Bulletin T.CXXXIII de l'Académie serbe des sciences et des arts − 2006
Classe des Sciences mathématiques et naturelles
Sciences mathématiques, № 31

ON POSITIVITY PROPERTIES OF FUNDAMENTAL CARDINAL POLYSPLINES[1]

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(Presented at the 1st Meeting, held on February 24, 2006)

A b s t r a c t. Polysplines on strips of order p are natural generalizations of univariate splines. In [3] and [4] interpolation results for cardinal polysplines on strips have been proven. In this paper the following problems will be addressed: (i) positivity of the fundamental polyspline on the strip $[-1,1]\times\mathbb{R}^n$*, and (ii) uniqueness of interpolation for polynomially bounded cardinal polysplines.*

AMS Mathematics Subject Classification (2000): 41A82
Key Words: Cardinal splines, L-splines, fundamental spline, polyharmonic functions, polysplines

1. *Introduction*

A function $f : U \to \mathbb{C}$ defined on an open subset U of the euclidean space $\mathbb{R}^{n+1}$ is polyharmonic of order p if it is $2p$ times continuously differentiable

[1] This paper was presented at the Conference GENERALIZED FUNCTIONS 2004, Topics in PDE, Harmonic Analysis and Mathematical Physics, Novi Sad, September 22–28, 2004

[2] The research of the author is partially supported by Dirección General de Investigación (Spain) under grant BFM2003–06335–C03–03.

and $\Delta^p f(x) = 0$ for all $x \in U$, where Δ^p is the p-th iterate of the Laplace operator $\Delta = \frac{\partial^2}{\partial x_1^2} + \ldots + \frac{\partial^2}{\partial x_{n+1}^2}$. A famous example in the area of interpolation with polyharmonic functions are the so-called *thin–plate splines* (and more generally, polyharmonic splines) which are linear combinations of translates of the function φ defined by

$$\varphi(x) = |x|^2 \log |x| ; \tag{1}$$

it is well known that (1) is the fundamental solution of the biharmonic operator Δ^2 in $\mathbb{R}^2$. Since the appearance of the fundamental work of Duchon [8] such "splines" have been used by numerous authors for interpolation purposes in the multivariate case, see, for example, the papers of W. Madych and S. Nelson [18], K. Jetter [10], and the recent monograph [5]. In all these examples one interpolates data prescribed on a (finite or countable) set of discrete points.

An alternative and completely different "data concept" is provided by the notion of polyspline, introduced by O. Kounchev in [11], and extensively discussed in [12]. Polysplines distinguish from the widely spread data principle and allow to interpolate functions prescribed on surfaces of codimension 1; for a concrete application see [17]. As in [3],[4] and [15] we consider here the case that data functions are prescribed on parallel equidistant hyperplanes. Let us recall that a function $S : \mathbb{R}^{n+1} \to \mathbb{C}$ is a *cardinal polyspline of order p on strips,* when S is a $2p-2$ times continuously differentiable function on $\mathbb{R}^{n+1}$ which is polyharmonic of order p on the strips $(j, j+1) \times \mathbb{R}^n$, $j \in \mathbb{Z}$, where as usually (a, b) denotes the open interval in $\mathbb{R}$ with endpoints a, b, and $\mathbb{Z}$ is the set of all integers. Note that for $n = 0$ (with the identification $\mathbb{R}^0 = \{0\}$ and $\mathbb{R} \times \{0\} = \mathbb{R}$) a cardinal polyspline of order p on strips is just a cardinal spline on the real line $\mathbb{R}$ of degree $2p-1$ (hence of order $2p$), as discussed by I. Schoenberg in his celebrated monograph [21] (or [22]). In passing, let us remark that in the recent paper [15] it has been proved that the cardinal polysplines on strips occur as a natural limit of polyharmonic splines considered on the lattice $\mathbb{Z} \times a\mathbb{Z}^n$ when the positive number $a \longrightarrow 0$, and an estimate of the rate of convergence has been given in [16]. A discussion of wavelet analysis of cardinal polysplines can be found in [12] and [13].

In the first section we recall briefly the main results about interpolation with polysplines presented by A. Bejancu, O. Kounchev and the author in [4] (for the case $p = 2$ see [3]). An important tool are so-called fundamental cardinal polysplines which can be seen as the multivariate analog of the *fundamental cardinal spline* $L^0 : \mathbb{R} \to \mathbb{R}$ which is by definition the unique cardinal spline which has exponential decay and the interpolation property

$$L^0(0) = 1 \text{ and } L^0(j) = 0 \text{ for } j \in \mathbb{Z}, j \neq 0. \tag{2}$$

We call a polyspline L_f a *fundamental cardinal polyspline with respect to the data function* $f : \mathbb{R}^n \to \mathbb{C}$ if

$$L_f(0, y) = f(y) \text{ and } L_f(j, y) = 0 \text{ for } j \in \mathbb{Z} \setminus \{0\}, y \in \mathbb{R}^n \tag{3}$$

and if there exists $C > 0$ and $\varepsilon > 0$ such that $|L_f(t, y)| \leq Ce^{-\varepsilon|t|}$ for all $y \in \mathbb{R}^n, t \in \mathbb{R}$. The existence of fundamental cardinal polysplines is guaranteed by Theorem 2, and the reader may take formula (9) as a defining formula.

It is a well-known fact that the fundamental cardinal spline L^0 defined in (2) is non-negative on the unit interval $[-1, 1]$, see [7]. One aim of this paper is to discuss the question whether the fundamental cardinal polyspline $L_f : \mathbb{R}^{n+1} \to \mathbb{C}$ is non-negative on the strip $[-1, 1] \times \mathbb{R}^n$ for any non-negative integrable function $f : \mathbb{R}^n \to [0, \infty)$. Unfortunately, we have not been able to give a positive answer to this question, although numerical experiments support this conjecture. However, in the second section we shall prove that the non-negativity of L_f on $[-1, 1] \times \mathbb{R}^n$ for any non-negative integrable function $f : \mathbb{R}^n \to [0, \infty)$ is equivalent to the positive definiteness of a certain family of functions $\xi \longmapsto L^\xi(t)$ where t ranges over $[-1, 1]$. Here L^ξ is the fundamental cardinal L-spline $L^\xi : \mathbb{R} \to \mathbb{R}$ (cf. [19] and [3] for definition and details) which can be written as

$$L^\xi(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{its} \frac{1}{\left(s^2 + |\xi|^2\right)^p S_p(s, \xi)} ds, \tag{4}$$

where

$$S_p(s, \xi) := \sum_{k \in \mathbb{Z}} \frac{1}{\left((s + 2\pi k)^2 + |\xi|^2\right)^p}. \tag{5}$$

In the third section we shall show that for the special, and much simpler, case $p = 1$ the fundamental cardinal polyspline L_f is non-negative on the strip $[-1, 1] \times \mathbb{R}^n$ for any non-negative integrable function $f : \mathbb{R}^n \to [0, \infty)$. Moreover we give a simplified formula for the fundamental cardinal polyspline L_f in the case $p = 1$.

The last section is devoted to the question under which conditions interpolation with cardinal polysplines on strips is unique. A simple example shows that even for the case $p = 1$ there is no uniqueness if we do not impose some growth conditions. The author believes that for polynomially bounded polysplines interpolation is unique; in the last section it is proved that this is true for the case $p = 1$. It is hoped that the results presented here motivate further research on the subject.

Let us recall some terminology and notation: the Fourier transform of an integrable function $f : \mathbb{R}^n \to \mathbb{C}$ is defined by

$$\widehat{f}(\xi) := \int_{\mathbb{R}^n} e^{-i\langle y,\xi\rangle} f(y)\, dy.$$

By $B_s(\mathbb{R}^n)$ we denote the set of all measurable functions $f : \mathbb{R}^n \to \mathbb{C}$ such that the integral

$$\|f\|_s := \int_{\mathbb{R}^n} \left|\widehat{f}(\xi)\right| (1 + |\xi|^s)\, d\xi \tag{6}$$

is finite (see Definition 10.1.6 in Hörmander [9], vol. 2). By $S(\mathbb{R}^n)$ we denote the Schwartz space of rapidly decreasing functions on $\mathbb{R}^n$, see [25, p. 19]. A function $f : \mathbb{R}^n \to \mathbb{R}$ is radially symmetric if $f(x)$ depends only on the Euclidean norm $|x| = \sqrt{x_1^2 + ... + x_n^2}$.

2. *Interpolation with Polysplines*

In this section we recall the interpolation theorem for cardinal polysplines of order p proved by A. Bejancu, O. Kounchev and the present author. As mentioned above, this result formally includes the theorem of I. Schoenberg about cardinal spline interpolation by setting $n = 0$. But it should be emphasized that the proof of Theorem 1 relys on results of Ch. Micchelli in [19] about cardinal interpolation with so-called L-splines which itself is a generalization of Schoenberg's theorem.

Theorem 1. *Let $\gamma \geq 0$ be fixed. Let integrable functions $f_j : \mathbb{R}^n \to \mathbb{C}$ be given such that $f_j \in B_{2p-2}(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$, and assume that the following growth condition holds*

$$\|f_j\|_{2p-2} \leq C(1 + |j|^\gamma) \qquad \textit{for all } j \in \mathbb{Z}. \tag{7}$$

Then there exists a polyspline S of order p on strips satisfying

$$S(j, y) = f_j(y) \qquad \textit{for } y \in \mathbb{R}^n, \quad j \in \mathbb{Z}, \tag{8}$$

as well as the growth estimate

$$|S(t, y)| \leq D(1 + |t|^\gamma) \qquad \textit{for all } y \in \mathbb{R}^n.$$

An important step in the proof of the last theorem is the following:

Theorem 2. *Let $f \in L_1(\mathbb{R}^n) \cap B_{2p-2}(\mathbb{R}^n)$ and define L^ξ as in (4). Then the function L_f defined by*

$$L_f(t,y) := \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i\langle y,\xi\rangle} \widehat{f}(\xi) L^\xi(t) \, d\xi \tag{9}$$

is a polyspline of order p such that

$$\begin{cases} L_f(0,y) = f(y) & \text{for } y \in \mathbb{R}^n, \\ L_f(j,y) = 0 & \text{for } y \in \mathbb{R}^n, \quad \text{for all } j \neq 0. \end{cases}$$

There exists a constant $C > 0$ and $\eta > 0$ such that for every multi-index $\alpha \in \mathbb{N}_0^{n+1}$ with $|\alpha| \leq 2(p-1)$, the decay estimate

$$\left| \frac{\partial^\alpha}{\partial x^\alpha} L_f(x) \right| \leq C e^{-\eta|t|} \|f\|_{|\alpha|} \tag{10}$$

holds for all $x = (t,y) \in \mathbb{R}^{n+1}$.

Theorem 1 can be deduced from Theorem 2 by considering the Lagrange-type representation

$$S(t,y) = \sum_{j=-\infty}^{\infty} L_{f_j}(t-j,y). \tag{11}$$

Details and proofs can be found in [4] and [3]. In this paper we shall make use only of formula (9) which can be taken as a definition for L_f. What we need in this paper is the following fact which also shows that (9) is well-defined.

Theorem 3. *There exist constants $C > 0$ and $\eta > 0$, such that for all $t \in \mathbb{R}$, $\xi \in \mathbb{R}^n$ the following estimate holds:*

$$\left| L^\xi(t) \right| \leq C e^{-\eta|t|}. \tag{12}$$

A proof for $p = 2$ can be found in [3], and for arbitrary p in [4].

3. *A conjecture about positivity of the fundamental spline*

Recall that a function $g : \mathbb{R}^n \to \mathbb{C}$ is *positive definite* if for all $y_1, ..., y_N \in \mathbb{R}^n$ and for all complex numbers $c_1, ..., c_N$ the inequality

$$\sum_{k,l=1}^{N} c_l \overline{c_k} g\left(y_k - y_l\right) \geq 0$$

holds; for properties of positive definite functions we refer to [23], cf. also the nice introduction [24]. It is well-known that the product of two positive definite functions is positive definite. Moreover it is elementary to see that the Fourier transform $\widehat{g}$ of a non-negative function $g \in L_1\left(\mathbb{R}^n\right)$ is positive definite. Conversely, if $g \in L_1\left(\mathbb{R}^n\right)$ is positive definite then the Fourier transform is a non-negative function on $\mathbb{R}^n$ (Theorem of Mathias).

Properties of the fundamental cardinal spline $L^0 : \mathbb{R} \to \mathbb{R}$ have been investigated by de Boor and Schoenberg in [7]. One particularly nice property is that L^0 has an alternating sign on the intervals $(k, k+1)$ for $k \in \mathbb{N}_0$, i.e., that

$$(-1)^k L^0\left(x+k\right) \geq 0$$

for all $k \in \mathbb{N}_0, x \in (0,1)$. Numerical experiments have lead us to formulate the following conjecture:

Conjecture 4. *Let $f \in L_1\left(\mathbb{R}^n\right) \cap B_{2p-2}\left(\mathbb{R}^n\right)$. If f is non-negative then the fundamental polyspline L_f has an alternating sign on the strips $(k, k+1) \times \mathbb{R}^n$ for $k \in \mathbb{N}_0$, i.e., that*

$$(-1)^k L_f\left(t+k, y\right) \geq 0$$

for all $k \in \mathbb{N}_0, t \in (0,1)$ and $y \in \mathbb{R}^n$.

Note that for $k = 0$ the conjecture implies that $L_f\left(t, y\right) \geq 0$ for all $(t, y) \in [-1, 1] \times \mathbb{R}^n$. The following result shows that the latter property is equivalent to the positive definiteness of the function $\xi \longmapsto L^{\xi}\left(t\right)$ for each $t \in [-1, 1]$. Note that this formulation is independent of the data function f.

Theorem 5. *Let $t \in \mathbb{R}$ be fixed. Then the following statements are equivalent*

(i) The function $\xi \longmapsto L^{\xi}\left(t\right)$ is positive definite.

(ii) For each non-negative $f \in L_1\left(\mathbb{R}^n\right)$ such that $\widehat{f} \in L_1\left(\mathbb{R}^n\right)$ the fundamental cardinal polyspline L_f is non-negative on $\{t\} \times \mathbb{R}^n$.

(iii) For each non-negative, radially symmetric function $f \in S\left(\mathbb{R}^n\right)$ the function L_f is non-negative on $\{t\} \times \mathbb{R}^n$.

P r o o f. For (i) $\rightarrow$ (ii) let $f \in L_1(\mathbb{R}^n)$ be non-negative, clearly then $\widehat{f}$ is positive definite. By assumption, $\xi \longmapsto L^{\xi}(t)$ is positive definite. By the above remarks the function $\xi \longmapsto \widehat{f}(\xi) L^{\xi}(t)$ is positive definite. Since by Theorem 3 the function $\xi \longmapsto L^{\xi}(t)$ is bounded, we know that $\xi \longmapsto \widehat{f}(\xi) L^{\xi}(t)$ is integrable. By the theorem of Mathias (see [24, p. 412]) the (inverse) Fourier transform is non-negative, i.e., that for all $y \in \mathbb{R}^n$

$$L_f(t, y) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i\langle y,\xi\rangle} \widehat{f}(\xi) L^{\xi}(t)\, d\xi \geq 0.$$

The implication (ii) $\rightarrow$ (iii) is trivial.

Let us show that (iii) $\rightarrow$ (i). We use arguments from the proof of Bochner's theorem in [1, p. 196]: Let us define $f_\delta(y) := e^{-\frac{1}{2}\delta|y|^2}$ which is radially symmetric and in the Schwartz class. By Theorem 3 the function $\xi \longmapsto L^{\xi}(t)$ is bounded. Hence g_ε defined by $g_\varepsilon(\xi) := L^{\xi}(t) e^{-\varepsilon|\xi|^2}$ is integrable for any $\varepsilon > 0$. Parseval's identity yields

$$\int_{\mathbb{R}^n} f_\delta(y)\, \widehat{g_\varepsilon}(y)\, dx = \int_{\mathbb{R}^n} \widehat{f_\delta}(\xi)\, g_\varepsilon(\xi)\, d\xi. \tag{13}$$

On the other hand, assumption (iii) implies that

$$L_{f_\varepsilon}(t, y) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i\langle y,\xi\rangle} e^{-\varepsilon|\xi|^2} L^{\xi}(t)\, d\xi \geq 0.$$

Thus $\widehat{g_\varepsilon}(y) = (2\pi)^n L_{f_\varepsilon}(t, -y) \geq 0$ for all $y \in \mathbb{R}^n$. So we obtain from (13) that

$$\int_{\mathbb{R}^n} e^{-\frac{1}{2}\delta|y|^2} \widehat{g_\varepsilon}(y)\, dy = \left| \int_{\mathbb{R}^n} e^{-\frac{1}{2}\delta|y|^2} \widehat{g_\varepsilon}(y)\, dy \right| \leq M \int_{\mathbb{R}^n} \widehat{f_\delta}(\xi)\, d\xi, \tag{14}$$

where M is a constant such that $|g_\varepsilon(\xi)| \leq M$ for all $\xi \in \mathbb{R}^n$ and for all $0 < \varepsilon \leq 1$. Since

$$\int_{\mathbb{R}^n} \widehat{f_\delta}(\xi)\, d\xi = (2\pi)^n f_\delta(0) \leq (2\pi)^n$$

we conclude from (14) and Fatou's lemma that $\widehat{g_\varepsilon}$ is integrable. Now the inversion formula

$$g_\varepsilon(\xi) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i\langle y,\xi\rangle} \widehat{g_\varepsilon}(y)\, dy. \tag{15}$$

shows that $g_\varepsilon(\xi) := L^\xi(t)\, e^{-\varepsilon|\xi|^2}$ is positive definite. Then $L^\xi(t) = \lim_{\varepsilon\to 0} g_\varepsilon(\xi)$ for each $\xi \in \mathbb{R}^n$ (and fixed t), and since the pointwise limit of positive definite functions is again positive definite, it follows that $\xi \longmapsto L^\xi(t)$ is positive definite. □

4. *Positivity of fundamental cardinal polysplines on* $[-1,1]\times\mathbb{R}^n$ *for* $p=1$.

Recall that a function $g : \mathbb{R}^n \to \mathbb{C}$ vanishes at infinity if for each $\varepsilon > 0$ there exists a compact subset K of $\mathbb{R}^n$ such that $|f(x)| < \varepsilon$ for all $x \in \mathbb{R}^n \setminus K$. Now we want to prove

Theorem 6. *Let $p = 1$. Let $f \in L_1(\mathbb{R}^n)$ such that $\widehat{f} \in L_1(\mathbb{R}^n)$. If f is non-negative then L_f defined in (9) is a non-negative function on $\mathbb{R}^{n+1}$.*

P r o o f. From the definition of L_f and L^ξ it follows that

$$L_f(t,y) = \frac{1}{(2\pi)^{n+1}} \int\limits_{\mathbb{R}^n} \int_{-\infty}^{\infty} e^{i\langle y,\xi\rangle} e^{its} \frac{\widehat{f}(\xi)}{\left(s^2 + |\xi|^2\right) S_1(s,\xi)} ds d\xi.$$

Further it can be shown that $(\xi, s) \longmapsto \widehat{f}(\xi) / \left(s^2 + |\xi|^2\right) S_1(s,\xi)$ is integrable. The Lemma of Riemann-Lebesgue (see [25, p. 2]) shows that $L_f : \mathbb{R}^{n+1} \to \mathbb{C}$ vanishes at infinity. Now the next theorem applied to L_f and $j \in \mathbb{Z}$, shows that L_f is a non-negative function. □

Theorem 7. *Let $S : \mathbb{R}^{n+1} \to \mathbb{C}$ be a cardinal polyspline of order 1 on strips which vanishes at infinity and let $j \in \mathbb{Z}$. If*

$$S(j,y) \geq 0 \text{ and } S(j+1,y) \geq 0 \text{ for all } y \in \mathbb{R}^n$$

then S is non-negative on $[j, j+1] \times \mathbb{R}^n$.

P r o o f. Let $\varepsilon > 0$ be arbitrary. Since S vanishes at infinity we can find $R > 0$ such that $|S(t,y)| < \varepsilon$ if $|t| > R$ or $|y| > R$. Define $G_R = [j, j+1] \times \{y \in \mathbb{R}^n : |y| \leq R+1\}$. Then $S(t,y) \geq -\varepsilon$ for (t,y) in the boundary of G_R. Since S is a harmonic function in the interior of G_R and continuous on G_R the minimum principle yields that $S(t,y) \geq -\varepsilon$ for all $(t,y) \in G_R$. Hence $S(t,y) \geq -\varepsilon$ for given $(t,y) \in G_R$. Since $\varepsilon > 0$ is arbitrary we obtain $S(t,y) \geq 0$ and the proof is accomplished. □

In the rest of this section we want to give an explicit formula for L^{ξ} in the case that $p=1$ (see (16)) which clearly leads to a simpler formula for fundamental cardinal polysplines, see formula (17). From formula (16) one can see that $\xi \longmapsto L^{\xi}(t)$ is positive definite for each $t \in [-1,1]$, so one obtains with Theorem 5 a second proof that L_f is non-negative on $[-1,1] \times \mathbb{R}^n$ for a non-negative data function $f \in L^1(\mathbb{R}^n)$ such that $\widehat{f} \in L^1(\mathbb{R}^n)$. Unfortunately, for $p \geq 2$ we do not have simple formulas for L^{ξ}.

Let us compute $S_p(s,\xi)$ defined in (5) for $p=1$. An application of Poisson's summation formula (see [6, p. 204]) shows that

$$2\sum_{k\in\mathbb{Z}} \frac{y}{y^2+(x+2\pi k)^2} = \sum_{k\in\mathbb{Z}} e^{-|k|y} e^{ikx} = \frac{1-e^{-2y}}{1-2e^{-y}\cos x + e^{-2y}}.$$

We apply this to $x := s$ and $y := |\xi| > 0$ and obtain for S_p defined in (5) with $p=1$

$$S_1(s,\xi) = \frac{1-e^{-2|\xi|}}{2|\xi|\left(1-2e^{-|\xi|}\cos s + e^{-2|\xi|}\right)}.$$

Hence we obtain

$$L^{\xi}(t) = \frac{1}{\pi}\frac{|\xi|}{1-e^{-2|\xi|}} \int_{-\infty}^{\infty} e^{its} \frac{1-2e^{-|\xi|}\cos s + e^{-2|\xi|}}{s^2+|\xi|^2} ds.$$

Since $2e^{its}\cos s = e^{its}\left(e^{is}+e^{-is}\right) = e^{i(t+1)s} + e^{is(t-1)}$ we see that $L^{\xi}(t)$ is equal to

$$\frac{|\xi|}{\pi}\frac{1+e^{-2|\xi|}}{1-e^{-2|\xi|}} \int_{-\infty}^{\infty} \frac{e^{its}}{s^2+|\xi|^2} ds - \frac{|\xi|}{\pi}\frac{e^{-|\xi|}}{1-e^{-2|\xi|}} \int_{-\infty}^{\infty} \frac{e^{i(t+1)s}+e^{is(t-1)}}{s^2+|\xi|^2} ds.$$

Since $\int_{-\infty}^{\infty} e^{its}\frac{1}{s^2+|\xi|^2} ds = \frac{\pi}{|\xi|} e^{-|t|\cdot|\xi|}$ a straightforward computation shows that

$$L^{\xi}(t) = \frac{1}{e^{|\xi|}-e^{|-\xi|}}\left[\left(e^{|\xi|}+e^{-|\xi|}\right)e^{-|t|\cdot|\xi|} - e^{-|t+1|\cdot|\xi|} - e^{-|t-1|\cdot|\xi|}\right].$$

If $t \geq 1$ one obtains easily $L^{\xi}(t) = 0$. For $0 \leq t \leq 1$ one has

$$L^{\xi}(t) = \frac{e^{|\xi|(1-t)} - e^{-(1-t)\cdot|\xi|}}{e^{|\xi|}-e^{-|\xi|}} = \frac{\sinh(|\xi|(1-t))}{\sinh|\xi|}.$$

We now summarize the result:

Corollary 8. *Let $p = 1$. For $|t| \geq 1$ the function L^{ξ} vanishes and for $0 \leq t \leq 1$*

$$L^{\xi}(t) = \frac{\sinh(|\xi|(1-t))}{\sinh|\xi|}. \tag{16}$$

In case $\xi = 0$ the function $t \longmapsto L^0(t)$ is a linear spline and $L^0(t) = 1 - t$ for $0 \leq t \leq 1$.

Now Theorem 2 for $p = 1$ can be read as follows:

Theorem 9. *Let $f \in L_1(\mathbb{R}^n)$ such that $\widehat{f} \in L_1(\mathbb{R}^n)$. Then there exists a continuous function $L_f : \mathbb{R}^{n+1} \to \mathbb{C}$ which is harmonic in $(-1,0) \times \mathbb{R}^n$ and $(0,1) \times \mathbb{R}^n$ such that*

$$L_f(0,y) = f(y)$$

for $y \in \mathbb{R}^n$, and it vanishes for all $(t,y) \in \mathbb{R}^{n+1}$ with $|t| \geq 1$. Further for $0 \leq t \leq 1$

$$L_f(t,y) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i\langle y,\xi\rangle} \widehat{f}(\xi) \frac{\sinh(|\xi|(1-t))}{\sinh|\xi|} d\xi. \tag{17}$$

The fundamental linear interpolation spline has nice symmetry properties around $x = \frac{1}{2}$. In the following we want to formulate a symmetry property for cardinal polysplines of order 1. Formula (17) suggests that we have to use the addition theorem for $\sinh x = \frac{1}{2}(e^x - e^{-x})$:

$$\sinh x - \sinh y = 2\cosh\frac{x+y}{2} \sinh\frac{x-y}{2}. \tag{18}$$

Proposition 10. *For $0 \leq s \leq t \leq 1$ the following relation holds*

$$L^{\xi}(s) = L^{\xi}(t) + 2L^{\xi}\left(1 - \frac{t-s}{2}\right) \cosh\frac{(2-s-t)|\xi|}{2}. \tag{19}$$

P r o o f. Put $x = (1-s)|\xi|$ and $y = (1-t)|\xi|$ in (18): then $x + y = (2-s-t)|\xi|$ and $x - y = (t-s)|\xi|$ and we have

$$\sinh[(1-s)|\xi|] - \sinh[(1-t)|\xi|] = 2\cosh\frac{(2-s-t)|\xi|}{2} \sinh\frac{(t-s)|\xi|}{2}. \tag{20}$$

Now divide (20) by $\sinh|\xi|$ and use formula (16). □

As an illustration put $s = \frac{1}{2} - \delta$ and $t = \frac{1}{2} + \delta$ in (19). Then

$$L^{\xi}\left(\frac{1}{2} - \delta\right) - L^{\xi}\left(\frac{1}{2} + \delta\right) = 2\cosh\left(\frac{1}{2}|\xi|\right) \cdot L^{\xi}(1-\delta)$$

Multiply (19) with $\widehat{f}(\xi)\, e^{i\langle y,\xi\rangle}$ and integrate with respect to $d\xi$. Then (17) implies that for an integrable function f the following formula holds:

$$L_f(\frac{1}{2}-\delta, y) - L_f(\frac{1}{2}+\delta, y) = \frac{2}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i\langle y,\xi\rangle} \widehat{f}(\xi) \cosh(\frac{1}{2}|\xi|) L^{\xi}(1-\delta)\, d\xi.$$

5. *Uniqueness of interpolation for polynomially bounded polysplines for $p = 1$*

In this section we want to prove uniqueness results for interpolation: suppose that S_1 and S_2 are two polysplines interpolating the same data. It is clear that $S_2 - S_1$ vanishes on $\{j\} \times \mathbb{R}^n$ for all $j \in \mathbb{Z}$. We would like to conclude that $S_2 - S_1 = 0$. The following simple example shows that we have to impose some conditions on the interpolation polysplines even in the case $p = 1$ in order to obtain uniqueness:

Example 11. *There exists a harmonic function f on $\mathbb{R}^2$ which vanishes on all hyperplanes $\{j\} \times \mathbb{R}$, $j \in \mathbb{Z}$ without being identically zero, namely*

$$f(t, y) = \sin \pi t \cdot e^{\pi y}.$$

As mentioned in the introduction we believe that interpolation is unique if we assume that S is polynomially bounded, i.e., that there exists a polynomial $p(x)$ such that

$$|S(x)| \leq |p(x)|$$

for all $x \in \mathbb{R}^{n+1}$.

In the following we shall prove this for $p = 1$. In the case that S_1 and S_2 vanish at infinity we could use Theorem 7 applied to $S_2 - S_1$ and $S_1 - S_2$: then $S_2 - S_1$ and $S_1 - S_2$ are non-negative functions on the whole space, hence $S_2 - S_1 = 0$.

Instead of the minimum principle we will use the Schwarz reflection principle for harmonic functions (see e.g., [2, p. 66]) in order to prove uniqueness. Reflection principles for polyharmonic functions have been investigated by several authors and we refer to [20] for a nice introduction. However, it seems that the latter results can not be used for a proof of uniqueness of interpolation for polysplines of order $p > 1$.

Proposition 12. *Suppose that $S : \mathbb{R}^{n+1} \to \mathbb{C}$ is a cardinal polyspline of order 1 on strips with $S(j, y) = 0$ for all $j \in \mathbb{Z}$ and $y \in \mathbb{R}^n$. Then there* *exists a harmonic function* $h : \mathbb{R}^{n+1} \to \mathbb{C}$ *such that*

$$\begin{aligned} h(t,y) &= S(t,y) \text{ for } t \in (0,1) \text{ and } y \in \mathbb{R}^n, && (21)\\ h(j,y) &= 0 \text{ for } j \in \mathbb{Z} \text{ and } y \in \mathbb{R}^n, && (22) \end{aligned}$$

and for each natural number N

$$\max_{|y|\le N, t\in\mathbb{R}} |h(t,y)| \le \max_{|y|\le N, 0\le t\le 1} |S(t,y)|. \tag{23}$$

P r o o f. Clearly S is a harmonic function on the strip $(0,1)\times\mathbb{R}^n$, and it is continuous on the closure of the strip. By the Schwarz reflection principle, S can be extended to a continuous function S_1 on $[-1,1]\times\mathbb{R}^n$ by defining

$$S_1(-t,y) = -S(t,-y) \text{ for } t \in [-1,0]$$

which is harmonic on $(-1,1)\times\mathbb{R}^n$. Further $S_1(-1,y) = -S(1,-y) = 0$ for all $y \in \mathbb{R}^n$, so S_1 vanishes on the boundary of the new strip $[-1,0]\times\mathbb{R}^n$ and clearly the maximum of $|h|$ on $\{(t,y) : |y| \le N, -1 \le t \le 0\}$ can be estimated by

$$\max_{|y|\le N, -1\le t\le 0} |S_1(t,y)| \le \max_{|y|\le N, 0\le t\le 1} |S(t,y)|.$$

Now apply the same procedure to $S_1 : [-1,0]\times\mathbb{R}^n$ at the hyperplane $\{-1\}\times\mathbb{R}^n$, obtaining an extension S_2 on $[-2,0]\times\mathbb{R}^n$ of S_1 with

$$\max_{|y|\le N, -2\le t\le -1} |S_2(t,y)| \le \max_{|y|\le N, -1\le t\le 0} |S_1(t,y)| \le \max_{|y|\le N, 0\le t\le 1} |S(t,y)|.$$

Proceed in this way for negative $j \in \mathbb{Z}$, then for positive $j \in \mathbb{Z}$ and we arrive at a harmonic function $h : \mathbb{R}^{n+1} \to \mathbb{C}$ with the desired properties. □

Theorem 13. *Let* $S : \mathbb{R}^{n+1} \to \mathbb{C}$ *be a cardinal polyspline of order* 1 *on strips vanishing on the affine hyperplanes* $\{j\}\times\mathbb{R}^n$, $j \in \mathbb{Z}$. *If* S *is polynomially bounded then* S *is identically zero.*

P r o o f. By Proposition 12 there exists a harmonic function $h : \mathbb{R}^{n+1} \to \mathbb{C}$ with (21), (22) and (23). Since S is polynomially bounded, (23) implies that h is polynomially bounded. It follows that h is a harmonic *polynomial*, see [2, p. 41]. A polynomial $h(t,y)$ which vanishes on the hyperplanes $\{j\}\times\mathbb{R}^{n+1}$ for all $j \in \mathbb{Z}$ is identically zero: the equation $h(0,y) = 0$ for all $y \in \mathbb{R}^n$ implies that the (finite) Taylor expansion of $h(t,y)$ contains only non-trivial summands where the variable t occurs. Hence $h(t,y) = t \cdot h_1(t,y)$

with a polynomial h_1. Similarly, $h_1(1,y) = 0$ for all $y \in \mathbb{R}^n$ implies that $h_1(t,y) = (t-1)\,h_2(t,y)$. Hence we can write

$$h(t,y) = t\,(t-1)\,....\,(t-m)\,h_m(t,y)\,.$$

If m is bigger than the total degree of h we obtain a contradiction, showing that h must be zero. By (21) we conclude that S must be zero on $(0,1)\times\mathbb{R}^n$. In order to show that S is zero on $\mathbb{R}^{n+1}$ consider the polyspline S_j defined by $S_j(t,y) = S(t-j,y)$ for $(t,y) \in \mathbb{R}^{n+1}$, $j \in \mathbb{Z}$. By the above, S_j is zero in $(0,1)\times\mathbb{R}^n$. Hence S must be zero on $(j,j+1)\times\mathbb{R}^n$. □

Corollary 14. *Interpolation with polynomially bounded cardinal polysplines of order* 1 *on strips is unique.*

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