. Then if f has the q.a.b at ∞ with respect to $k^\alpha L(k)$, it follows that*

$$\frac{F(kt)}{k^{\alpha+m} L(k)} \to C f_{\alpha+m+1} \ \ in\ \mathcal{S}'_p \ \ for\ p > \alpha + m + 1 \ \ as\ k \to \infty. \qquad (3.1)$$

In the case of q.a.b. at 0^+

$$\frac{F(t/k)}{(1/k)^{\alpha+m}L(1/k)} \to Cf_{\alpha+m+1} \ \ \textit{in } \mathcal{S}'_p \ \ \textit{for } p > \alpha+m+1 \ \ \textit{as } k \to \infty.$$

If $f \in \mathcal{M}'(r)$, and $f = t^{-r}D^m F$ then $F \in \mathcal{S}'_{r+m+1}$. For given $z \in C\setminus(-\infty,0]$, we denote by $A(z)$ the space of all $\eta \in C^\infty$, such that $\eta \in A(z)$, if $0 \leq \eta \leq 1$, $\eta(t) = 1$ for $t > -\epsilon, \eta(t) = 0$ for $t < -2\epsilon$, where $\varepsilon > 0$ is arbitrary if $z \notin (0,\infty)$ and $z \in (0,\infty)$ then we choose ε such that $0 < 2\varepsilon < z$. Clearly, for a given $z \in C\setminus(-\infty,0]$ and every $\eta \in A(z)$

$$R \ni t \to \eta(t)(t+z)^{-r-m-1} \in S_p \text{ for } p < r+m+1. \tag{3.2}$$

Let $f \in \mathcal{M}'(r)$. we have $(x > 0, t > 0)$,

$$\Gamma(r+1)(T_{r+1}f)(tx) = x(r+t)\int_0^\infty \Gamma(r+2)(T_{r+2}f)(xu)du$$

and if

$$\Gamma(r+2)(T_{r+2})(x) \sim x^{-(r-\alpha)-1}L(x) \tag{3.3}$$

as $x \to \infty (x \to 0^+)$, then

$$\Gamma(r+1)(T_{r+1}f)(x) \sim \frac{(r+1)}{(1-\alpha)}x^{-(r-\alpha)}L(x) \text{ as } x \to \infty (x \to 0^+).$$

Now we are ready to prove.

Theorem 2. *Let f have the closed q.a.e at ∞ of order (α, L) and of length ℓ related to $k^{\alpha-\ell}L_0(k)$ (see the notation in Definition 3). Let $r > \alpha, r \in R\setminus(-N)$. Then*

(i) $f \in \mathcal{M}'(r), (f_L)_{\alpha+1} \in \mathcal{M}'(r)$;

(ii) If we put $\Gamma(r+1)T_{r+1}(f_L)_{\alpha+1}(x) = S_{\alpha,L}(x)$, *then for* $L \sim \tilde{L}, S_{\alpha,L}(x) \sim S_{\alpha,\tilde{L}}(x) \sim \dfrac{\Gamma(r-\alpha)}{\Gamma(r+1)}x^{a-r}L(x), x \to \infty$;

(iii) $$\Gamma(r+1)(T_{r+1}f)(x) - C\frac{\Gamma(r-\alpha)}{\Gamma(r+1)}x^{\alpha-r}L(x) = 0(x^{\alpha-\ell-r}L_0(x)), \tag{3.4}$$ $x \to \infty$.

P r o o f. We shall prove the theorem by using the similar idea in the proof of the main theorem in [9]. Obviously, (i) follows from (1.2).

(ii) Let $\beta < r-1, x \in R, L \in \sum_\infty$. Let m be the smallest element from N_0 such that $\beta + m > -1$. Then

$$\Gamma(r+1)(T_{r+1}(f_L)_{\beta+1})(x) = (r+1)_m \int_0^\infty \frac{f_{\beta+m+1}(t)L(t)}{(x+t)^{r+m+1}}dt$$

$$= (r+1)_m < f_{\beta+m+1}(t)L(t), \frac{\eta(t)}{(x+t)^{r+m+1}} >, \eta \in A(x),$$

where $< f_{\beta+m+1}(t)L(t), \frac{\eta(t)}{(x+t)^{r+m+1}} >$ is observed as a pair from (S'_{r+m}, S_{r+m}). Obviously this pair does not depend on $\eta \in A(x)$. Since $r+m > \beta+m+1$, we have

$$\frac{\Gamma(r+1)T_{r+1}(f_L)_{\beta+1}(kx)}{k^{\beta-r}L(k)} =$$

$$= (r+1)_m \Big\langle f_{m+\beta+1}L(t), \frac{\eta(t)}{k^{\beta+m+1}L(k)(x+(t/k))^{r+m+1}} \Big\rangle$$

$$= (r+1)_m \Big\langle \frac{f_{\beta+m+1}(kt)L(kt),}{k^{\beta+m}L(k)}, \frac{\eta(kt)}{(x+t)^{r+m+1}} \Big\rangle.$$

If $k \to \infty$ from equation (3.1) it follows

$$\frac{\Gamma(r+1)T_{r+1}(f_L)_{\beta+1}(kx)}{k^{\beta-r}L(k)} \to \Big\langle f_{\beta+m+1}(t), \frac{\eta(t)}{(x+t)^{r+m+1}} \Big\rangle =$$

$$= \frac{(r+1)_m}{\Gamma(\beta+m+1)} \int_0^\infty \frac{t^{\beta+m}}{(x+t)^{r+m+1}} \cdot dt = \frac{\Gamma(r-\beta)}{\Gamma(r+1)} x^{\beta-r}.$$

On putting $x=1$ we obtain that (ii) holds for all $\alpha < r-1$. Let us suppose that $r-1 \le \beta < r$. Then by the same arguments given above, we have

$$\Gamma(r+2)T_{r+2}(f_L)_{\beta+1}(x) \sim \frac{\Gamma(r+1-\beta)}{\Gamma(r+2)} x^{\beta-r-1}L(x), x \to \infty.$$

Now we completed the proof of result (ii).

(iii) We can assume that $\alpha < r-1$ because if $r-1 \le \alpha < r$ we have as in (ii), to observe $\Gamma(r+2)T_{r+2}(f_L)_{\beta+1}$ firstly and after that to use (3.3). Since $f - C(f_L)_{\alpha+1} \in S'_{r+m}$, equation (3.1) implies that

$$\left\{\frac{\Gamma(r+1)(T_{r+1}f)(xk) - \mathcal{C}(\Gamma(r+1)T_{r+1}(f_L)_{\alpha+1})(kx)}{x^{\alpha-\ell-r}L_0(x)}\right\}$$

$$=< \{f(kt) - C(f_L)(kt)\}/(k^{\alpha-\ell}L_0(k)), \eta(t)(x+t)^{-r-m-1} > \to 0$$

as $k \to \infty$. On putting $x = 1$ the assertion (iii) follows. The similar assertion holds for closed q.a.e. at 0^+ but with more restrictive assumptions.

Theorem 3. *Let f have the closed q.a.e. at 0^+ of order (α, L) and of length ℓ with respect to $(1/k)^{\alpha+\ell}L_0(1/k)$. If $(\alpha+\ell) < r$ and $f \in \mathcal{M}'(r)$, then*

$$\Gamma(r+1)(T_{r+1}f)(x) - \mathcal{C}\frac{\Gamma(r-\alpha)}{\Gamma(r+1)}x^{\alpha-r}L(x) = 0(x^{\alpha+\ell-r}L_0(x)), x \to 0, \quad (3.5)$$

P r o o f. The proof of this theorem is very similar to the proof of Theorem 2. We only notice that we must observe firstly $S_{r+1}f$ and after that to use (3.3). From $f \in M'(r)$ we have $F \in S'_{r+m+1}$ and this implies that we have to observe the dual pair $(S'_{r+m+1}S_{r+m+1})$ $\quad(\eta(t)(x+t)^{-r-m-2} \in S_{r+m+1}$ as a function of $t)$.

4. *The uniform Behaviour of* $T_{r+1}(f)$

Let F be a continuous function with supp $F \subset [0,\infty), r > \alpha > -1$ and $F(x) \sim x^\alpha$ as $x \to \infty$. Denote by $\lambda_{a,\epsilon}, a > 0, \epsilon > 0$, a subset of C defined by $\Lambda_{a,\epsilon} = \{a + \mathrm{R}e^{i\phi}; \mathrm{R} \geq 0, -\pi+\varepsilon \leq \phi \leq \pi - \varepsilon\}$. If $z = a + \mathrm{R}e^{i\phi} \in \Lambda_{a,\varepsilon}$ and $t \in [0,\infty)$, then we have

$$|z+t|^{r+1} > (\frac{1-\cos\phi}{2})^{(r+1)/2}(\mathrm{R}+a+t)^{r+1}. \quad (4.1)$$

This follows from the elementary inequalities:

$$\begin{aligned}(a+t)^2 + 2(a+t)\mathrm{R}\cos\phi + \mathrm{R}^2 &\geq (a+t)^2 - 2(a+t)\mathrm{R}\cos\varepsilon + \mathrm{R}^2\\ &= (a+t)^2 + \mathrm{R}^2 + ((a+t)^2+\mathrm{R}^2)\cos\varepsilon - (a+t+\mathrm{R})^2\cos\varepsilon\\ &\geq ((a+t)^2+\mathrm{R}^2)(1+\cos\varepsilon) - 2((a+t)^2+\mathrm{R}^2)\cos\varepsilon\\ &= ((a+t)^2+\mathrm{R}^2)(1-\cos\varepsilon).\end{aligned}$$

Assumptions on F imply $F(x) < C(1+x^\alpha), x \geq 0$ For $z \in \Lambda_{a,\epsilon}$ and suitable C_1

$$\Big|\int_0^\infty \frac{F(t)}{(z+t)^{r+1}}dt\Big| \leq C(\frac{2}{1-\cos\varepsilon})^{(r+1)/2}\int_0^\infty \frac{(1+t^\alpha)}{(\mathrm{R}+a+t)^{r+1}}dt$$

$$\leq C_1\Big[\frac{1}{r}+\frac{\Gamma(r-\alpha)\Gamma(\alpha+1)}{\Gamma(r+1)}\Big]\Big[\frac{1}{a+\mathrm{R}}\Big]^{r-\alpha}. \tag{4.2}$$

So, we have proved the following

Lemma 1. *Let F satisfy the conditions given above. The function $z^{r-\alpha}\Gamma(r+1)(T_{r+1}F)(z)$ is bounded in $\Lambda_{a,\epsilon}, a>0, \epsilon>0$.*

We use this lemma for the proof the following theorem.

Theorem 4. *Let f satisfy the conditions of Theorem 2 and let all the slowly varying functions in Theorem 2 be equal to 1. Then*

i) $A_{f,r}(z)=\Gamma(r+1)(T_{r+1}f)(x)-C\dfrac{\Gamma(r-\alpha)}{\Gamma(r+1)}\dfrac{z^{\alpha-r}}{z^{\alpha-\ell-r}}$ is bounded analytic function in any $\Lambda_{a,\varepsilon},\ a>0,\varepsilon>0$;

(ii) $A_{f,r}(z)$ converges uniformly to zero in $\Lambda_{a,\varepsilon}$ when $|z|\to\infty$.

P r o o f: (i) It follows from the Structural theorem (cf. 2.1) and Lemma 1.

(ii) It follows from (i) and Theorem 2 which enable us to use the Montel theorem $[1, p.5]$.

Theorem 5. *Let f satisfy the conditions of Theorem 3 and let all the solwly varying functions in Theorem 3 be equal to 1. Let*

$$A_{f,r}(z)=\{\Gamma(r+1)(T_{r+1}f)(z)-\frac{\Gamma(r-\alpha)}{\Gamma(r+1)}z^{\alpha-r}\}/z^{\alpha+\ell-r}.$$

Then

(i) $A_{f,r}(z)$ is a bounded function in $\Lambda_{0,\varepsilon}\cap B(0,R_0),\ \varepsilon>0,\ R_0>0$, where $B(0,R_0)=\{z;|z|<R_0\}$;

(ii) $A_{f,r}(z)$ converges uniformly to zero in $\Lambda_{0,\varepsilon}$ when $|z|\to 0$.

For the proof of this theorem, we need

Lemma 2. *Let $F\in L^1_{loc}, \mathrm{supp}F\subset[0,\infty), r>\alpha>-1$,*

$$F(x)\sim x^\alpha, x\to 0^+,\ \textit{and}\ \int_0^\infty |F(t)(z+t)^{-r-1}|dt<\infty,$$

$z\in\Lambda_{0,\varepsilon}\cap B(0,R_0)$. Then $z^{r-\alpha}\Gamma(r+1)(T_{r+1}F)(z)$ is bounded in $\Lambda_{0,\varepsilon}\cap B(0,R_0),\ R_0>0$.

P r o o f of Theorem 4:

(i) From the Structural theorem and Lemma 2 it follows that $A_{f,r}(z)$ is bounded in $\Lambda_{0,\varepsilon} \cap B(0, R_0)$.

(ii) Let $\tilde{A}_{f,r}(z) = A_{f,r}(1/z), z \in C \setminus (-\infty, 0]$. The function $A_{f,r}(w), w \in \Lambda_{0,\varepsilon} \cap \{w : |w| > 1/R_0\}, \varepsilon > 0, R_0 > 0$ is analytic and bounded. As well, we have $\tilde{A}_{f,r}(x) \to 0$ as $x \to \infty$. This implies that the same assertions hold for $A_{f,r}$ in the domain $\Lambda_{a,\epsilon} \cap \{w; |w| > 1/R_0\}, a > 0, R_0 > 0$. So by the Montel theorem it follows that $\tilde{A}_{f,r}(z)$ converges uniformly to 0, in $\Lambda_{a,\varepsilon}$ as $|z| \to \infty$. Further on this implies that $A_{f,r}(z)$ converges uniformly to 0 in $\Lambda_{0,\varepsilon}$ as $|z| \to \infty$ and so that the assertion (ii) holds.

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