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POSITIVITY IN TWISTED CONVOLUTION ALGEBRA AND FOURIER MODULATION SPACES[1]

J. TOFT

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A b s t r a c t. Let $\mathcal{W}^{p,q}$ be the Fourier modulation space $FM^{p,q}$ and let $*_\sigma$ be the twisted convolution. If $a \in D'$ such that $(a *_\sigma \varphi, \varphi) \ge 0$ for every $\varphi \in C_0^\infty$, and $\chi \in S$ such that $\chi(0) \neq 0$, then we prove that $\chi a \in \mathcal{W}^{p,\infty}$ iff $a \in \mathcal{W}^{p,\infty}$. We also present some extensions to the case when weighted Fourier modulation spaces are used.

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0. *Introduction*

The aim of the paper is to discuss positivity in the twisted convolution algebra (the $*_\sigma$-algebra) in background of Fourier modulation spaces. At the same time we give a new proof of Bochner-Schwartz theorem in the case of twisted convolutions. (See also Proposition 2.8 in [To3].)

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A motivation for studying positivity and algebraic properties in the $*_\sigma$-algebra is the close relation for such properties between the $*_\sigma$-algebra, operator theory and pseudo-differential calculus. Such questions were briefly investigated in [To1], [To2] and [To3]. In the present paper we continue the analysis in [To3] in terms of Fourier modulation spaces. The modulation spaces were introduced in time-frequency analysis by Feichtinger in [Fe]. Later on they have been used in certain problems in pseudo-differential calculus as well. (See [G], [L], [Ta], [Te] and [To5] and the references therein.) A reason for this is that informations concerning regularity as well as growing and decay properties can be easily obtained when using such spaces.

In order to be more specific, we give some necessary definitions. Let W be a symplectic vector space of dimension $2n < \infty$ with the symplectic form σ. (The reader who is not familar with symplectic vector spaces may consider W as $\mathbf{R^{2n}}$ and $\sigma(X,Y) = \langle y,\xi\rangle - \langle x,\eta\rangle$ when $X = (x,\xi) \in \mathbf{R^{2n}}$ and $Y = (y,\eta) \in \mathbf{R^{2n}}$.)

Then the twisted convolution $*_\sigma$ is defined by the formula

$$(a *_\sigma b)(X) \equiv (2/\pi)^{n/2} \int a(X-Y)b(Y)e^{2i\sigma(X,Y)}\, dY\ , \tag{0.1}$$

when $a, b \in L^1(W)$. Here and in what follows we use the standard notation for the usual functions and distribution spaces, see e. g., [H]. The definition of $*_\sigma$ extends in different ways. It extends for example to a continuous and bilinear mapping from $D'(W) \times C_0^\infty(W)$ to $D'(W)$.

We are concerned with the set $S'_+(W)$ of positive elements in the $*_\sigma$-algebra, i. e., the set of all $a \in D'(W)$ such that $(a *_\sigma \varphi, \varphi) \geq 0$ for every $\varphi \in C_0^\infty(W)$. Here $(a,\varphi) \equiv \langle a, \overline{\varphi}\rangle$, where $\langle\,\cdot\,,\,\cdot\,\rangle$ denotes the duality between elements in appropriate function and distribution spaces, and their duals. (A motivation for using S'_+ instead of D'_+ is given by Corollary 2.4 below.)

It might seem hard to find common structures for positivity results in the $*_\sigma$-algebra, but there are indeed such ones. In fact, in many situations the following principle holds. (See [To3].)

Assume that $a \in S'_+$. If a satisfies a certain regularity or boundedness property at the origin, then a and its Fourier transform $\widehat{a}$ satisfy the same regularity or boundedness property everywhere.

In the present paper we prove that this principle holds when certain types of Fourier modulation spaces are involved.

1. *Preliminaries*

In this section we make some necessary preparations. First we discuss some algebraic and continuous properties for the twisted convolution. From the definitions it follows that any L^p-estimate which are valid when usual convolutions are involved, also holds if these convolutions are replaced by twisted convolutions. For example, Young's inequality holds for twisted convolution. Moreover, many algebraic properties which are true for the usual convolution also hold for the twisted convolution. For example, if $a, b, c \in S(W)$, then

$$a *_\sigma (b *_\sigma c) = (a *_\sigma b) *_\sigma c, \quad (a *_\sigma b, c) = (a, c *_\sigma \widetilde{b}),$$

where $\widetilde{b}(X) = \overline{b(-X)}$.

In contrast to the usual convolution there is a canonical one to one correspondence between the twisted convolution algebra and operator algebras on $L^2(V)$, leading to that further properties are available for the twisted convolution algebra. For example, if $p \in [1, 2]$, then any $L^p(W)$ is an algebra under twisted convolution. (See Corollary 1.4.3 in [To1] or Proposition 1.4 in [To2].)

Proposition 1.1 below is a restatement of certain results in [To2] and [To3]. (See also [To1].) The proof is therefore omitted. Here $F_\sigma a$ denotes the symplectic Fourier transform which is defined by the formula

$$(F_\sigma a)(X) = \pi^{-n} \int a(Y) e^{2i\sigma(X,Y)} \, dY, \quad a \in L^1(W).$$

Then F_σ is a homeomorphism on $S(W)$ which in a usual way extends to a homeomorphism on $S'(W)$, and to a unitary operator on $L^2(W)$.

Proposition 1.1. *The following are true:*

1. *the map* $(\varphi, a, \psi) \mapsto \varphi *_\sigma a *_\sigma \psi$ *is sequentially continuous from* $S(W) \times S'(W) \times S(W)$ *to* $S(W)$*, and from* $C_0^\infty(W) \times D'(W) \times C_0^\infty(W)$ *to* $C^\infty(W)$*;*

2. *if* $a \in S'(W)$ *and* $b \in S(W)$*, then* $F_\sigma(a *_\sigma b) = (F_\sigma a) *_\sigma b$*;*

3. *if* $p \in [1, 2]$*, then* $*_\sigma$ *on* $S(W)$ *extends uniquely to a continuous multiplication on* $L^p(W)$*.*

Remark 1.2. As remarked in the introduction, there are close relations between positivity in pseudo-differential calculus and positivity in the twisted convolution algebra. In fact, let $a \in S(\mathbf{R^{2n}})$. Then the Weyl operator for a is defined by the formula

$$a^w(x,D)f(x) = (2\pi)^{-n} \iint a((x+y)/2,\xi)f(y)e^{i\langle x-y,\xi\rangle}\, dyd\xi,$$

where $f \in S(\mathbf{R^n})$. Then $a^w(x,D)$ is continuous on S, and the definition of $a^w(x,D)$ extends to any $a \in S'(\mathbf{R^{2n}})$ at which $a^w(x,D)$ is continuous from S to S'.

If $a \in S'(\mathbf{R^{2n}})$, then $a^w(x,D)$ is positive semi-definite as an operator on S, if and only if $F_\sigma a \in S'_+$. (See [To1]–[To3].)

Next we discuss modulation spaces and start to consider appropriate conditions for the involved weight functions. Let ω and v be positive and measureable functions on the vector space V of finite dimension. Then ω is called *v-moderate* if there is a constant $C > 0$ such that $\omega(x+y) \le C\omega(x)v(y)$ for every $x, y \in V$. We say that ω is a *moderate function* if $\omega(x+y) \le C\omega(x)\omega(y)$. The set of all positive functions ω on V such that ω is v-moderate for some polynomial v on V is denoted by $P(V)$. Also let $P_0(V)$ be the set of all $\omega_0 \in P(V)$ such that for every multi-index α, there is a constant C_α such that $|\partial^\alpha \omega_0| \le C_\alpha \omega_0$. If $\omega \in P(V)$ and $\varphi \in C_0^\infty \setminus 0$ is non-negative, then it follows that $\omega_0 = \omega * \varphi \in P_0(V)$ and that $C^{-1}\omega \le \omega_0 \le C\omega$ for some constant $C > 0$. (See [To5] or [To6].)

Next we give examples on weight functions which are of particular interest. For any $s, t \in \mathbf{R}$ and $x \in \mathbf{R^n}$, let $\tau_s(x) = \langle x\rangle^s$, where $\langle x\rangle = (1+|x|^2)^{1/2}$. Also let $\tau_{s,t} = \tau_t \otimes \tau_s$. Then

$$\tau_s(x,y) = (1+|x|^2+|y|^2)^{s/2} \quad \text{and} \quad \tau_{s,t}(x,y) = (1+|x|^2)^{t/2}(1+|y|^2)^{s/2}.$$

It follows that $\tau_s \in P_0$, and that τ_s is $\tau_{|s|}$-moderate.

Assume that $\chi \in S(W)\setminus 0$, $p, q \in [1,\infty]$, $\omega \in P(W \times W)$ and $a \in S'(W)$, and let $(\tau_X\chi)(Y) = \chi(Y-X)$. Then we set

$$\begin{aligned} \|a\|_{M^{p,q}_{(\omega)}} &\equiv \Big(\int \Big(\int |F_\sigma(a\,\tau_X\chi)(Y)\omega(X,Y)|^p\, dX\Big)^{q/p} dY\Big)^{1/q}, \\ \|a\|_{\mathcal{W}^{p,q}_{(\omega)}|} &\equiv \Big(\int \Big(\int |F_\sigma(a\,\tau_X\chi)(Y)\omega(X,Y)|^p\, dY\Big)^{q/p} dX\Big)^{1/q} \end{aligned} \tag{1.1}$$

(with obvious interpretation when $p = \infty$ and/or $q = \infty$). Note here that $\|a\|_{\mathcal{W}^{p,q}_{(\omega)}}$ and $\|a\|_{M^{p,q}_{(\omega)}}$ may attain the value $+\infty$.

Definition 1.3. *Assume that $\chi \in S(W) \setminus 0$, $p, q \in [1, \infty]$, and $\omega \in P(W \times W)$. Then the modulation space $M^{p,q}_{(\omega)}(W)$ consists of all $a \in S'(W)$ such that $\|a\|_{M^{p,q}_{(\omega)}} < \infty$. In the same way, $\mathcal{W}^{p,q}_{(\omega)}(W)$ consists of all $a \in S'(W)$ such that $\|a\|_{\mathcal{W}^{\surd,\text{II}}_{(\omega)}} < \infty$.*

If $\omega = 1$, then the notation $\mathcal{W}^{p,q}$ is used instead of $\mathcal{W}^{p,q}_{(\omega)}$. Moreover, if $p = q$, then the notations $\mathcal{W}^{p}_{(\omega)}$ and $\mathcal{W}^{p}$ are used instead of $\mathcal{W}^{p,p}_{(\omega)}$ and $\mathcal{W}^{p,p}$, respectively. If in addition $W = \mathbf{R^{2n}}$, then we set $\mathcal{W}^{p,q}_{s} = \mathcal{W}^{p,q}_{(\omega)}$ when $\omega(X,Y) = \tau_s(X,Y)$, and $\mathcal{W}^{p,q}_{s,t} = \mathcal{W}^{p,q}_{(\omega)}$ when $\omega(X,Y) = \tau_{s,t}(X,Y)$. Such spaces are common in many situations. (See [G] and the references therein.)

If $\omega(X,Y)$ is a weight function on $W \times W$ which is constant with respect to X or Y, then we set $\omega(X,Y) = \omega(Y)$ and $\omega(X,Y) = \omega(X)$, respectively. In this situation, ω is sometimes considered as a function on W instead of a function on $W \times W$.

The convention using parenthesis, when weight functions are involved in the definition of function spaces, are also used in other situations. For example, if $\omega \in P(W)$, then $L^p_{(\omega)}(W)$ consists of all measurable functions a such that $a\,\omega \in L^p(W)$.

The space $\mathcal{W}^{p,q}_{(\omega)}$ is a Fourier modulation space. In fact, by Parseval's formula it follows that

$$F_\sigma(\widehat{a}\,\tau_Y(F_\sigma\check{\chi}))(X) = e^{2i\sigma(X,Y)}F_\sigma(a\,\tau_X\chi)(Y), \tag{1.2}$$

where $\check{\chi}(X) = \chi(X)$. This in turn implies that

$$\mathcal{W}^{p,q}_{(\omega)} = F_\sigma M^{p,q}_{(\omega_0)} \quad \text{when} \quad \omega_0(Y,X) = \omega(X,Y). \tag{1.3}$$

Here it is essential that the definitions of $M^{p,q}_{(\omega)}$ and $\mathcal{W}^{p,q}_{(\omega)}$ are independent of the choice of $\chi \in S \setminus 0$. This is a consequence of the following proposition, where some important properties for modulation spaces are recalled. Here and in what follows, p' denotes the conjugate exponent for p, i.e., $1/p + 1/p' = 1$.

Proposition 1.4. *Assume that $\omega, \omega_1, \omega_2 \in P(W \times W)$ and $p, q \in [1, \infty]$. Then the following are true:*

1. *$M^{p,q}_{(\omega)}(W)$ and $\mathcal{W}^{p,q}_{(\omega)}(W)$ are Banach spaces which are independent of the choices of $\chi \in S(W) \setminus 0$ in (1.1). Moreover, different choices of χ give rise to equivalent norms;*

2. *if* $p_1, p_2, q_1, q_2 \in [1,\infty]$ *such that* $p_1 \le p_2$, $q_1 \le q_2$ *and* $\omega_2 \le C\omega_1$ *for some constant* C*, then*

$$S(W) \hookrightarrow \mathcal{W}^{p_1,q_1}_{(\omega_1)}(W) \hookrightarrow \mathcal{W}^{p_2,q_2}_{(\omega_2)}(W) \hookrightarrow S'(W) \ .$$

Moreover, $\mathcal{W}^{1,p} \subseteq C(W) \cap L^\infty$*;*

3. *assume that* $1 \le q_1 \le \min(p,p')$ *and* $\max(p,p') \le q_2 \le \infty$*, and that in addition* $\omega(X,Y) = \omega(Y)$*. Then*

$$\mathcal{W}^{p,q_1}_{(\omega)} \hookrightarrow F_\sigma(L^p_{(\omega)}) \hookrightarrow \mathcal{W}^{p,q_2}_{(\omega)} \quad \textit{and} \quad \mathcal{W}^{p,q}_{(\omega)} \cap E' = F_\sigma(L^p_{(\omega)}) \cap E' \ ;$$

4. *let* $\omega_0(X,Y) = \omega(Y,X)$*, and assume in addition that* $q \le p$*. Then* $F_\sigma \mathcal{W}^{p,q}_{(\omega)} \hookrightarrow \mathcal{W}^{q,p}_{(\omega_0)}$*. In particular,* F_σ *is a homeomorphism on* $\mathcal{W}^p$*.*

P r o o f. The assertions (1), (2) and (4) follow from [G] and (1.3) (see also [Fe], or [To5] and the references therein). The assertion (3) is an immediate consequence of Proposition 4.3 and Theorem 5.5 in [To6], and (1.3). □

Remark 1.5. Assume that $\chi \in C_0^\infty$ and that $\omega \in P(W \times W)$, and let $\widetilde{W}^{p,q}_{(\omega)}$ be the set of all $a \in D'(W)$ such that $\|a\|_{\mathcal{W}^{p,q}_{(\omega)}} < \infty$. Then it follows that $\widetilde{\mathcal{W}}^{p,q}_{(\omega)} \subseteq S'$. Consequently, $\widetilde{\mathcal{W}}^{p,q}_{(\omega)} = \mathcal{W}^{p,q}_{(\omega)}$ in view of Proposition 1.4.

We finish the section with the following proposition which concerns multiplications and differentiations of elements in Fourier modulation spaces. We refer to [To6] for the proof. Here and in what follows, if $\omega \in P_0(W)$, then $\omega(D)$ is the linear and continuous operator on $S'(W)$ which means a multiplication by ω on the symplectic Fourier transform side.

Proposition 1.6. *Assume that* $p, q \in [1,\infty]$*,* $\omega \in P(W \times W)$*,* $\omega_0 \in P_0(W)$*, and set*

$$\omega_1(X,Y) = \omega(X,Y)\omega_0(X) \quad \textit{and} \quad \omega_2(X,Y) = \omega(X,Y)\omega_0(Y).$$

Then $a \mapsto a\,\omega_0$ *is a homeomorphism from* $\mathcal{W}^{p,q}_{(\omega_1)}$ *to* $\mathcal{W}^{p,q}_{(\omega)}$*, and* $a \mapsto \omega_0(D)a$ *is a homemorphism from* $\mathcal{W}^{p,q}_{(\omega_2)}$ *to* $\mathcal{W}^{p,q}_{(\omega)}$*.*

2. *Positive elements in the $*_\sigma$-algebra*

In this section we prove that if $\omega \in P$ is appropriate, $a \in S'_+$ and $\chi\, a \in \mathcal{W}^{p,\infty}_{(\omega)}$ for some $\chi \in C_0^\infty$ such that $\chi(0) \neq 0$, then $a \in \mathcal{W}^{p,\infty}_{(\omega)}$. We use these results to prove that $S'_+ \subseteq S'$.

In the following proposition we recall some facts for elements in $C_+(W) \equiv S'_+(W) \cap C(W)$, the set of positive elements in $*_\sigma$-algebra which at the same time are continuous functions. The proof is omitted since the result is a consequence of Proposition 1.10, Theorem 3.3 and Corollary 3.7 in [To3].

Proposition 2.1. *Assume that $a \in S'_+(W)$. Then the following are true:*

1. *if $\varphi \in C_0^\infty(W)$, then $\widetilde{\varphi} *_\sigma a *_\sigma \varphi \in C_+ \cap C^\infty$. Moreover, if in addition $\int \varphi\, dX = (\pi/2)^{n/2}$ and $\varphi_\varepsilon = \varepsilon^{-2n}\varphi(\,\cdot\,/\varepsilon)$, then $\widetilde{\varphi}_\varepsilon *_\sigma a *_\sigma \varphi_\varepsilon \to a$ in D' as $\varepsilon \to 0$;*
2. *if a is a continuous function near the origin, then $a \in C_+$;*
3. *if $a \in C_+$, then $a \in L^2$ vanishes at infinity, and $|a(X)| \le a(0)$;*
4. *$a \in C_+$ if and only if $a = \widetilde{\psi} *_\sigma \psi$ for some $\psi \in L^2$;*
5. *if $b \in C_0^\infty \cap C_+$, then $(a, b) \ge 0$.*

As a consequence of Proposition 1.1 (4) and Proposition 2.1, it follows that if $a \in C_+$, then $(\varphi, \psi)_a \equiv (a *_\sigma \varphi, \psi)$ is a semi-scalar product on L^2. Hence Cauchy-Schwartz inequality holds, i. e.,

$$|(\varphi, \psi)_a|^2 \le (\varphi, \varphi)_a (\psi, \psi)_a.$$

The investigations in Section 4 in [To3] as well as in the present section depend on the following result. (Cf. Proposition 4.6 in [To3].) Here a slightly different proof is given.

Proposition 2.2. *Assume that $a \in S'_+(W)$ and that $\chi \in C_+(W) \cap C_0^\infty(W)$. Then $F_\sigma(a\,\chi)$ is a non-negative function. If $u = \big(F_\sigma(a\,\chi)\big)^{1/2}$ and $X, Y \in W$, then*

$$|F_\sigma(a\, \tau_Y \chi)(X)| \le u(X+Y)u(X-Y) \tag{2.1}$$

P r o o f. By Proposition 2.1 and a simple argument of approximation, it suffices to prove the result in the case $a \in C_+ \cap C^\infty$.

By straight-forward computations it follows that $\overline{\chi} \in C_+$, and that S'_+ is invariant under multiplication by exponentials. Hence Proposition 2.1 (5) gives that $F_\sigma(a\,\chi)(X) = (e^{2i\sigma(X,\,\cdot\,)}a, \overline{\chi})$ is non-negative for every $X \in W$. This proves the first part of the proposition.

Next we prove (2.1). By Proposition 2.1, there is a function $\psi \in L^2$ such that $\overline{\chi} = \widetilde{\psi} *_\sigma \psi$. Then $\overline{\chi}_Y = \widetilde{\psi}_Y *_\sigma \phi_Y$, where $\psi_Y = \psi(\,\cdot\, + Y)$ and $\phi_Y = e^{-2i\sigma(Y,\,\cdot\,)}\psi$. From the fact that $e^{2i\sigma(X,\,\cdot\,)}a \in S'_+$ for every $X \in W$, an application of Cauchy-Schwartz inequality gives

$$\begin{aligned} |F_\sigma(a\,\tau_Y\chi)(X)|^2 &= |(e^{2i\sigma(X,\,\cdot\,)}a, \widetilde{\psi}_Y *_\sigma \phi_Y)|^2 \\ &\le (e^{2i\sigma(X,\,\cdot\,)}a, \widetilde{\psi}_Y *_\sigma \psi_Y)(e^{2i\sigma(X,\,\cdot\,)}a, \widetilde{\phi}_Y *_\sigma \phi_Y). \end{aligned} \tag{2.2}$$

By simple calculations it follows that

$$\widetilde{\psi}_Y *_\sigma \psi_Y = e^{2i\sigma(Y,\,\cdot\,)}\overline{\chi}, \quad \text{and} \quad \widetilde{\phi}_Y *_\sigma \phi_Y = e^{-2i\sigma(Y,\,\cdot\,)}\overline{\chi}.$$

This implies that

$$(e^{2i\sigma(X,\,\cdot\,)}a, \widetilde{\psi}_Y *_\sigma \psi_Y) = (e^{2i\sigma(X-Y,\,\cdot\,)}a, \overline{\chi}) = F_\sigma(a\,\chi)(X-Y),$$

and similarly $(e^{2i\sigma(X,\,\cdot\,)}a, \widetilde{\phi}_Y *_\sigma \phi_Y) = F_\sigma(a\,\chi)(X+Y)$. This proves (2.2), and the result follows. The proof is complete. □

The following result is now an immediate consequence of Proposition 1.4 (3), Proposition 2.2 and Hölder's inequality.

Theorem 2.3. *Assume that $\omega_0 \in P(W \times W)$ such that $\omega_0(X,Y) = \omega_0(Y)$, and set*

$$\omega_1(X,Y) = (\omega_0(X-Y)\omega_0(X+Y))^{1/2}.$$

Also assume that $a, \chi \in S'_+(W)$ such that $\chi \in C_0^\infty \setminus 0$ and $a\,\chi \in \mathcal{W}^{p,\infty}_{(\omega_0)}$ for some $p \in [1,\infty]$. Then $a \in \mathcal{W}^{p,\infty}_{(\omega_1)}$.

As a consequence of Theorem 2.3 we obtain Bochner-Schwartz theorem in the case when the usual convolutions are replaced by twisted convolutions.

Corollary 2.4. *The set $S'_+(W)$ is contained in $S'(W)$.*

P r o o f. Assume that $a, \chi \in S'_+$ such that $\chi \in C_0^\infty \setminus 0$. Then $a\,\chi \in E'$. Hence $F_\sigma(a\,\chi) \in L^p_{(\omega_0)}$ for some $\omega_0 \in P$ which in turn implies that $a \in \mathcal{W}^{p,\infty}_{(\omega)}$ for some $\omega \in P$ by Proposition 1.4 (3) and Theorem 2.3. Since $\mathcal{W}^{p,\infty}_{(\omega)} \subset S'$, the result follows. □

By Corollary 2.4, it follows that Proposition 2.2 also holds under the weaker assumption for χ that $\chi \in S'_+ \cap S$. Moreover, by (1.2) it follows that (2.1) is equivalent to

$$|F_\sigma(\widehat{a}\,\tau_Y(F_\sigma\check{\chi}))(X)| \le u(X+Y)u(Y-X). \tag{2.1$'$}$$

Theorem 2.3 may now be improved in the following way.

Theorem 2.3$'$ *Assume that ω_0 and ω_1 are the same as in Theorem 2.3, and let $\omega_2(X,Y) = \omega_1(Y,X)$. Also assume that $a, \chi \in S'_+(W)$ such that $\chi \in S$ and $\chi(0) \neq 0$ and $a\,\chi \in \mathcal{W}^{p,\infty}_{(\omega_0)}$ for some $p \in [1,\infty]$. Then $a \in \mathcal{W}^{p,\infty}_{(\omega_1)}$ and $F_\sigma a \in \mathcal{W}^{p,\infty}_{(\omega_2)}$.*

P r o o f. Let Ω be an open neighbourhood of the origin such that $\chi(X) \neq 0$ as $X \in \Omega$, and let $\chi_0 = \widetilde{\psi} *_\sigma \psi$, where $\psi \in C_0^\infty(W) \setminus 0$. Then $0 \neq \chi_0 \in C_+ \cap C_0^\infty$, and the support of χ_0 is contained in Ω, provided ψ is chosen with sufficiently small support. This gives

$$a\,\chi_0 = (\chi_0/\chi)(a\,\chi) \in S \cdot \mathcal{W}^{p,\infty}_{(\omega_0)} \subseteq \mathcal{W}^{p,\infty}_{(\omega_0)}. \tag{2.3}$$

Here the last step is a consequence of Theorem 5.5 in [To6], and the fact that multiplications are replaced by convolutions on the Fourier transform. Hence Theorem 2.3 and (2.1)$'$ show that $a \in \mathcal{W}^{p,\infty}_{(\omega_1)}$ and $F_\sigma a \in \mathcal{W}^{p,\infty}_{(\omega_2)}$. The proof is complete. □

Theorem 2.5. *Assume that $v_0 \in P(W \times W)$ such that $v_0(X,Y) = v_0(Y)$ is an even and moderate function, $v(X,Y) = v_0(X)v_0(Y)$, $p \in [1,\infty]$, $a \in S'_+(W)$, and $\chi \in S(W)$ such that $\chi(0) \neq 0$. Then the following conditions are equivalent:*

1. *$F_\sigma(a\,\chi) \in L^p_{(1/v_0)}(W)$;*
2. *$a\,\chi \in \mathcal{W}^{p,\infty}_{(1/v_0)}(W)$;*
3. *$a\,\chi \in \mathcal{W}^{p,\infty}_{(1/v)}(W)$;*
4. *$a \in \mathcal{W}^{p,\infty}_{(1/v)}(W)$;*

5. $a, F_\sigma a \in \mathcal{W}^{p,\infty}_{(1/v)}(W)$.

Before the proof we observe that if v_0 is even and moderate, then

$$(v_0(X-Y)v_0(X+Y))^{1/2} \le Cv_0(X)v_0(Y), \tag{2.4}$$

for some constant C.

P r o o f. Let χ_0 be the same as in the proof of Theorem 2.3′. Then $F_\sigma(a\chi_0) \in L^p_{(1/v_0)}$ (cf. (2.3)). This in turn implies that $a\chi_0 \in \mathcal{W}^{p,\infty}_{(1/v_0)}$ by Proposition 1.4 (3). Hence Theorem 2.3′ and (2.4) show that (5) holds. By similar arguments it follows that (2) implies (5).

The implication (5) ⇒ (4) is obvious, and the implications (4) ⇒ (2) and (4) ⇒ (1) follow immediately from Theorem 5.5 in [To6] and the inequality

$$\|F_\sigma(a\chi)\|_{L^p_{(1/v_0)}} \le C\|a\|_{\mathcal{W}^{p,\infty}_{(1/v)}},$$

respectively.

When proving the equivalence (2) ⇔ (3) we may assume that $v_0 \in P_0$. Then the assertion follows immediately from Proposition 1.6 and the fact that the map $\varphi \mapsto v_0\,\varphi$ is a homeomorphism on S. The proof is complete. □

Corollary 2.6 *Assume that $s \in \mathbf{R}$, $p \in [1,\infty]$, $a \in S'_+(\mathbf{R^{2n}})$, and $\chi \in S(\mathbf{R^{2n}})$ such that $\chi(0) \neq 0$. Then the following are true:*

1. *if $s \le 0$, then $a\,\chi \in \mathcal{W}^{p,\infty}_{s,0}$ if and only if $a \in \mathcal{W}^{p,\infty}_{s,s}$;*
2. *if $s \ge 0$ and $a\,\chi \in \mathcal{W}^{p,\infty}_{s,0}$, then $a \in \mathcal{W}^{p,\infty}_{s^{1/4},s^{1/4}}$.*

P r o o f. The assertion (1) follows by letting $v_0 = \tau_s$ in Theorem 2.5. The assertion (2) follows from Theorem 2.3′ and the fact that for some constant C we have $\langle X\rangle\langle Y\rangle \le C\langle X-Y\rangle^2\langle X+Y\rangle^2$. The proof is complete. □

Remark 2.7 Assume that $a \in S'_+$, $\chi \in S$ such that $\chi(0) \neq 0$, and that $F_\sigma(a\,\chi) \in L^1$. Then Theorem 2.5 shows that $a \in \mathcal{W}^{1,\infty}$. This is also a consequence of Theorem 1.5 in [To4].

In fact, from the assumptions it follows that $a\,\chi$ is a continuous function near the origin. Hence Proposition 2.1 shows that a belongs to the set of all symbols such that their corresponding Weyl operators are trace-class operators on L^2. The result now follows since the latter set is contained in $\mathcal{W}^{1,\infty}$ in view of Theorem 1.5 in [To4].

Finally we have the following result, parallel to Theorem 2.3′.

Proposition 2.8 *Assume that* $p, q \in [1, \infty]$ *and* $p_0 \in [2^{-1}, \infty]$ *satisfy* $p \le q$ *and* $1/p + 1/q = 1/p_0$. *Assume also that* v_0, χ *and* a *are the same as in Theorem 2.5, and set* $v(X,Y) = (v_0(X-Y)v_0(X+Y))^{1/2}$. *If* $F_\sigma(a\,\chi) \in L^{p_0}_{(1/v_0)}(W)$, *then* $a, F_\sigma a \in \mathcal{W}^{p,q}_{(1/v)}(W)$.

P r o o f. It is no restriction to assume that $\chi \in C_+ \cap C_0^\infty$. We use the same notations as in Proposition 2.2. Let $h = (u/v_0^{1/2})^p$ and $r = q/p \ge 1$. Then for some constant C, Young's inequality gives

$$\begin{aligned}\|a\|_{\mathcal{W}^{p,q}_{(1/v)}} &= \Big(\int \Big(\int |F_\sigma(a\,\tau_X\chi)(Y)/v(X,Y)|^p \, dY \Big)^{q/p} dX \Big)^{1/q} \\ &\le C(\|h * h\|_{L^r})^{1/p} \le C'(\|h\|_{L^{2p_0/p}})^{2/p} = \|F_\sigma(a\,\chi)\|_{L^{p_0}_{(1/v_0)}}.\end{aligned}$$

The result is now a consequence of Proposition 1.4. □

Corollary 2.9 *Let* v_0, v, a *and* χ *be the same as in Proposition 2.8, and assume that* $F_\sigma(a\,\chi) \in L^{1/2}_{(1/v_0)}$ *Then* $a, F_\sigma a \in \mathcal{W}^1_{(1/v)}$.

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Department of Mathematics and Systems Engineering
Växjö University
351 95 Växjö
Sweden
e-mail: `joachim.toft@vxu.se`