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THE CIRCLE AND DIVISOR PROBLEM

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A b s t r a c t. New proofs for the classical bounds

$$P(x) \ll x^{1/3}, \quad \Delta(x) \ll x^{1/3}\log x$$

are given. Here $P(x)$ denotes the error term in the classical circle, and $\Delta(x)$ in the classical divisor problem.

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Recently S.D. Miller and W. Schmid [7] gave a proof of the bound

$$P(x) \;\ll_\varepsilon\; x^{1/3+\varepsilon} \tag{1}$$

in the classical circle problem. Here as usual $P(x) = \sum_{n\le x} r(n) - \pi x$, $r(n)$ is the number of representations of the natural number n as a sum of two integer squares, and $\varepsilon > 0$ denotes arbitrarily small numbers. The bound (1) was obtained by the use of the Voronoi summation formula

$$\sum_{a\le n\le b}{}' r(n)f(n) = \pi\int_a^b f(x)\,\mathrm{d}x + \sum_{n=1}^\infty r(n)\int_a^b f(x)J_0(2\pi\sqrt{xn}\,)\,\mathrm{d}x, \tag{2}$$

where $f(x)$ is a suitable smooth function, J is the Bessel function, and $\sum'$ denotes that at $n = a$ and $n = b$ the summand is to be halved if a or b is an integer. In [7], (2) was proved by a two-dimensional Poisson summation formula, but it may be proved analogously like the classical Voronoi formula (see e.g., [2, Chapter 3]). Then, on p. 20, the authors say: "With more effort, one can remove ε from these bounds (i.e. (1), and the analogous bound in the divisor problem), and get Voronoi's result

$$P(x) \ll x^{1/3}.\text{''} \tag{3}$$

The aim of this note is to give a new, simple proof of (3) by using (2). We start by noting that, in view of the non-negativity of $r(n)$, we have

$$\sum_{n=1}^{\infty} f_-(n)r(n) \le \sum_{X<n\le 2X} r(n) \le \sum_{n=1}^{\infty} f_+(n)r(n), \tag{4}$$

where $f_-(x)$ is a smooth, non-negative function supported in $[X, 2X]$ such that $f(x) = 1$ for $x \in [X + G,\, 2X - G]$ $(X^\varepsilon \le G \le \sqrt{X})$, while similarly $f_+(x)$ is supported in $[X-G,\, 2X+G]$ and satisfies $f(x) = 1$ for $x \in [X, 2X]$. If henceforth we denote by $f(x)$ either $f_-(x)$ or $f_+(x)$, then $f^{(r)}(x) \ll_r G^{-r}$ $(r = 0, 1, 2, \dots)$, and by (2) we have

$$\sum_{n=1}^{\infty} f(n)r(n) = \pi X + O(G) + \sum_{n=1}^{\infty} r(n) \int_{X-G}^{2X+G} f(x)J_0(2\pi\sqrt{xn})\,\mathrm{d}x. \tag{5}$$

From the theory of Bessel functions we need only the relation (see e.g., N.N. Lebedev [6])

$$\frac{\mathrm{d}}{\mathrm{d}z}\,[z^\nu J_\nu(z)] = z^\nu J_{\nu-1}(z) \tag{6}$$

and the asymptotic expansion ($k \in \mathbb{N}$ is arbitrary, but fixed, and $|\arg z| \le \pi - \varepsilon$)

$$\begin{aligned} J_\nu(z) \;&= \left(\frac{2}{\pi z}\right)^{1/2} \cos(z - \tfrac{1}{2}\nu\pi - \tfrac{1}{4}\pi) \left(\sum_{j=0}^{k} c_j(\nu) z^{-2j} + O(|z|^{-2k-2})\right) \\ &\quad + \left(\frac{2}{\pi z}\right)^{1/2} \sin(z - \tfrac{1}{2}\nu\pi - \tfrac{1}{4}\pi) \left(\sum_{j=1}^{k} d_j(\nu) z^{1-2j} + O(|z|^{-2k-1})\right), \end{aligned} \tag{7}$$

with suitable constants $c_j(\nu)$, $d_j(\nu)$. By using (7) and the first derivative test (see e.g., [2, Lemma 2.1]) it is seen that the integral on the right-hand side of (5) is $\ll X^{1/4}n^{-3/4}$, hence for $Y \geq 2$ we have

$$\sum_{n\le Y} r(n)\int_{X-G}^{2X+G} f(x)J_0(2\pi\sqrt{xn})\,\mathrm{d}x \ll \sum_{n\le Y} r(n)X^{1/4}n^{-3/4} \ll (XY)^{1/4}$$

on using partial summation and $\sum_{n\le x} r(n) \ll x$. Furthermore, by using (6) (with $\nu = 1, 2$), performing two integrations by parts and noting that the support of f'' has measure $\ll G$, we obtain that

$$\begin{aligned}&\sum_{n>Y} r(n)\int_{X-G}^{2X+G} f(x)J_0(2\pi\sqrt{xn}\,)\,\mathrm{d}x\\ &= \sum_{n>Y} \frac{r(n)}{\pi^2 n}\int_{X-G}^{2X+G} f''(x)xJ_2(2\pi\sqrt{xn}\,)\,\mathrm{d}x\\ &\ll \sum_{n>Y} r(n)n^{-5/4}G^{-1}X^{3/4} \ll X^{3/4}G^{-1}Y^{-1/4}.\end{aligned}$$

Therefore (5) yields

$$\begin{aligned}\sum_{n=1}^{\infty} f(n)r(n) &= \pi X + O(G) + O((XY)^{1/4}) + O(X^{3/4}G^{-1}Y^{-1/4})\\ &= \pi X + O(X^{1/3})\end{aligned}$$

on choosing $G = Y = X^{1/3}$. From (4) and the above estimates we have then

$$\sum_{X<n\le 2X} r(n) = \pi X + O(X^{1/3}), \tag{8}$$

and (3) follows from (8) on replacing X by $2^{-j}X$ and summing over $j = 1, 2, \ldots$.

An analogous reasoning gives also the classical bound

$$\Delta(X) \ll X^{1/3}\log X, \quad \Delta(X) = \sum_{n\le X} d(n) - X(\log X + 2\gamma - 1),$$

in the Dirichlet divisor problem, where $d(n)$ is the number of divisors of n and $\gamma = -\Gamma'(1)$ is Euler's constant. This follows with the use of the corresponding Voronoi formula (see [2, Chapter 3]) for the divisor function, instead of (2). Namely we have

$$\sum_{a\le n\le b}{}' d(n)f(n) = \int_a^b (\log x + 2\gamma)f(x)\,\mathrm{d}x + \sum_{n=1}^\infty d(n)\int_a^b f(x)\alpha(xn)\,\mathrm{d}x, \quad (9)$$

where $0 < a < b < \infty$, $f(x) \in C^2[a,b]$, and in standard notation of Bessel functions

$$\alpha(x) = 4K_0(4\pi\sqrt{x}) - 2\pi Y_0(4\pi\sqrt{x}) \quad (10)$$

$$= -\sqrt{2}x^{-1/4}\Big(\sin(4\pi\sqrt{x}-\pi/4) - (32\pi)^{-1}\cos(4\pi\sqrt{x}-\pi/4)\Big) + O(x^{-5/4}).$$

In case we are dealing with a Voronoi formula for a multiplicative function which is not necessarily non-negative, then often one can use the theorem of P. Shiu [8] on sums of multiplicative functions in short intervals. For example, such is the function $a(n)$, the n-th Fourier coefficient of a holomorphic cusp form with respect to the full modular group, which is a normalized eigenfunction for the Hecke operators (see e.g., M. Jutila [4] for Voronoi-type formulas for these functions). More generally, Voronoi-type formulas for smooth functions f can be obtained for a wide class of arithmetic functions (e.g., the so-called $\mathcal{S}$-class of A. Selberg; see Kaczorowski-Perelli [5] for an comprehensive account). The author [3] obtained such a formula for the divisor function $d_r(n)$, generated by $\zeta^r(s)$, $r \in \mathbb{N}$. The key to obtaining Voronoi-type formulas in such situations is the relation

$$\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} F(s)G(s)\,\mathrm{d}s = \int_0^\infty f(x)g(x)\,\mathrm{d}x$$

which holds (see e.g., E.C. Titchmarsh [9]) under suitable conditions if $F(s)$ and $G(s)$ are Mellin transforms of $f(x)$ and $g(x)$, respectively. The ideas used in this note in proving (1) work well also in the general case, enabling one to get bounds for corresponding error terms without the "ε"-factor. The simple (explicit) form of the right-hand side of (2) (and (9)-(10)) is due to the fact that

$$\frac{2^{s-p-1}\Gamma(\frac12 s)}{\Gamma(p-\frac12 s+1)} \qquad (0 < \Re\mathrm{e}\, s = \sigma < p + \tfrac32)$$

is the Mellin transform of $x^{-p}J_p(x)$, and a corresponding Mellin pair exists also in the case of the Voronoi formula (9). In the general case the function appearing in the summation formula is a 'generalized' Bessel function (see e.g., J.L. Hafner [1]), and its asymptotics may be found by the method developed in Hafner's paper or by the author in [3].

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