Bulletin T.CXXXIII de l'Académie serbe des sciences et des arts − 2006
Classe des Sciences mathématiques et naturelles
Sciences mathématiques, № 31

GENERALIZED SOLUTIONS TO SINGULAR INITIAL-BOUNDARY HYPERBOLIC PROBLEMS WITH NON-LIPSHITZ NONLINEARITIES[1]

IRINA KMIT

(Presented at the 1st Meeting, held on February 24, 2006)

A b s t r a c t. We prove the existence and uniqueness of global generalized solutions in a Colombeau algebra of generalized functions to semilinear hyperbolic systems with nonlinear boundary conditions. Our analysis covers the case of non-Lipschitz nonlinearities both in the differential equations and in the boundary conditions. We admit strong singularities in the differential equations as well as in the initial and boundary conditions.

AMS Mathematics Subject Classification (2000): 35L50, 35L67, 35D05
Key Words: Colombeau algebra, semilinear hyperbolic system, nonlinear boundary condition

1. *Introduction*

We study existence and uniqueness of global generalized solutions to mixed problems for semilinear hyperbolic systems with nonlinear nonlocal

[1]This paper was presented at the Conference GENERALIZED FUNCTIONS 2004, Topics in PDE, Harmonic Analysis and Mathematical Physics, Novi Sad, September 22–28, 2004

boundary conditions. Specifically, in the domain $\Pi = \{(x,t)\,|\,0 < x < l, t > 0\}$ we study the following problem:

$$(\partial_t + \Lambda(x,t)\partial_x)U = F(x,t,U), \qquad (x,t) \in \Pi \tag{1}$$

$$U(x,0) = A(x), \qquad x \in (0,l) \tag{2}$$

$$\begin{aligned} U_i(0,t) &= H_i(t,V(t)), \quad k+1 \le i \le n, \qquad t \in (0,\infty) \\ U_i(l,t) &= H_i(t,V(t)), \quad 1 \le i \le k, \qquad t \in (0,\infty)\,, \end{aligned} \tag{3}$$

where U, F, and A are real n-vectors, $\Lambda = \mathrm{diag}(\Lambda_1,\dots,\Lambda_n)$ is a diagonal matrix, $\Lambda_1,\dots,\Lambda_k < 0$, $\Lambda_{k+1},\dots,\Lambda_n > 0$ for some $1 \le k \le n$, and $V(t) = (U_1(0,t),\dots, U_k(0,t), U_{k+1}(l,t),\dots,U_n(l,t))$. Due to the conditions imposed on Λ, the system (1) is non-strictly hyperbolic. Note also that the boundary of Π is not characteristic. We will denote $H = (H_1,\dots,H_n)$.

Special cases of (1)–(3) arise in laser dynamics [7, 20, 21] and chemical kinetics [22].

All the data of the problem are allowed to be strongly singular, namely, they can be of any desired order of singularity. This entails nonlinear superpositions of distributions in the right-hand sides of (1)–(3), including compositions of the singular initial data and the singular characteristic curves. To tackle this complication, we use the framework of the Colombeau algebra of generalized functions $\mathcal{G}(\overline{\Pi})$ [1, 16]. We show that all superpositions appearing here are well defined in $\mathcal{G}(\overline{\Pi})$.

We establish a positive existence-uniqueness result in $\mathcal{G}(\overline{\Pi})$ for the problem (1)–(3) with strongly singular initial data and with nonlinearities of the following type (more detailed description is given in Section 3): The functions F and H may be non-Lipschitz with less than quadratic growth in U and V.

For different aspects of the subject we refer the reader to sources [3, 4, 10, 11, 12, 14, 15, 16, 17]. The essential assumption made on F in papers [12, 16] is that $\mathrm{grad}_U F$ is globally bounded uniformly over (x,t) varying in any compact set. In contrast to [12, 16], in [11] we investigated the problem (1)–(3) with Colombeau-Lipschitz nonlinearities in (1) and (3). This means that the functions F and H are Lipschitz with Colombeau generalized numbers as Lipschitz constants and therefore their gradients are not globally bounded.

M. Nedeljkov and S. Pilipović [14, 15] deal with Cauchy problems for semilinear hyperbolic systems (1) with F slowly increasing at the infinity. The nonlinear term is replaced by a suitable regularization F_ε having a bounded gradient with respect to U for every fixed ε and converging to F as $\varepsilon \to 0$. The regularized system is solved in $\mathcal{G}(\mathbb{R}^2)$. Moreover, in [14] the components of Λ are allowed to be 1-tempered generalized functions. The authors replace Λ by a regularization which is a 1-tempered generalized function of bounded growth and solve the regularized problem.

T. Gramchev [3, 4] investigates weak limits for semilinear hyperbolic systems and nonlinear superpositions for strongly singular distributions appearing in these systems. He establishes an optimal link between the singularity of the initial data and the growth of the nonlinear term. Weak limits of strongly singular Cauchy problems for semilinear hyperbolic systems with bounded, sublinear, and superlinear growth are investigated in [2, 9, 18, 19].

In the present paper we develop some results of [10] and [12] to the case of non-Lipschitz nonlinearities in (1) and (3). In Section 2 we compile some facts about Colombeau algebra of generalized functions. In Section 3 we state and prove our main result.

2. *Preliminaries*

In this section we summarize the relevant material on the full version of Colombeau algebras of generalized functions.

Let $\Omega \subset \mathbb{R}^n$ be a domain in $\mathbb{R}^n$. By $\mathcal{G}(\Omega)$ and $\mathcal{G}(\overline{\Omega})$ we denote the full version of Colombeau algebra of generalized functions over Ω and $\overline{\Omega}$, respectively. To define $\mathcal{G}(\Omega)$ and $\mathcal{G}(\overline{\Omega})$, we first introduce the mollifier spaces used to parametrize the regularizing sequences of generalized functions. Given $q \in \mathbb{N}_0$, denote

$$\mathcal{A}_q(\mathbb{R}) = \Big\{\varphi \in \mathcal{D}(\mathbb{R}) \;\Big|\; \int \varphi(x)\,dx = 1, \int x^k \varphi(x)\,dx = 0 \text{ for } \; 1 \le k \le q\Big\},$$

$$\mathcal{A}_q(\mathbb{R}^n) = \Big\{\varphi(x_1, \dots, x_n) = \prod_{i=1}^{n} \varphi_0(x_i) \;\Big|\; \varphi_0 \in \mathcal{A}_q(\mathbb{R})\Big\}.$$

Set

$$\mathcal{E}(\overline{\Omega}) = \{u : \mathcal{A}_0 \times \overline{\Omega} \to \mathbb{R} \;\Big|\; u(\varphi, .) \in \mathrm{C}^\infty(\overline{\Omega}) \;\; \forall \varphi \in \mathcal{A}_0(\mathbb{R})\}.$$

We define the algebra of moderate elements $\mathcal{E}_M(\overline{\Omega})$ to be the subalgebra of $\mathcal{E}(\overline{\Omega})$ consisting of the elements $u \in \mathcal{E}(\overline{\Omega})$ such that

$$\forall K \subset \overline{\Omega} \text{ compact}, \forall \alpha \in \mathbb{N}_0^n, \exists N \in \mathbb{N} \text{ such that } \;\forall \varphi \in \mathcal{A}_N(\mathbb{R}^n)$$

$$\exists C > 0, \exists \eta > 0 \text{ with } \;\sup_{x \in K} |\partial^\alpha u(\varphi_\varepsilon, x)| \le C\varepsilon^{-N}, \;\; 0 < \varepsilon < \eta,$$

where $\varphi_\varepsilon(x) = 1/\varepsilon^n\varphi(x/\varepsilon)$. The ideal $\mathcal{N}(\overline{\Omega})$ (see [5]) consists of all $u \in \mathcal{E}_M(\overline{\Omega})$ such that

$$\forall K \subset \overline{\Omega} \text{ compact}, \exists N \in \mathbb{N} \text{ such that } \ \forall q \geq N, \forall \varphi \in \mathcal{A}_q(\mathbb{R}^n)$$

$$\exists C > 0, \exists \eta > 0 \text{ with } \ \sup_{x\in K} |u(\varphi_\varepsilon, x)| \leq C\varepsilon^{q-N}, \quad 0 < \varepsilon < \eta.$$

Finally,

$$\mathcal{G}(\overline{\Omega}) = \mathcal{E}_M(\overline{\Omega})/\mathcal{N}(\overline{\Omega}).$$

This is an associative and commutative differential algebra. The algebra $\mathcal{G}(\Omega)$ on an open set Ω is constructed in the same manner, with Ω in place of $\overline{\Omega}$. Note that $\mathcal{G}(\Omega)$ admits a canonical embedding of $\mathcal{D}'(\Omega)$. We will use the notation $U = [(u(\varphi, x))_{\varphi\in\mathcal{A}_0(\mathbb{R}^n)}]$ for the elements U of $\mathcal{G}(\Omega)$ with the representative $u(\varphi, x)$.

One of the advantages of using the Colombeau algebra of generalized functions $\mathcal{G}$ lies in the fact that in a variety of important cases the division by generalized functions, in particular the division by discontinuous functions and measures, is defined in $\mathcal{G}$. Complete description of the cases when the division is possible in the full version of Colombeau algebras is given by the following criterion of invertibility [10] (the criterion of invertibility for the special version of Colombeau algebras $\mathcal{G}_s(\Omega)$ is proved in [6]):

Theorem 1. *Let $U \in \mathcal{G}(\Omega)$ (resp. $U \in \mathcal{G}(\overline{\Omega})$). Then the following two conditions are equivalent:*
(i) U is invertible in $\mathcal{G}(\Omega)$ (resp. in $\mathcal{G}(\overline{\Omega})$), i.e., there exists $V \in \mathcal{G}(\Omega)$ (resp. $V \in \mathcal{G}(\overline{\Omega})$) such that $UV = 1$ in $\mathcal{G}(\Omega)$ (resp. in $\mathcal{G}(\overline{\Omega})$).
(ii) For each representative $(u(\varphi, x))_{\varphi\in\mathcal{A}_0(\mathbb{R}^n)}$ of U and each compact set $K \subset \Omega$ (resp. $K \subset \overline{\Omega}$) there exists $p \in \mathbb{N}$ such that for all $\varphi \in \mathcal{A}_p(\mathbb{R}^n)$ there is $\eta > 0$ with $\inf_K |u(\varphi_\varepsilon, x)| \geq \varepsilon^p$ for all $0 < \varepsilon < \eta$.

3. *Existence and uniqueness of a Colombeau generalized solution*

We will need a notion of a generalized function whose growth is more restrictive than the $1/\varepsilon$-growth (as in the definition of $\mathcal{E}_M$).

Definition 2. *([10]) Let $\Omega \subset \mathbb{R}^n$ be a domain in $\mathbb{R}^n$. Given a function $\gamma : (0, 1) \mapsto (0, \infty)$, we say that an element $U \in \mathcal{G}(\Omega)$ (resp. $U \in \mathcal{G}(\overline{\Omega})$) is* locally of γ-growth, *if it has a representative $u \in \mathcal{E}_M(\Omega)$ (resp. $u \in \mathcal{E}_M(\overline{\Omega})$) with the following property:*

For every compact set $K \subset \Omega$ (resp. $K \subset \overline{\Omega}$) there is $N \in \mathbb{N}$ such that for every $\varphi \in \mathcal{A}_N(\mathbb{R}^n)$ there exist $C > 0$ and $\eta > 0$ with $\sup_{x\in K} |u(\varphi_\varepsilon, x)| \le C\gamma^N(\varepsilon)$ for $0 < \varepsilon < \eta$.

Let $K \subset \mathbb{R}^m$ be a compact. Let $U(x,y)$ and $V(x,y)$, as functions of x, are in $\mathcal{G}(K)$ for each $y \in \mathbb{R}^n$. We will say that U is bounded by V and write $U \le V$ if U and V have representatives $u(\cdot, y) \in \mathcal{E}_M(K)$ and $v(\cdot, y) \in \mathcal{E}_M(K)$, respectively, satisfying the following property for some $N \in \mathbb{N}$: For every $\varphi \in \mathcal{A}_N(\mathbb{R}^n)$ there exists $\eta > 0$ such that $|u(\varphi_\varepsilon, x, y)| \le v(\varphi_\varepsilon, x, y)$ for all $x \in K$, $y \in \mathbb{R}^n$, and $0 < \varepsilon < \eta$.

We will write $F(x,y) \in \mathrm{C}^\infty_y(\mathbb{R}^n; \mathcal{G}(\overline{\Pi}))$ if F is C^∞ with respect to $y \in \mathbb{R}^n$ and $\partial^\alpha_y F(\cdot, y) \in \mathcal{G}(\overline{\Pi})$ for every $\alpha \in \mathbb{N}^n_0$ and each $y \in \mathbb{R}^n$. Here $\partial^\alpha_y = \frac{\partial^{\alpha_1+\ldots+\alpha_n}}{\partial^{\alpha_1}_{y_1}\ldots\partial^{\alpha_n}_{y_n}}$.

We now make assumptions on the initial data of the problem (1)–(3). Let $\gamma(\varepsilon)$ be a function from $(0,1)$ to $(0,\infty)$ such that

$$\gamma(\varepsilon)^{\gamma^N(\varepsilon)} = O\left(\frac{1}{\varepsilon}\right) \tag{4}$$

for each $N \in \mathbb{N}$. Assume that

1. $\Lambda(x,t) \in (\mathcal{G}(\overline{\Pi}))^n$, $A(x) \in (\mathcal{G}[0,l])^n$.
2. Λ_i for $i \le n$ are locally of γ-growth on $\overline{\Pi}$ and invertible on $\overline{\Pi}$ (see Theorem 1).
3. $\partial_x \Lambda_i$ for $i \le n$ are locally of γ-growth on $\overline{\Pi}$.
4. $F(x,t,y) \in (\mathrm{C}^\infty_y(\mathbb{R}^n; \mathcal{G}(\overline{\Pi}))^n$, $H(t,z) \in (\mathrm{C}^\infty_z(\mathbb{R}^n; \mathcal{G}[0,\infty)))^n$.
5. For every compact set $K \subset \overline{\Pi}$, $i \le n$, and $\alpha \in \mathbb{N}^{n+2}_0$, the function $D^\alpha F_i(x,t,y)$ is bounded by a polynomial in $\mathcal{G}(K)[y]$ (polynomials over y with coefficients in $\mathcal{G}(K)$).
6. For every compact set $K \subset [0,\infty)$, $i \le n$, and $\alpha \in \mathbb{N}^{n+1}_0$, the function $D^\alpha H_i(t,z)$ is bounded by a polynomial in $\mathcal{G}(K)[z]$.
7. $\operatorname{supp} A_i(x) \subset (0,l)$ and $\operatorname{supp} H_i(t,0) \subset (0,\infty)$ for $i \le n$.

Assumptions imposed on Λ_i allow them to be strongly singular and, even more, to have any desired order of singularity. Assumptions 4–6 state that, given $U \in (\mathcal{G}(\overline{\Pi}))^n$ and $V \in (\mathcal{G}[0,\infty))^n$, $F(x,t,U)$ and $H(t,V)$ are well defined in the Colombeau algebra $\mathcal{G}$. The last assumption ensures the compatibility of (2) and (3) of any desired order.

Given $T > 0$, denote

$$\Pi^T = \{(x,t) \,|\, 0 < x < l, 0 < t < T\}.$$

To state the main result of the paper, we suppose additionally that at least one of the following two assumptions holds.

Assumption 8.

a) $H(t,V)$ is smooth in t, V and the mapping $V \mapsto \nabla_V H(t,V)$ is globally bounded, uniformly over t varying in compact subsets of $[0,\infty)$;

b) Given $T > 0$, there exists C_F such that for all $1 \le i \le n$ we have

$$|\nabla_y F_i(x,t,y)| \le C_F \log\log D(x,t,y),$$

where $D(x,t,y) \in \mathcal{G}(\overline{\Pi}^T)[y]$.

Assumption 9.

a) Given $T > 0$, there exists C_H such that for all $1 \le i \le n$ we have

$$|\nabla_z H_i(t,z)| \le C_H (\log\log B(t,z))^{1/4},$$

where $B(t,z) \in \mathcal{G}[0,T][z]$.

b) Assumptions 2 and 3 are true with $\gamma(\varepsilon) = O((\log\log 1/\varepsilon)^{1/4})$;

c) Given $T > 0$, there exists C_F such that for all $1 \le i \le n$ we have

$$|\nabla_y F_i(x,t,y)| \le C_F (\log\log D(x,t,y))^{1/4},$$

where $D(x,t,y) \in \mathcal{G}(\overline{\Pi}^T)[y]$.

Theorem 3. *Assume that Assumption 8 or 9 is true. Under Assumptions 1–7 where the function γ is specified by (4), the problem (1)–(3) has a unique solution $U \in \mathcal{G}(\overline{\Pi})$.*

Set

$$E_U(\alpha_1,\alpha_2;T) = \max\Big\{|\partial_x^{\alpha_1}\partial_t^{\alpha_2} U_i(x,t)| \,\Big|\, (x,t) \in \overline{\Pi}^T, 1 \le i \le n\Big\}, \quad E_F(\alpha_1,\alpha_2)$$

$$= \max\Big\{|\partial_x^{\alpha_1}\partial_t^{\alpha_2} F_i(x,t,y)| \,\Big|\, (x,t,y) \in \overline{\Pi}^T \times \{y : |y| \le E_U(0,0;T)\}, 1 \le i \le n\Big\},$$

$$E_H(\alpha) = \max\Big\{|\partial_t^\alpha H_i(t,z)| \,\Big|\, (t,z) \in [0,T]\times\{z : |z| \le E_U(0,0;T)\}, 1 \le i \le n\Big\},$$

$$L_{\nabla F}(U) = \max\Big\{|\nabla_U F_i(x,t,U(x,t))| \,:\, (x,t) \in \overline{\Pi}^T, 1 \le i \le n\Big\},$$

$$L_{\nabla H}(V) = \max\Big\{|\nabla_V H_i(t,V(t))| \,:\, t \in [0,T], 1 \le i \le n\Big\}.$$

Simplifying the notation, we drop the dependence of $E_F(\alpha_1,\alpha_2)$, $E_H(\alpha)$, $L_{\nabla F}(U)$ and $L_{\nabla H}(V)$ on T. Note that T will be a fixed positive number.

To prove the theorem, we need the following lemma.

Lemma 4. *Assume that the initial data Λ, F, A, and H are smooth with respect to all their arguments and satisfy Assumption 7, $\nabla_y F(x,t,y)$ is bounded on $K \times \mathbb{R}^n$ for every compact $K \subset \overline{\Pi}$, and $\nabla_z H(t,z)$ is bounded on $K \times \mathbb{R}^n$ for every compact $K \subset [0,\infty)$. Then, given $T > 0$, the problem (1)–(3) has a unique smooth solution U in $\overline{\Pi}^T$ satisfying the following a priori estimates:*

$$E_U(0,0;T) \le P_{1,0}\Big(\frac{1}{1-q_0t_0}, n, L_{\nabla H}(V)\Big)$$
$$\times P_{2,0}\Big(\max_{x\in[0,l],1\le i\le n}|A_i(x)|, \max_{(x,t)\in\overline{\Pi}^T,1\le i\le n}|F_i(x,t,0)|, \max_{t\in[0,T],1\le i\le n}|H_i(t,0)|\Big) \tag{5}$$

and

$$E_U(m,0;T) \le P_{1,m}\Big(\frac{1}{1-q_mt_m}, n, L_{\nabla H}(V)\Big)$$
$$\times P_{2,m}\Big(n, \max_{x\in[0,l],1\le i\le n}|A_i^{(m)}(x)|, \max_{0\le\alpha_1+\alpha_2\le m-1} E_{\Lambda^{-1}}(\alpha_1,\alpha_2;T),$$
$$\max_{0\le\alpha_1+\alpha_2\le m} E_\Lambda(\alpha_1,\alpha_2;T), \max_{1\le|\beta|+\alpha_1+\alpha_2\le m} E_{\partial_U^\beta F}(\alpha_1,\alpha_2), \max_{1\le|\beta|+\alpha_1\le m} E_{\partial_V^\beta H}(\alpha_1),$$
$$L_{\nabla F}(U), L_{\nabla H}(V), \max_{1\le\alpha_1+\alpha_2\le m-1} E_U(\alpha_1,\alpha_2;T)\Big), \quad m \in \mathbb{N}, \tag{6}$$

where

$$q_m = (nL_{\nabla F}(U) + mE_\Lambda(1,0;T))(1 + nL_{\nabla H}(V)), \quad m \in \mathbb{N}_0,$$

$$t_m \le \min\{L/E_\Lambda(0,0;T), 1/q_m\}, \quad m \in \mathbb{N}_0,$$

$P_{1,m}$ is a polynomial of degree $3\lceil T/t_m\rceil$ with all coefficients identically equal to 1, $P_{2,m}$ is a polynomial whose degree depends on m but neither on T nor on t_m and whose coefficients are positive constants depending only on m and T.

The lemma directly follows from the proof of Theorem 2.1 in [11]. Note that similar global a priori estimates for $E_U(\alpha_1, \alpha_2; T)$, where $\alpha_1 + \alpha_2 \le m$, follow from the estimates (5) and (6) as well as from the system (1) and its suitable differentiations.

Proof of the theorem. The classical smooth solution to the problem (1)–(3) satisfying estimates (5) and (6) in $\overline{\Pi}^T$ for any $m \in \mathbb{N}_0$ and $T > 0$ can be constructed by the sequential approximation method. We now use this solution to construct a representative of the Colombeau solution. According to the assumptions of the theorem, we consider all the initial data as elements of the corresponding Colombeau algebras. We choose representatives λ, a, f, and h of Λ, A, F, and H, respectively, with the properties required in the theorem. Let $\phi = \varphi \otimes \varphi \in \mathcal{A}_0(\mathbb{R}^2)$. Consider a prospective representative $u = u(\phi, x, t)$ of U which is the classical smooth solution to the problem (1)–(3) with the initial data $\lambda(\phi, x, t)$, $a(\varphi, x)$, $f(\phi, x, t, u(\phi, x, t))$, $h(\varphi, t, v(\varphi, t))$, where $v(\varphi, t) = (u_1(\phi, 0, t), \dots, u_k(\phi, 0, t),\ u_{k+1}(\phi, l, t), \dots, u_n(\phi, l, t))$. For the existence part of the proof, we have to show that $u \in \mathcal{E}_M$, i.e., to obtain moderate growth estimates of $u(\phi_\varepsilon, x, t)$ in $\overline{\Pi}^T$ for any $T > 0$ in terms of the regularization parameter ε. Set $f_\varepsilon(x, t, y) = f(\phi_\varepsilon, x, t, y)$, $h_\varepsilon(t, z) = h(\varphi_\varepsilon, t, z)$, and $\lambda_\varepsilon(x, t) = \lambda(\phi_\varepsilon, x, t)$.

In the proof we will use a modified notion of $\mathcal{E}_M(\overline{\Pi})$. Namely, let $u \in \mathcal{E}_M(\overline{\Pi})$ iff $u \in \mathcal{E}(\overline{\Pi})$ and for every compact set $K \subset \overline{\Pi}$ there is $N \in \mathbb{N}$ such that for every $\varphi \in \mathcal{A}_N(\mathbb{R}^n)$ there exists $\eta > 0$ with $\sup\limits_{x \in K} |u(\phi_\varepsilon, x, t)| \le \gamma^N(\varepsilon)$ for all $0 < \varepsilon < \eta$.

Fix an arbitrary $T > 0$. Fix $N \in \mathbb{N}$ to be so large that for all $\varphi \in \mathcal{A}_N(\mathbb{R})$ there exists $\varepsilon(\varphi)$ such that for all $\varepsilon < \varepsilon(\varphi)$ the following conditions are true:

(a) The moderate estimate (see the definition of $\mathcal{E}_M$) holds for $a(\varphi_\varepsilon, x)$, $f(\phi_\varepsilon, x, t, 0)$, and $h(\varphi_\varepsilon, t, 0)$.

(b) The invertibility estimate (see Theorem 1) holds for $\lambda(\phi_\varepsilon, x, t)$.

(c) The local-γ-growth estimate (see Definition 2) holds for $\lambda(\phi_\varepsilon, x, t)$ and $\partial_x \lambda(\phi_\varepsilon, x, t)$.

(d) For all $i \le n$, all $y, z \in \mathbb{R}^n$, and some $C > 0$ the following estimates are true: $|\nabla_y f_{i,\varepsilon}(x, t, y)| \le C_F \log\log d(\phi_\varepsilon, x, t, y)$ and $|\nabla_z h_{i,\varepsilon}(t, z)| \le C$ if Assumption 8 is fulfilled or $|\nabla_y f_{i,\varepsilon}(x, t, y)| \le C_F \sup\limits_{(x,t) \in \overline{\Pi}^T} (\log\log d(\phi_\varepsilon, x, t, y))^{1/4}$ and $|\nabla_z h_{i,\varepsilon}(t, z)| \le C_F \sup\limits_{(x,t) \in \overline{\Pi}^T} (\log\log b(\varphi_\varepsilon, t, z))^{1/4}$ if Assumption 9 is fulfilled, where b and d are representatives of B and D, respectively.

(e)The moderate estimate holds for the coefficients of the polynomial

$d(\phi_\varepsilon, x, t, y)$ if Assumption 8 is fulfilled or for the coefficients of the polynomials $d(\phi_\varepsilon, x, t, y)$ and $b(\varphi_\varepsilon, t, z)$ if Assumption 9 is fulfilled.

Given $\varphi \in \mathcal{A}_N(\mathbb{R})$, denote by $p_{1,m}(\varphi)$, $p_{2,m}(\varphi)$, $q_m(\varphi)$, and $t_m(\varphi)$ the value of, respectively, $P_{1,m}$, $P_{2,m}$, q_m, and t_m, where $U(x,t)$, $\Lambda(x,t)$, $A(x)$, $F(x,t,U(x,t))$, $H(t,V(t))$, $L_{\nabla F}(U)$, $L_{\nabla H}(V)$ are replaced by their representatives $u(\phi,x,t)$, $\lambda(\phi,x,t)$, $a(\varphi,x)$, $f(\phi,x,t,\ u(\phi,x,t))$, $h(\varphi,t,v(\varphi,t))$, $L_{\nabla f}(u)$, and $L_{\nabla h}(v)$, respectively. On the account of (5) and (6), it suffices to prove the moderate estimates for $p_{1,m}(\varphi_\varepsilon)$ and $p_{2,m}(\varphi_\varepsilon)$ for all $m \in \mathbb{N}_0$.

From the condition (a) and the description of $P_{2,0}$ given in Lemma 4 it follows that $[(p_{2,0}(\varphi))_{\varphi\in\mathcal{A}_0(\mathbb{R})}]$ is a Colombeau generalized number and hence has the moderateness property. This means that there exists $N_1 \le N$ such that for all $\varphi \in \mathcal{A}_{N_1}(\mathbb{R})$ there is $0 < \eta(\varphi) < \varepsilon(\varphi)$ with

$$|p_{2,0}(\varphi_\varepsilon)| \le \varepsilon^{-N_1}, \quad 0 < \varepsilon < \eta(\varphi). \tag{7}$$

Note that any $U \in \mathcal{G}(\overline{\Pi})$ has the following property: there exists $N_2 \in \mathbb{N}$ such that for all $\varphi \in \mathcal{A}_{N_1+N_2}(\mathbb{R})$ there is $\varepsilon_0(\varphi) \le \eta(\varphi)$, where the value of $\eta(\varphi)$ is the same as in (7), with

$$\sup_{\overline{\Pi}^T} |u(\phi_\varepsilon, x, t)| \le \varepsilon^{-N_1-N_2}, \quad 0 < \varepsilon < \varepsilon_0(\varphi), \tag{8}$$

with the constant N_1 being the same as in (7). Obviously, any increase of N_2 and any decrease of $\varepsilon_0(\varphi)$ will keep this property true. This will allow us to adjust the values of N_2 and $\varepsilon_0(\varphi)$ according to our purposes.

Set $u_\varepsilon(x,t) = u(\phi_\varepsilon, x, t)$ and $v_\varepsilon(t) = (u_{1,\varepsilon}(0,t), \dots, u_{k,\varepsilon}(0,t), u_{k+1,\varepsilon}(l,t), \dots, u_{n,\varepsilon}(l,t))$. Given $\varphi \in \mathcal{A}_{N_1+N_2}(\mathbb{R})$, let us consider the estimates (5) and (6) with $p_{1,m}(\varphi_\varepsilon)$ and $p_{2,m}(\varphi_\varepsilon)$ in place of $P_{1,m}$ and $P_{2,m}$, respectively, where $0 < \varepsilon < \eta(\varphi)$ and the value of $\eta(\varphi)$ is the same as in (7). On the account of these estimates, we will obtain the existence once we prove the following assertion:

(ι) given $m \in \mathbb{N}_0$, a positive integer $N(m)$, where $N(0) = N_2$, can be chosen so that for all $\varphi \in \mathcal{A}_{N_1+N_2}(\mathbb{R})$ there exists $\varepsilon_m(\varphi)$ such that

$$\Big[2n(1+L_{\nabla h_\varepsilon}(v_\varepsilon))\Big]^{6T(1+L_{\nabla h_\varepsilon}(v_\varepsilon))(L_{\nabla f_\varepsilon}(u_\varepsilon)+mE_{\lambda_\varepsilon}(1,0;T))+2T/lE_{\lambda_\varepsilon}(0,0;T)+1} \le \varepsilon^{-N(m)} \tag{9}$$

for all $0 < \varepsilon < \varepsilon_m(\varphi)$, provided $u(\phi, x, t) \in \mathcal{E}(\overline{\Pi}^T)$ satisfies the inequality (8).

Indeed, take $t_m(\varphi_\varepsilon) = 1/2 \min\{L/E_{\lambda_\varepsilon}(0,0;T), 1/q_m(\varphi_\varepsilon)\}$. When the assertion ($\iota$) is fulfilled, then the moderate estimates for $p_{1,m}(\varphi_\varepsilon)$ follow from the fact that the left-hand side of (9) is an upper bound for $p_{1,m}(\varphi_\varepsilon)$. By (5), (7), and (9) for $m = 0$, we have the zero-order moderate estimate (8) for $u_\varepsilon(x,t)$. Moderate estimates of order $m \geq 1$ for $u_\varepsilon(x,t)$ are easy to obtain, using induction on m, estimates (6) and (9), assumptions imposed on the initial data, and the fact that $p_{1,m}(\varphi_\varepsilon)$ are polynomials whose degree do not depend on ε (see Lemma 4).

Let us prove Assertion (ι) under Assumption 8. Recall that at this point N_2 is a constant whose exact value will be fixed below. Fix $\varphi \in \mathcal{A}_{N_1+N_2}(\mathbb{R})$. By (8) and Assumption 8, there exists $N_3 \in \mathbb{N}$ for which the estimate

$$L_{\nabla f_\varepsilon}(u_\varepsilon) \leq C_F \log\log d(\phi_\varepsilon, x, t, u_\varepsilon) \leq C_F \log\log \varepsilon^{-N_3}, \quad 0 < \varepsilon < \varepsilon_0(\varphi),$$

is true. Furthermore, there exist constants $C > 0$ and $k_i(m) \in \mathbb{N}$ such that the left hand side of (9) is bounded from above by

$$C^{k_1(m)(\log\log\varepsilon^{-N_3}+\gamma^{N+1}(\varepsilon))} \leq e^{k_2(m)\log\log\varepsilon^{-N_3}}\gamma(\varepsilon)^{k_1(m)\gamma^{N+1}(\varepsilon)}$$

$$\leq e^{\log(\log\varepsilon^{-N_3})^{k_2(m)}}\varepsilon^{-k_3(m)} \leq (\log\varepsilon^{-N_3})^{k_2(m)}\varepsilon^{-k_3(m)} \leq N_3^{k_2(m)}\varepsilon^{-k_2(m)-k_3(m)},$$

where $0 < \varepsilon < \varepsilon_m(\varphi)$. Set $N(m) = 2k_2(m) + k_3(m)$ and $\varepsilon_m(\varphi) = \min\{\eta(\varphi), N_3^{-k_2(m)}\}$. It is important to note that the $k_2(0)$ and $k_3(0)$ can be fixed so that the estimates with $m = 0$ hold for all N_2 and all φ. This makes the values $N_2 = N(0)$ and $\varepsilon_0(\varphi)$, which we just fixed, well defined. Assertion (ι) now follows from the fact that φ is an arbitrary function in $\mathcal{A}_{N_1+N_2}(\mathbb{R})$.

Let us now prove Assertion (ι) under Assumption 9. Following the same scheme as above, fix $\varphi \in \mathcal{A}_{N_1+N_2}(\mathbb{R})$, where N_2 will be specified below. By (8) and Assumption 9, there exist $N_3, N_4 \in \mathbb{N}$ such that the following estimates are true:

$$L_{\nabla f_\varepsilon}(u_\varepsilon) \leq C_F \log\log d(\phi_\varepsilon, x, t, u_\varepsilon) \leq C_F \log\log(\varepsilon^{-N_3}), \quad 0 < \varepsilon < \varepsilon_0(\varphi),$$

$$L_{\nabla h_\varepsilon}(v_\varepsilon) \leq C_H \log\log b(\varphi_\varepsilon, t, v_\varepsilon) \leq C_H \log\log(\varepsilon^{-N_4}), \quad 0 < \varepsilon < \varepsilon_0(\varphi).$$

Furthermore, there exist $C > 0$ and $k(m) \in \mathbb{N}$ such that the left hand side of (9) is bounded from above by

$$\Big[C\log\log(\varepsilon^{-N_4})\Big]^{1/2k(m)(\log\log\varepsilon^{-N_3-N_4})^{1/2}}$$

$$\leq \exp\Big\{k(m)\log(\log(\log(\varepsilon^{-N_4}))^C)^{1/2}(\log\log\varepsilon^{-N_3-N_4})^{1/2}\Big\}$$

$$\leq \exp\Big\{k(m)\log(\log\varepsilon^{-N_3-N_4})^C\Big\} = (\log\varepsilon^{-N_3-N_4})^{Ck(m)}$$

$$= \Big((N_3 + N_4)\log\varepsilon^{-1}\Big)^{Ck(m)} \le (N_3 + N_4)^{\lceil Ck(m)\rceil}\varepsilon^{-\lceil Ck(m)\rceil},$$

where $0 < \varepsilon < \varepsilon_m(\varphi)$. We now set $N_2 = 2\lceil Ck(m)\rceil$ and $\varepsilon_m(\varphi) = \min\{\eta(\varphi), (N_3+N_4)^{-\lceil Ck(m)\rceil}\}$. Note that C and $k(0)$ can be fixed so that the estimates with $m = 0$ hold for all N_2 and all φ. This makes the values $N_2 = N(0)$ and $\varepsilon_0(\varphi)$ well defined. Assertion (ι) now follows from the fact that φ is an arbitrary function in $\mathcal{A}_{N_1+N_2}(\mathbb{R})$.

Since $T > 0$ is arbitrary, the existence part of the proof is complete.

The proof of the uniqueness part follows the same scheme. The only difference is that now we consider the problem with respect to the difference $U - W$ of two Colombeau solutions U and W. We hence have the problem (1)–(3) with the right hand sides M_2,

$$\int_0^1 \nabla_U F(x,t,\sigma U + (1-\sigma)W)\, d\sigma \cdot (U - W) + M_1,$$

and

$$\int_0^1 \nabla_V H(t,\sigma V + (1-\sigma)V_W)\, d\sigma \cdot (V - V_W) + M_3,$$

in (2), (1), (3), respectively. Here $M_i \in \mathcal{N}$ and V_W are equal to V if we replace U by W. We apply the estimate (5) to the difference $U - W$. From the existence part of the proof we see that the first factor in the right-hand side of (5) has the moderateness property. Since the second factor is negligible, the uniqueness follows. □

Example 5. Let $n = 1$ and

$$F(x,t,U) = (1+A^2(x,t)+B^2(x,t)U^2)^{1/2}\log\log\left(1 + C^2(x,t) + D^2(x,t)U^2\right)^{1/2},$$

where $A, B, C, D \in \mathcal{G}(\overline{\Pi})$. Then

$$\partial_U F(x,t,U) = \frac{1 + B^2 U}{(1 + A^2 + B^2U^2)^{1/2}} \log\log\left(1 + C^2 + D^2U^2\right)^{1/2}$$

$$+\frac{D^2U(1 + A^2 + B^2U^2)^{1/2}}{\log\left(1 + C^2 + D^2U^2\right)^{1/2}(1 + C^2 + D^2U^2)}.$$

The function $F(x,t,U)$ is non-Lipschitz and satisfies Assumption 8(b).

Remark 6. *Theorem 3 shows that, whatsoever singularity of the initial data of our problem and whatsoever nonlinearities of F and H allowed by Assumption 8 (or 9), the problem (1)–(3) has a unique solution in the Colombeau algebra $\mathcal{G}(\overline{\Pi})$.*

REFERENCES

[1] J. F. C o l o m b e a u, *Elementary Introduction to New Generalized Functions*, North-Holland, Amsterdam 1985.

[2] F. D e m e n g e l, J. R a u c h, *Measure valued solutions of asymptotically homogeneous semilinear hyperbolic systems in one space dimension,* Proc. Edinburgh Math. Soc. **33** (1990), 443–460.

[3] T. G r a m c h e v, *Semilinear hyperbolic systems with singular initial data,* Monatshefte Math. **112** (1991), 99–113.

[4] T. G r a m c h e v, *Nonlinear maps in spaces of distributions,* Math. Zeitschr. **209** (1992), 101–114.

[5] M. G r o s s e r, E. F a r k a s, M. K u n z i n g e r, R. S t e i n b a u e r, *On the foundations of nonlinear generalized functions I and II,* Mem. Am. Math. Soc. **153** (2001), No. 729.

[6] M. G r o s s e r, M. K u n z i n g e r, M. O b e r g u g g e n b e r g e r, R. S t e i n b a u e r, *Geometric Theory of Generalized Functions.* Kluwer Academic Publishers, Dordrecht 2001.

[7] F. J o c h m a n n, L. R e c k e, *Well-posedness of an initial boundary value problem from laser dynamics,* Math. Models and Methods in Applied Sciences **12** (1999), No. 4, 593–606.

[8] G. H ö r m a n n, M. V. d e H o o p, *Microlocal analysis and global solutions of some hyperbolic equations with discontinuous coefficients,* Acta Appl. Math. **67** (2001), 173–224.

[9] I. K m i t, *Delta waves for a strongly singular initial-boundary hyperbolic problem with integral boundary condition,* J. of Analysis and its Applications **24** (2005), No. 1, 29–74.

[10] I. K m i t, *Initial-boundary problems for semilinear hyperbolic systems with singular coefficients,* Nonlin. Boundary Value Probl., to appear.

[11] I. K m i t, *Generalized solutions to hyperbolic systems with nonlinear conditions and strongly singular data,* Integral Transforms and Special Functions (2005), accepted.

[12] I. K m i t, G. H ö r m a n n, *Semilinear hyperbolic systems with nonlocal boundary conditions: reflection of singularities and delta waves,* J. of Analysis and its Applications **20** (2001), No. 3, 637–659.

[13] F. L a f o n, M. O b e r g u g g e n b e r g e r, *Generalized solutions to symmetric hyperbolic systems with discontinuous coefficients: the multidimensional case,* J. Math. Anal. Appl. **160** (1991), 93–106.

[14] M. N e d e l j k o v, S. P i l i p o v i ć,*A note on a semilinear hyperbolic system with generalized functions as coefficients,* Nonlin. Analysis, Theory, Methods and Appl. **30** (1997), 41–46.

[15] M. N e d e l j k o v, S. P i l i p o v i ć, *Generalized solution to a semilinear hyperbolic system with a non-Lipschitz nonlinearity,* Mh. Math. **125** (1998), 255–261.

[16] M. O b e r g u g g e n b e r g e r, *Multiplication of Distributions and Applications to Partial Differential Equations*, volume **259** of Pitman Research Notes in Mathematics, Longman, 1992.

[17] M. O b e r g u g g e n b e r g e r, *Hyperbolic systems with discontinuous coefficients: generalized solutions and a transmission problem in acoustics,* J. Math. Anal. Appl. **142** (1989), 452–467.

[18] M. O b e r g u g g e n b e r g e r, *Weak limits of solutions to semilinear hyperbolic systems,* Math. Ann. **274** (1986), 599–607.

[19] J. R a u c h, M. R e e d, *Nonlinear superposition and absorption of delta waves in one space dimension,* Funct. Anal. **73** (1987), 152–178.

[20] J. S i e b e r, L. R e c k e, K. S c h n e i d e r, *Dynamics of multisection semiconductor lasers,* J. Math. Sci. (New York) **124** (2004), No. 5, 5298–5309.

[21] B. T r o m b o r g, H. E. L a s s e n, H. O l e s e n, *Travelling wave analysis of semiconductor lasers,* IEEE J. of Quant. El. **30** (1994), No. 5, 939–956.

[22] T. I. Z e l e n j a k, *On stationary solutions of mixed problems arising in studying of some chemical processes,* Differential Equations **2** (1966), No. 2, 205–213.

Institute for Applied Problems of Mechanics and Mathematics
Ukrainian Academy of Sciences
Naukova St. 3b
79060 Lviv
Ukraine
E-mail: kmit@ov.litech.net