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GRAPH CONNECTIVITY AND WIENER INDEX

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A b s t r a c t. The graphs with a given number n of vertices and given (vertex or edge) connectivity k, having minimum Wiener index are determined. In both cases this is $K_k + (K_1 \cup K_{n-k-1})$, the graph obtained by connecting all vertices of the complete graph K_k with all vertices of the graph whose two components are K_{n-k-1} and K_1.

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1. *Introduction*

Let G be a connected graph whose vertex and edge sets are $V(G)$ and $E(G)$, respectively. The *distance* between two vertices u and v, denoted by $d_G(u,v)$, is the length of a shortest path between u and v. The *Wiener index* of G, denoted by $W(G)$, is defined by

$$W(G) = \sum_{u,v \in V(G)} d_G(u,v) \ .$$

Details on the extensive (mathematical) research on the Wiener index can be found in the reviews [2, 3] and in the references cited therein. It should be noted that the Wiener index has significant applications in chemistry [5, 7].

Among connected graphs of order n, the path P_n has maximum Wiener index [4]. Therefore in the class of 1-connected and 1-edge–connected graphs of order n, P_n has the maximal W-value. Plesnik [6] showed that in the class of 2-connected and 2-edge–connected graphs of order n, the cycle C_n has maximal W-value. To the authors' best knowledge, other relations between Wiener index and graph connectivity have not been established so far. Our aim is to contribute towards filling this gap, and to determine the minimum Wiener indices of graphs with prescribed number of vertices and connectivity, as well as to characterize the corresponding graphs.

Recall that if G is a connected graph on n vertices, different from the complete graph K_n, then the *connectivity* (or more precisely: the *vertex–connectivity*) of G is equal to k if all subgraphs of G, obtained by deleting from G fewer than k vertices are connected, and a subgraph obtained by deleting from G exactly k vertices is disconnected. One also says that G is k-connected. The vertex–connectivity–concept is not applicable to the complete graph. If k is the connectivity of G, then $1 \leq k \leq n-2$.

Analogously, if G is a connected graph on n vertices, then the *edge–connectivity* of G is equal to k if all subgraphs of G, obtained by deleting from G fewer than k edges are connected, and a subgraph obtained by deleting from G exactly k edges is disconnected. If k is the edge–connectivity of G, then $1 \leq k \leq n-1$, with $k = n-1$ if and only if $G = K_n$.

Let G and H be two graphs with $V(G) \cap V(H) = \emptyset$. By $G \cup H$ we denote the disjoint union of G and H. The *join* of G and H, denoted by $G + H$, is the graph with vertex set $V(G+H) = V(G) \cup V(H)$ and edge set $E(G+H) = E(G) \cup E(H) \cup \{uv | u \in V(G), v \in V(H)\}$.

For terminology and notations not defined here, we refer to the book [1].

2. *Minimum Wiener Index of Graphs with Connectivity or Edge–connectivity k*

Let G be a connected graph on n vertices. It is clear that the Wiener index is minimal if and only if $G = K_n$, in which case, $W(G) = n(n-1)/2$. In what follows we investigate when a graph with a given vertex– or edge–connectivity has minimum Wiener index.

Theorem 1. *Let G be a k-connected, n-vertex graph, $1 \leq k \leq n-2$. Then*

$$W(G) \geq \frac{1}{2}\, n(n+1) - (k+1) \ .$$

Equality holds if and only if $G = K_k + (K_1 \cup K_{n-k-1})$.

P r o o f. Let G_{min} be the graph that among all graphs on n vertices and connectivity k has minimum Wiener index. Since the connectivity of G_{min} is k, there is a vertex–cut $X \subset V(G_{min})$, such that $|X| = k$. Denote the components of $G_{min} - X$ by $G_1, G_2, \ldots, G_\omega$. Then each of the subgraphs $G_1, G_2, \ldots, G_\omega$ must be complete. Otherwise, if one of them would not be complete, then by adding an edge between two nonadjacent vertices in this subgraph we would arrive at a graph with the same number of vertices and same connectivity, but smaller Wiener index, a contradiction.

It must be $\omega = 2$. Otherwise, by adding an edge between a vertex from one component and a vertex from another component $G_1, G_2, \ldots, G_\omega$, if $\omega > 2$, then the resulting graph would still have connectivity k, but its Wiener index would decrease, a contradiction. So, $G_{min} - X$ has two components G_1 and G_2. By a similar argument, we conclude that any vertex in G_1 and G_2 is adjacent to any vertex in X.

Denote the number of vertices of G_1 by n_1 and that of G_2 by n_2. Then $n_1 + n_2 + k = n$ and by direct calculation we get

$$\begin{aligned} W(G_{min}) &= \frac{1}{2}\, n_1(n_1-1) + \frac{1}{2}\, n_2(n_2-1) + \frac{1}{2}\, k(k-1) + k(n_1+n_2) + 2\, n_1\, n_2 \\ &= \frac{1}{2}\, n^2 + \left(k - \frac{1}{2}\right) n + \frac{1}{2}\, k(k-1) + n_1\, n_2 \end{aligned}$$

which for fixed n and k is minimum for $n_1 = 1$ or $n_2 = 1$. This in turn means that $G_{min} = K_k + (K_1 \cup K_{n-k-1})$. Direct calculation yields

$$W(G_{min}) = \frac{1}{2}\, n(n+1) - (k+1)$$

which completes the proof. □

The edge–connectivity version for Theorem 1 is also valid. Here the case $k = n-1$ needs not be considered, since the only $(n-1)$-edge–connected graph is K_n.

Theorem 2. *Let G be a k-edge–connected, n-vertex graph,* $1 \leq k \leq n-2$. *Then*

$$W(G) \geq \frac{1}{2}\, n(n-1) + (n-k-1) \ .$$

Equality holds if and only if $G = K_k + (K_1 \cup K_{n-k-1})$.

P r o o f. Let now G_{min} denote the graph that among all graphs with n vertices and edge-connectivity k has minimum Wiener index. Let X be an edge–cut of G_{min} with $|X| = k$. Then $G_{min} - X$ has two components, G_1 and G_2. Both G_1 and G_2 must be complete graphs. Let $|V(G_1)| = n_1$ and $|V(G_2)| = n_2$, $n_1 + n_2 = n$.

Denote the set of the end-vertices of the edges of X in G_1 by S, and that in G_2 by T. Let $|V(G_1 - S)| = a_1$ and $|V(G_2 - T)| = a_2$.

There are

$$\frac{1}{2}\, n_1(n_1 - 1) + \frac{1}{2}\, n_2(n_2 - 1) + k = |E(G_{min})|$$

pairs of vertices at distance 1, and $a_1\, a_2$ pairs of vertices at distance of 3. All other vertex pairs, namely

$$\binom{n}{2} - |E(G_{min})| - a_1\, a_2$$

are at distance 2. Consequently,

$$\begin{aligned} W(G_{min}) &= \left[\frac{1}{2}\, n_1(n_1 - 1) + \frac{1}{2}\, n_2(n_2 - 1) + k\right] \\ &\quad + 2\left[\binom{n}{2} - \left(\frac{1}{2}\, n_1(n_1 - 1) + \frac{1}{2}\, n_2(n_2 - 1) + k\right) - a_1\, a_2\right] + 3\,[a_1\, a_2] \\ &= \frac{1}{2}\, n(n-1) - k + n_1\, n_2 + a_1\, a_2 \end{aligned}$$

which for fixed n and k is minimum for $n_1 = 1$, $a_1 = 0$ or $n_2 = 1$, $a_2 = 0$. This, as before, implies $G_{min} = K_k + (K_1 \cup K_{n-k-1})$. $\square$

3. *Discussion*

Theorems 1 and 2 establish that the graph with n vertices, connectivity k, and minimum Wiener index is same in the case of vertex– and edge–connectivity. One may wonder whether Theorem 1 implies Theorem 2, or vice versa. It appears (at least within the present considerations) that the proofs of these two theorems are independent.

As already mentioned, the 1– and 2-connected graphs with maximum Wiener indices are known. The natural question at this point is to ask for k-connected ($k \geq 2$), n-vertex graphs having maximum W. This problem seems to be much more difficult, and, at this moment, we cannot offer any solution of it, not even for the case $k = 3$.

Another related question is whether n-vertex, k-vertex–connected and n-vertex, k-edge–connected graphs with maximum W differ at all, and if yes, for which values of k and n.

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