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INEQUALITIES WHICH INCLUDE q-INTEGRALS[1]

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A b s t r a c t. *The main problem in analyzing inequalities which include q-integrals is the fact that q-integral of a function over an interval $[a, b]$ $(0 < a < b)$ is defined by the difference of two infinite sums. Thus defined q-integral properties must include the points outside of interval of integration.*

In this paper, we will signify to some directions for solving this problem and derive some inequalities which are analogues to well-known ones in standard integral calculus.

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1. *Introduction*

In the fundamental books about q-calculus [3],[4] the q-integral of the function f over the interval $[0, b]$ is defined by

$$I_q(f; 0, b) = \int_0^b f(x) d_q x = b(1-q) \sum_{n=0}^{\infty} f(bq^n) q^n \quad (0 < q < 1). \tag{1}$$

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If f is integrable over $[0,b]$, then

$$\lim_{q\nearrow 1} I_q(f;0,b) = \int_0^b f(x)\ dx = I(f;0,b).$$

Our attention was pulled up by the definition of integral over the interval $[a,b]$. Namely, generally accepted definition is

$$I_q(f;a,b) = \int_a^b f(x)d_qx = \int_0^b f(x)d_qx - \int_0^a f(x)d_qx \quad (0<q<1). \tag{2}$$

For example, in that case the values of q-integrals of the polynomials over $[a,b]$ are very similar to well-known ones in the standard integral calculus. But, problem is what will happen if f is defined in $[a,b]$ and if it is not defined in $[0,a]$.

In this paper we specify two ways to overcome the mentioned problem. The first one is the restriction of the q-integral over $[a,b]$ to a finite sum whose number of the elements directly depends on a, b and q (see [2]). The second one is indicated in [6] and it means introduction of the definition of the q-integral of the Riemann type.

2. *The q-integrals, correlations and properties*

Let a, b and q be some real numbers such that $0<a<b$ and $q\in(0,1)$.

Beside the q-integrals defined by (1) and (2), we will consider two other types of the q-integrals.

In the paper [2], H. Gauchman has introduced *the restricted q-integral*

$$G_q(f;a,b) = \int_a^b f(x)\ d_q^G x = b(1-q)\sum_{k=0}^{n-1} f(bq^k)q^k \quad (a=bq^n). \tag{3}$$

Let us notice that lower bound of integral is $a=bq^n$, i.e., it is tied by chosen q,b and positive integer n.

In the paper [6], we have introduced *Riemann-type q-integral* by

$$R_q(f;a,b) = \int_a^b f(t)d_q^R t = (b-a)(1-q)\sum_{k=0}^{\infty} f(a+(b-a)q^k)q^k. \tag{4}$$

This definition includes only points within the interval of the integration.

The different types of the q-integral defined by (1)–(4) can be denoted in the unique way by $J_q(\,\cdot\,; a_{(J)}, b)$, where J can be G, I or R. Interval of the integration $E_{(J)} = [a_{(J)}, b]$ of q-integral $J_q(\,\cdot\,; a_{(J)}, b)$ depends on its type:

$a_{(G)} = bq^n$, $n \in \mathbb{N}$, for $G_q(\,\cdot\,; a, b)$;

$a_{(I)} = 0$ for $I_q(\,\cdot\,; 0, b)$;

$a_{(I)}$ and $a_{(R)}$ are arbitrary numbers $a \in [0, b]$ for $I_q(\,\cdot\,; a, b)$ and $R_q(\,\cdot\,; a, b)$.

We can say that a real function f is *q-integrable* on $[0, b]$ or $[a, b]$ if the series in (1) and (2) converge. In the similar way, we say that f is *qR-integrable* on $[a, b]$ if the series in (4) converges. From now on, it will be assumed that the function f is q-integrable on $[0, b]$ (qR-integrable on $[a, b]$) whenever $I_q(f; 0, b)$ or $I_q(f; a, b)$ $(R_q(f; a, b))$ appears in the formula.

In this research it is convenient to define the operators

$$\begin{array}{ll} \widehat{\ } : f \mapsto \widehat{f}, & \widehat{f}(x) = f(a + (b-a)x), \\ \widetilde{\ } : f \mapsto \widetilde{f}, & \widetilde{f}(x) = bf(bx) - af(ax), \\ \breve{\ } : f \mapsto \breve{f}, & \breve{f}(x) = f(bx) - f(ax), \end{array}$$

such that associate the functions defined on $[0, 1]$ to the function defined on $[a, b]$. Notice that, for $x \in [0, 1]$, it is

$$\widehat{(fg)}(x) = \widehat{f}(x)\ \widehat{g}(x), \quad \widetilde{(fg)}(x) = \frac{1}{b-a}(\widetilde{f}(x)\widetilde{g}(x) - ab\ \breve{f}(x)\breve{g}(x)). \tag{5}$$

The correlations between the q-integrals defined by (1)–(4) are given in the following lemma.

Lemma 2.1. *If the real function f is q-integrable on $[0, b]$ or qR-integrable on $[a, b]$, $0 < a < b$, then it holds*

$$I_q(f; 0, b) = \lim_{n\to\infty} G_q(f; bq^n, b), \tag{6}$$

$$I_q(f; a, b) = I_q(\widetilde{f}; 0, 1), \tag{7}$$

$$R_q(f; a, b) = (b-a) I_q(\widehat{f}; 0, 1). \tag{8}$$

P r o o f. The relation (6) is evident because $G_q(f; bq^n, b)$, $n \in \mathbb{N}$, are the partial sums of the series $I_q(f; 0, b)$. The equality (7) is valid according to

$$I_q(f; a, b) = (1-q) \sum_{k=0}^{\infty} (bf(bq^k) - af(aq^k))q^k = I_q(\widetilde{f}; 0, 1).$$

Finally, for (8), according to the definition we have

$$R_q(f;a,b) = (b-a)(1-q)\sum_{k=0}^{\infty} f(a+(b-a)q^k)q^k = (b-a)I_q(\widehat{f};0,1). \quad \square$$

The mentioned connections can be used to derive the inequalities for all types of the q-integrals. By (6), the inequalities for the infinite sum $I_q(f;0,b)$ can be derived in the limit process from this one for $G_q(f;a,b)$, defined by the finite sum. Using (7) and (8), the integrals $I_q(f;a,b)$ and $R_q(f;a,b)$ can be considered as the q-integrals over $[0,1]$. Nevertheless, the results for $I_q(f;a,b)$ are quite rough because the points outside of the interval of the integration (i.e., points on $[0,a]$) are included.

According to (5) and Lemma 2.1, the following integral relations are valid:

$$R_q(fg;a,b) = (b-a)I_q(\widehat{(fg)};0,1) = (b-a)I_q(\widehat{f}\,\widehat{g};0,1), \tag{9}$$

$$I_q(fg;a,b) = I_q(\widetilde{(fg)};0,1) = \frac{1}{b-a}\Big(I_q(\widetilde{f}\,\widetilde{g};0,1) - ab\; I_q(\breve{f}\,\breve{g};0,1)\Big). \tag{10}$$

At last, let us remind on some definitions and terms from q-calculus.

The q-natural number is defined by

$$[n]_q = \frac{1-q^n}{1-q} = 1+q+\cdots+q^{n-1}, \qquad n \in \mathbb{N}.$$

The function $f:[a,b]\to\mathbb{R}$ is called *q-increasing* (*q-decreasing*) on $[a,b]$ if $f(qx)\le f(x)$ ($f(qx)\ge f(x)$) whenever $x\in[a,b]$ and $qx\in[a,b]$. It is easy to see that if the function f is increasing (decreasing), then it is q-increasing (q-decreasing) for $0<q<1$.

3. *q-Chebyshev inequality*

In this section we give the q-analogues of Chebyshev inequality for the monotonic functions (see [5], pp. 239.). The discrete case of this inequality is used in [2] for the restricted q-integrals. We derive its variants for the rest of the q-integrals.

Theorem 3.1. *Let $f,g: E_{(J)}\to\mathbb{R}$ be two real functions, both q-decreasing or both q-increasing. If $J_q(\,\cdot\,;a_{(J)},b)$ is the q-integral defined by* (1), (3) *or* (4), *it holds*

$$J_q(fg;a_{(J)},b) \ge \frac{1}{b-a_{(J)}} J_q(f;a_{(J)},b)\; J_q(g;a_{(J)},b).$$

P r o o f. For $J_q(\,\cdot\,; a_{(J)}, b) = G_q(\,\cdot\,; a, b)$, $a = bq^n$, the inequality is proved in [2]. So, the inequalities

$$G_q(fg; bq^n, b) \geq \frac{1}{b - bq^n} G_q(f; bq^n, b)\ G_q(g; bq^n, b)$$

are valid for all $n = 1, 2, \ldots$. When $n \to \infty$, using (6) we get the desired inequality for $J_q(\,\cdot\,; a_{(J)}, b) = I_q(\,\cdot\,; 0, b)$. In the case $J_q(\,\cdot\,; a_{(J)}, b) = R_q(\,\cdot\,; a, b)$, from the q-monotonicity of the functions f and g on $[a, b]$ follows the q-monotonicity of the functions $\widehat{f}$ and $\widehat{g}$ on $[0, 1]$. Hence, we have

$$I_q(\widehat{f}\ \widehat{g}; 0, 1) \geq I_q(\widehat{f}; 0, 1)\ I_q(\widehat{g}; 0, 1).$$

According to (8) and (9) we get the required inequality. □

The Chebyshev inequality in the source form is not valid for $I_q(\,\cdot\,; a, b)$, where $0 < a < b$.

Example 3.1 For $f(x) = x^3$ and $g(x) = x^4$ on the interval $[1, 2]$ we have

$$I_q(x^3 \cdot x^4; 1, 2) - I_q(x^3; 1, 2) I_q(x^4; 1, 2) = 255 \frac{1 - q}{1 - q^8} - 465 \frac{(1 - q)^2}{(1 - q^4)(1 - q^5)},$$

wherefrom we conclude that the inequality holds only for $q > 1/2$, but it has opposite sign for $q < 1/2$.

Lemma 3.2. *Let the function $f : [0, b] \to \mathbb{R}$ be increasing and $0 < a < b$. If there exist two positive constants l and L such that $a^2/b^2 \leq l/L$ and for every $x, y \in [0, b]$ the inequality*

$$l \leq \frac{f(x) - f(y)}{x - y} \leq L$$

is valid, then the function $\widetilde{f} : [0, 1] \to \mathbb{R}$ is increasing too.

P r o o f. Under the conditions of the Lemma, for every $0 \leq x < y \leq b$ we have

$$l(y - x) \leq f(y) - f(x) \leq L(y - x).$$

Then it holds

$$\begin{aligned} \widetilde{f}(y) - \widetilde{f}(x) &= b(f(by) - f(bx)) - a(f(ay) - f(ax)) \\ &\geq (b^2 l - a^2 L)(y - x) \geq 0. \end{aligned}$$ □

Theorem 3.3. *Let $f, g : [0,b] \to \mathbb{R}$ be two real increasing functions. If there exist the constants l_f, L_f, l_g and L_g such that $a^2/b^2 \le l_f/L_f$, $a^2/b^2 \le l_g/L_g$ and*

$$l_f \le \frac{f(x)-f(y)}{x-y} \le L_f, \quad l_g \le \frac{g(x)-g(y)}{x-y} \le L_g$$

holds, then the inequalities are valid:

$$(a)\quad I_q(fg;a,b) \ge \frac{1}{b-a} I_q(f;a,b) I_q(g;a,b) - \frac{ab(b-a)}{[3]_q} L_f L_g$$

$$(b)\quad I_q(fg;a,b) \ge \frac{1}{b-a} I_q(f;a,b) I_q(g;a,b) - \frac{ab}{b-a}(f(b)-f(0))(g(b)-g(0)).$$

P r o o f. Suppose that f and g are both increasing on $[0,b]$. Then, according to Lemma 3.2, $\widetilde{f}$ and $\widetilde{g}$ are both increasing and hence q-increasing on $[0,1]$. With respect to (10) we can write

$$I_q(fg;a,b) = \frac{1}{b-a}\Big(I_q(\widetilde{f}\ \widetilde{g};0,1) - ab\ I_q(\breve{f}\ \breve{g};0,1)\Big).$$

Using Theorem 3.1, we have

$$I_q(\widetilde{f}\ \widetilde{g};0,1) \ge I_q(\widetilde{f};0,1)\ I_q(\widetilde{g};0,1),$$

wherefrom

$$I_q(fg;a,b) \ge \frac{1}{b-a}\Big(I_q(f;a,b)\ I_q(g;a,b) - abI_q(\breve{f}\ \breve{g};0,1)\Big). \tag{11}$$

(a) Under the conditions satisfied by the functions f and g on $[0,b]$, it holds

$$\begin{aligned} I_q(\breve{f}\ \breve{g};0,1) &= (1-q)\sum_{k=0}^{\infty}(f(bq^k)-f(aq^k))(g(bq^k)-g(aq^k))q^k \\ &\le (1-q)\sum_{k=0}^{\infty} L_f L_g (bq^k - aq^k)^2 q^k = L_f L_g (b-a)^2 \frac{1-q}{1-q^3} \end{aligned}$$

Substituting this estimation in (11), we get the first inequality.
(b) Since the functions f and g are increasing on $[0,b]$, it holds

$$I_q(\breve{f}\ \breve{g};0,1) \le (1-q)(f(b)-f(0))(g(b)-g(0))\sum_{k=0}^{\infty} q^k = (f(b)-f(0))(g(b)-g(0)),$$

what, with (11), gives the second inequality. $\square$

4. *q-Chebyshev functional and inequalities*

Let us denote by

$$\mathcal{T}(f,g;a,b)=\frac{1}{b-a}\int_a^b f(x)g(x)dx-\frac{1}{(b-a)^2}\int_a^b f(x)dx\ \int_a^b g(x)dx$$

the Chebyshev functional, i.e., the functional whose positivity for the monotonic functions proves Chebyshev inequality. Its q-analogues can be made for each of mentioned types of q-integrals:

$$\mathcal{T}_q^{(J)}(f,g;a_{(J)},b)=\frac{J_q(fg;a_{(J)},b)}{b-a_{(J)}}-\frac{J_q(f;a_{(J)},b)J_q(g;a_{(J)},b)}{(b-a_{(J)})^2},$$

where $J_q(\,\cdot\,;a_{(J)},b)$ is one of q-integrals defined by (1)–(4).

With respect to Lemma 2.1, it can be evince that

$$\mathcal{T}_q^{(I)}(f,g;0,b) = \lim_{n\to\infty}\mathcal{T}_q^{(G)}(f,g;bq^n,b), \tag{12}$$

$$\mathcal{T}_q^{(R)}(f,g;a,b) = \mathcal{T}_q^{(I)}(\widehat{f},\widehat{g};0,1), \tag{13}$$

$$\mathcal{T}_q^{(I)}(f,g;a,b) = \frac{1}{(b-a)^2}\Big(\mathcal{T}_q^{(I)}(\widetilde{f},\widetilde{g};0,1)-ab\ I_q(\check{f}\,\check{g};0,1)\Big). \tag{14}$$

In that way, the basic inequalities including the Chebyshev functional are derived for $\mathcal{T}_q^{(G)}(f,g;a,b)$, and then for the other variants.

Theorem 4.1. *Let $f,g:E_{(J)}\to\mathbb{R}$ be two real functions. If there exist the real constants m and M such that*

$$m(g(x)-g(y))\le f(x)-f(y)\le M(g(x)-g(y)),\qquad a_{(J)}\le x<y\le b,$$

is valid, then it holds the inequality

$$(m+M)\ \mathcal{T}_q^{(J)}(f,g;a_{(J)},b)\ge\mathcal{T}_q^{(J)}(f,f;a_{(J)},b)+mM\ \mathcal{T}_q^{(J)}(g,g;a_{(J)},b),$$

where $J_q(\,\cdot\,;a_{(J)},b)$ is the q-integral defined by (1), (3) *or* (4).

P r o o f. At the start, let $J_q(\,\cdot\,;a_{(J)},b)=G_q(\,\cdot\,;a,b)$, $a=bq^n$, $n\in\mathbb{N}$. Under the conditions satisfied by the functions f and g, the product

$$\Big(M(g(bq^i)-g(bq^j))-(f(bq^i)-f(bq^j))\Big)\Big((f(bq^i)-f(bq^j))-m(g(bq^i)-g(bq^j))\Big)$$

is nonnegative for $i, j = 0, 1, \dots, n-1$ $(i < j)$, i.e.,

$$\begin{aligned}(f(bq^i) - f(bq^j))^2 \quad &+ mM(g(bq^i) - g(bq^j))^2 \\ &\leq (m+M)(f(bq^i) - f(bq^j))(g(bq^i) - g(bq^j)).\end{aligned}$$

Multiplying by q^{i+j} $(i < j)$ and summing over i and j, we obtain

$$\begin{aligned}&\sum_{\substack{i,j=0\\ i<j}}^{n-1} (f(bq^i) - f(bq^j))^2 q^{i+j} + mM \sum_{\substack{i,j=0\\ i<j}}^{n-1} (g(bq^i) - g(bq^j))^2 q^{i+j} \\ &\leq (m+M) \sum_{\substack{i,j=0\\ i<j}}^{n-1} (f(bq^i) - f(bq^j))(g(bq^i) - g(bq^j)) q^{i+j}.\end{aligned}$$

Since (see [1],[5]) $\mathcal{T}_q^{(G)}(f, g; a, b)$ can be presented in the form

$$\mathcal{T}_q^{(G)}(f, g; a, b) = \left(\frac{b(1-q)}{b-a}\right)^2 \sum_{\substack{i,j=0\\ i<j}}^{n-1} (f(bq^i) - f(bq^j))(g(bq^i) - g(bq^j)) q^{i+j},$$

we have the required inequality.

For $J_q(\,\cdot\,; a_{(J)}, b) = I_q(\,\cdot\,; 0, b)$ it is enough to put $n \to \infty$ in the proved inequality for the previous case.

Finally, let $J_q(\,\cdot\,; a_{(J)}, b) = R_q(\,\cdot\,; a, b)$. Under the condition satisfied by the functions f and g on $[a, b]$, the functions $\widehat{f}$ and $\widehat{g}$ satisfy the same conditions on $[0, 1]$. Applying (13) and the proved inequality for $\mathcal{T}_q^{(I)}(\widehat{f}, \widehat{g}; 0, 1)$, we get the statement. $\square$

The upper inequality can be presented in the form

$$J_q((f-mg)(Mg-f); a_{(J)}, b) \geq \frac{1}{b - a_{(J)}} J_q(f-mg; a_{(J)}, b) J_q(Mg-f; a_{(J)}, b),$$

what is Chebyshev inequality for the monotonic functions $f-mg$ and $Mg-f$.

Theorem 4.2. *Let $f, g : E_{(J)} \to \mathbb{R}$ be two real functions. If there exist the real constants l and L such that*

$$\begin{aligned}l((b - a_{(J)})g(x) - J_q(g; a_{(J)}, b)) \quad &\leq (b - a_{(J)})f(x) - J_q(f; a_{(J)}, b) \\ &\leq L((b - a_{(J)})g(x) - J_q(g; a_{(J)}, b)),\end{aligned}$$

on $[a,b]$*, then it holds the inequality*

$$(l+L)\ \mathcal{T}_q^{(J)}(f,g;a_{(J)},b) \geq \mathcal{T}_q^{(J)}(f,f;a_{(J)},b) + lL\ \mathcal{T}_q^{(J)}(g,g;a_{(J)},b),$$

where $J_q(\,\cdot\,;a_{(J)},b)$ *is the* q*-integral defined by* (1), (3) *or* (4).

P r o o f. Let $J_q(\,\cdot\,;a_{(J)},b)$ be $G_q(\,\cdot\,;a,b)$, $a=bq^n$, or $I_q(\,\cdot\,;0,b)$. Then Chebyshev functional $\mathcal{T}_q^{(J)}$ can be represented (see [1]) in the form

$$\begin{aligned}&\mathcal{T}_q^{(J)}(f,g;a_{(J)},b)\\&= \frac{b(1-q)}{b-a_{(J)}}\sum_{i=0}^{N_{(J)}}\Big(f(bq^i)-\frac{J_q(f;a_{(J)},b)}{b-a_{(J)}}\Big)\Big(g(bq^i)-\frac{J_q(g;a_{(J)},b)}{b-a_{(J)}}\Big)q^i,\end{aligned}$$

where $N_{(G)}=n-1$ and $N_{(I)}=\infty$. Under the conditions satisfied by the functions f and g, the product

$$\begin{aligned}&\bigg(L\Big(g(bq^i)-\frac{J_q(g;a_{(J)},b)}{b-a_{(J)}}\Big)-\Big(f(bq^i)-\frac{J_q(f;a_{(J)},b)}{b-a_{(J)}}\Big)\bigg)\\&\qquad\times\bigg(\Big(f(bq^i)-\frac{J_q(f;a_{(J)},b)}{b-a_{(J)}}\Big)-l\Big(g(bq^i)-\frac{J_q(g;a_{(J)},b)}{b-a_{(J)}}\Big)\bigg)\end{aligned}$$

is nonnegative for $i=0,1,\dots,N_{(J)}$, i.e.,

$$\begin{aligned}&\Big(f(bq^i)-\frac{J_q(f;a_{(J)},b)}{b-a_{(J)}}\Big)^2+lL\Big(g(bq^i)-\frac{J_q(g;a_{(J)},b)}{b-a_{(J)}}\Big)^2\\&\qquad\leq (l+L)\Big(f(bq^i)-\frac{J_q(f;a_{(J)},b)}{b-a_{(J)}}\Big)\Big(g(bq^i)-\frac{J_q(g;a_{(J)},b)}{b-a_{(J)}}\Big).\end{aligned}$$

For $J_q(\,\cdot\,;a_{(J)},b)=G_q(\,\cdot\,;a,b)$, the desired inequality is obtained only by multiplying the upper inequalities by q^i and summing over i, $i=0,1,\dots,n-1$. In the case $J_q(\,\cdot\,;a_{(J)},b)=I_q(\,\cdot\,;0,b)$ it is needed to put $n\to\infty$.

For $J_q(\,\cdot\,;a_{(J)},b)=R_q(\,\cdot\,;a,b)$ it should be noticed that under the conditions satisfied by f and g on $[a,b]$ the inequality

$$l(\widehat{g}(x)-I_q(\widehat{g};0,1))\leq \widehat{f}(x)-I_q(\widehat{f};0,1)\leq L(\widehat{g}(x)-I_q(\widehat{g};0,1))$$

is valid for all $x\in[0,1]$. Hence, according to (13) and the proved inequality for the previous case, we get desired inequality. $\square$

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