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REGULARITY THEORY IN COLOMBEAU ALGEBRAS[1]

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A b s t r a c t. This paper introduces and discusses various notions of regularity of solutions to partial differential equations in Colombeau algebras. The aim is to suggest a number of approaches that allow to extend the regularity theory to nonlinear equations. A relation with the notion of delta waves is observed.

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1. *Introduction*

Consider a partial differential equation $P(u) = f$ where P is a linear or nonlinear partial differential operator. This paper addresses regularity of solutions in the Colombeau algebra $\mathcal{G}(\Omega)$ of generalized functions on open subsets $\Omega \subset \mathbb{R}^n$. That is, assuming that $u \in \mathcal{G}(\Omega)$ is a solution, we ask how

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the regularity or lack of regularity in the driving term f affects the regularity of the solution u; in the case of hyperbolic equations, we are also interested in the influence of the regularity of the initial data.

As opposed to regularity theory for solutions in the space of distributions $\mathcal{D}'(\Omega)$, the algebra of smooth functions $\mathcal{C}^\infty(\Omega)$ is not a suitable category for measuring regularity for Colombeau solutions. This is simply due to the fact that the ring of constants in $\mathcal{G}(\Omega)$ contains non-classical numbers (not belonging to $\mathbb{C}$), so that a homogeneous, linear equation $P(u) = 0$ always has non-classical, constant solutions. With the introduction of the subalgebra $\mathcal{G}^\infty(\Omega)$ of $\mathcal{G}(\Omega)$ in [21] it has been noticed early on that the key to regularity theory in the Colombeau setting lies in the asymptotic behavior of the nets of smooth functions representing the elements of $\mathcal{G}(\Omega)$. For *linear* equations, even with coefficients belonging to the Colombeau algebra $\mathcal{G}(\Omega)$, a large bulk of regularity results using $\mathcal{G}^\infty(\Omega)$ has been developed in the past decade. These results include conditions on $\mathcal{G}^\infty$-hypoellipticity of linear operators, microlocalization of the notion of $\mathcal{G}^\infty$-regularity (the $\mathcal{G}^\infty$-wave front set), propagation of $\mathcal{G}^\infty$-regularity along bicharacteristic curves, a calculus of pseudodifferential operators with Colombeau symbols and, more recently, generalized Fourier integral operators [2, 4, 5, 6, 7, 8, 11, 12, 13, 14, 15, 16, 17, 19, 22, 24].

However, though $\mathcal{G}^\infty(\Omega)$ is an algebra - hence invariant under polynomial mappings -, it is not invariant under arbitrary nonlinear mappings. In fact, given $u \in \mathcal{G}^\infty(\Omega)$, the superposition $F(u)$ generally no longer belongs to $\mathcal{G}^\infty(\Omega)$ even when F is a trigonometric function. Thus the $\mathcal{G}^\infty$-category is not suitable for a general regularity theory for nonlinear equations. It appears that a tighter control over the asymptotic behavior of the representing nets is required to construct a nonlinear regularity theory. Promising candidates for such a theory are the Colombeau-Zygmund spaces of [13], the spaces of generalized functions of limited growth of [4, 5], notions of subsheaf regularity introduced in [18] in the context of $(\mathcal{C}, \mathcal{E}, \mathcal{P})$-algebras, and the notions related to *slow scale nets*, introduced in [16] and applied to regularity theory for semilinear wave equations in [22].

One may state safely that nonlinear regularity theory in the Colombeau setting is in a stage of exploration. It is the aim of this paper to discuss and introduce a number of candidates for this purpose and to show in various instances how they allow to deduce global regularity statements as well as preliminary results about propagation of singularities. As will be explained in the respective sections, essential tests for suitability are: hypoellipticity for linear constant coefficient operators; propagation of regularity into the interior of the light cone of the linear wave equation; global preservation of regularity of the initial data for semilinear wave equations; propagation of regularity across singularity bearing characteristic curves for the semilinear wave equation in one space dimension.

The plan of the paper is as follows. In Section 2, the notions of Colombeau theory will be recalled that are needed later. Section 3 is devoted to the discussion of subalgebras of $\mathcal{G}(\Omega)$ which appear as possible candidates for measuring regularity in the nonlinear case. These subalgebras will be scrutinized with respect to their suitability for an investigation of hypoellipticity for linear constant coefficient operators as well as propagation of singularities for the linear wave equation in Section 4. Section 5 tests the subalgebras in the situation of semilinear wave equations in one space dimension. The question of global regularity will be addressed and a result on propagation of singularities is presented, which employs arguments from the theory of delta waves.

2. *Notation*

The theory of generalized functions of Colombeau [1] has been presented in numerous papers and monographs by now, so we will just collect some basic notions which we need without further explanation. For details we refer to [9].

Let Ω be an open subset of $\mathbb{R}^n$. The basic objects of the theory as we use it here are families $(u_\varepsilon)_{\varepsilon\in(0,1]}$ of smooth functions $u_\varepsilon \in \mathcal{C}^\infty(\Omega)$ for $0 < \varepsilon \le 1$. The following subalgebras are singled out:

Moderate families, denoted by $\mathcal{E}_{\mathrm{M}}(\Omega)$, are defined by the property

$$\forall K \Subset \Omega\, \forall \alpha \in \mathbb{N}_0^n\, \exists p \ge 0 : \sup_{x\in K} |\partial^\alpha u_\varepsilon(x)| = \mathcal{O}(\varepsilon^{-p}) \text{ as } \varepsilon \to 0. \tag{1}$$

Null families, denoted by $\mathcal{N}(\Omega)$, are defined by the property

$$\forall K \Subset \Omega\, \forall \alpha \in \mathbb{N}_0^n\, \forall q \ge 0 : \sup_{x\in K} |\partial^\alpha u_\varepsilon(x)| = \mathcal{O}(\varepsilon^{q}) \text{ as } \varepsilon \to 0. \tag{2}$$

The null families form a differential ideal in the collection of moderate families. The *Colombeau algebra* is the factor algebra

$$\mathcal{G}(\Omega) = \mathcal{E}_{\mathrm{M}}(\Omega)/\mathcal{N}(\Omega).$$

The Colombeau algebra on a closed half space $\mathbb{R}^n \times [0,\infty)$ is constructed in a similar way. The restriction of an element $u \in \mathcal{G}(\mathbb{R}^n \times [0,\infty))$ to the line

$\{t=0\}$ is defined on representatives by $u|_{\{t=0\}} =$ class of $(u_\varepsilon(\cdot,0))_{\varepsilon\in(0,1]}$. Similarly, restrictions of the elements of $\mathcal{G}(\Omega)$ to open subsets of Ω are defined on representatives, and $\Omega \to \mathcal{G}(\Omega)$ is a sheaf of differential algebras on $\mathbb{R}^n$. The space of compactly supported distributions is imbedded in $\mathcal{G}(\Omega)$ by convolution

$$\iota : \mathcal{E}'(\Omega) \to \mathcal{G}(\Omega),\ \iota(w) = \text{ class of } (w * (\varphi_\varepsilon)|_\Omega)_{\varepsilon\in(0,1]}, \tag{3}$$

where

$$\varphi_\varepsilon(x) = \varepsilon^{-n}\varphi\,(x/\varepsilon) \tag{4}$$

is obtained by scaling a fixed test function $\varphi \in \mathcal{S}(\mathbb{R}^n)$ of integral one with all moments vanishing. By the sheaf property, this can be extended in a unique way to an imbedding of the space of distributions $\mathcal{D}'(\Omega)$. This imbedding renders $\mathcal{C}^\infty(\Omega)$ a faithful subalgebra. In fact, given $f \in \mathcal{C}^\infty(\Omega)$, one can define a corresponding element of $\mathcal{G}(\Omega)$ by the constant imbedding $\sigma(f) =$ class of $(f_\varepsilon)_{\varepsilon\in(0,1]}$ with $f_\varepsilon \equiv f$ for all ε. Then $\iota(f) = \sigma(f)$ in $\mathcal{G}(\Omega)$.

If $u \in \mathcal{G}(\Omega)$ and F is a smooth function which is of at most polynomial growth at infinity, together with all its derivatives, the superposition $F(u)$ is a well-defined element of $\mathcal{G}(\Omega)$.

Families $(r_\varepsilon)_{\varepsilon\in(0,1]}$ of complex numbers such that $|r_\varepsilon| = \mathcal{O}(\varepsilon^{-p})$ as $\varepsilon \to 0$ for some $p \geq 0$ are called *moderate*, those for which $|r_\varepsilon| = \mathcal{O}(\varepsilon^q)$ for every $q \geq 0$ are termed *negligible*. The ring $\widetilde{\mathbb{C}}$ of Colombeau generalized numbers is obtained by factoring moderate families of complex numbers with respect to negligible families. When Ω is connected, $\widetilde{\mathbb{C}}$ coincides with the *ring of constants* in the differential algebra $\mathcal{G}(\Omega)$.

Regularity theory for linear equations has been based on the subalgebra $\mathcal{G}^\infty(\Omega)$ of *regular generalized functions* in $\mathcal{G}(\Omega)$. It is defined by those elements which have a representative satisfying

$$\forall K \Subset \Omega\, \exists p \geq 0\, \forall \alpha \in \mathbb{N}_0^n : \sup_{x\in K} |\partial^\alpha u_\varepsilon(x)| = \mathcal{O}(\varepsilon^{-p}) \ \text{ as } \varepsilon \to 0. \tag{5}$$

Observe the change of quantifiers with respect to formula (1); locally, all derivatives of a regular generalized function have the same order of growth in $\varepsilon > 0$.

3. *Measuring regularity*

We begin by introducing further subalgebras of $\mathcal{G}(\Omega)$ that will be potentially useful in the study of regularity in the nonlinear case; Ω is an open subset of $\mathbb{R}^n$ throughout. Following the general concepts of construction developed in [4, 5], we start with a set of sequences $\mathcal{R}$ of nonnegative reals.

Definition 1. *An element $u \in \mathcal{G}(\Omega)$ is called of* asymptotic $\mathcal{R}$-type, *if for some representative and every $K \Subset \Omega$ there is a sequence $(N_\ell)_{\ell \geq 0}$ belonging to $\mathcal{R}$ such that for all $\alpha \in \mathbb{N}_0^n$, $\|\partial^\alpha u_\varepsilon\|_{\mathrm{L}^\infty(K)} = \mathcal{O}(\varepsilon^{-N_{|\alpha|}})$ as $\varepsilon \to 0$. The corresponding subspace of $\mathcal{G}(\Omega)$ will be denoted by $\mathcal{G}^{\mathcal{R}}(\Omega)$.*

Under certain stability conditions on the set $\mathcal{R}$ of sequences detailed in [4, 5], $\mathcal{G}^{\mathcal{R}}(\Omega)$ forms a sheaf of differential subalgebras of $\mathcal{G}(\Omega)$. This is immediately checked in the following cases which we shall employ in the sequel.

Example 2. *(a) When $\mathcal{R} = \{0\}$, we obtain the differential subalgebra $\mathcal{G}^0(\Omega)$ represented by families $(u_\varepsilon)_{\varepsilon \in (0,1]}$ all whose derivatives remain locally uniformly bounded as $\varepsilon \to 0$.*

(b) When $\mathcal{R}$ consists of the set of bounded sequences, the resulting differential subalgebra is $\mathcal{G}^\infty(\Omega)$.

(c) When $\mathcal{R} = \{(N_\ell)_{\ell \geq 0} : \limsup_{\ell \to \infty} N_\ell/\ell < \infty\}$, we obtain a differential subalgebra denoted by $\mathcal{G}^{\mathcal{L}}(\Omega)$. The representatives of its elements have the following property: for every $K \Subset \Omega$ there are $a, b \geq 0$ such that for all $\alpha \in \mathbb{N}_0^n$, $\|\partial^\alpha u_\varepsilon\|_{\mathrm{L}^\infty(K)} = \mathcal{O}(\varepsilon^{-a|\alpha|-b})$ as $\varepsilon \to 0$.

The use of the algebra $\mathcal{G}^0(\Omega)$ for regularity purposes was proposed by [3]. Algebras of type (c) were introduced in [4]; a similar space (with a fixed) has also been considered by [28]. The following inclusion relations are obvious:

$$\mathcal{G}^0(\Omega) \subset \mathcal{G}^\infty(\Omega) \subset \mathcal{G}^{\mathcal{L}}(\Omega) \subset \mathcal{G}(\Omega).$$

Using the arguments of [10, Thm. 1.2.6] one can show that for any sequence $(N_\ell)_{\ell \geq 0}$ of real numbers there is an element $(u_\varepsilon)_{\varepsilon \in (0,1]} \in \mathcal{E}_{\mathrm{M}}(\mathbb{R})$ such that $\partial^\ell u_\varepsilon(0) = \varepsilon^{-N_\ell}$ for all $\ell \in \mathbb{N}_0$. Therefore, $\mathcal{G}^{\mathcal{L}}(\mathbb{R})$ is a proper subalgebra of $\mathcal{G}(\mathbb{R})$, and the other inclusions are proper as well. Following the notion of subsheaf regularity introduced in [18], we shall also consider the following subspace of $\mathcal{G}(\Omega)$.

Definition 3. *$\mathcal{L}^1_{\mathcal{G}}(\Omega)$ is the space of the elements of $\mathcal{G}(\Omega)$ with a representative $(u_\varepsilon)_{\varepsilon \in (0,1]}$ such that $\lim_{\varepsilon \to 0} u_\varepsilon$ exists in $\mathrm{L}^1_{\mathrm{loc}}(\Omega)$.*

As subalgebras of $\mathcal{G}(\Omega)$, all spaces from Ex. 2 are invariant under polynomial maps. Additional invariance properties of these algebras as well as the space $\mathcal{L}^1_{\mathcal{G}}(\Omega)$ under superposition by nonlinear maps are collected in the following result.

Proposition 4. *(a) $F(\mathcal{G}^0(\Omega)) \subset \mathcal{G}^0(\Omega)$ for every smooth function F.*

(b) $F(\mathcal{G}^{\mathcal{L}}(\Omega)) \subset \mathcal{G}^{\mathcal{L}}(\Omega)$ for every smooth function F which is bounded, together with all its derivatives.

(c) $F(\mathcal{L}^1_{\mathcal{G}}(\Omega)) \subset \mathcal{L}^1_{\mathcal{G}}(\Omega)$ for every smooth function F all whose derivatives grow at most polynomially at infinity and which is Lipschitz continuous with a global Lipschitz constant.

P r o o f. Take a representative $(u_\varepsilon)_{\varepsilon\in(0,1]}$ of an element u of one of the algebras under consideration. Apply the chain rule to $F(u_\varepsilon)$. In case (a), it is clear that all derivatives will remain bounded on compact sets for whatever smooth function F. To prove (b), assume that on some compact subset K of $\mathbb{R}^n$, $\|\partial^\alpha u_\varepsilon\|_{\mathrm{L}^\infty(K)} = \mathcal{O}(\varepsilon^{-a|\alpha|-b})$ as $\varepsilon \to 0$. If F is bounded together with all derivatives, it is quite obvious that $\|\partial^\alpha F(u_\varepsilon)\|_{\mathrm{L}^\infty(K)} = \mathcal{O}(\varepsilon^{-(a+b)|\alpha|})$ as $\varepsilon \to 0$, so $F(u)$ belongs to $\mathcal{G}^{\mathcal{L}}(\Omega)$. In case (c), the polynomial bounds guarantee that $F(u)$ is a well-defined element of $\mathcal{G}(\Omega)$. If u_ε converges to an element $w \in \mathrm{L}^1(K)$ on some compact set K, the estimate $|F(u_\varepsilon) - F(w)| \leq \mathrm{Lip_F}|\mathrm{u}_\varepsilon - \mathrm{w}|$ shows that $F(u_\varepsilon)$ converges to $F(w)$, as desired. □

The kind of regularity that is encapsulated in the subspaces above is described by the following assertions.

Proposition 5. *(a) $\mathcal{L}^1_{\mathcal{G}}(\Omega) \cap \mathcal{D}'(\Omega) = \mathrm{L}^1_{\mathrm{loc}}(\Omega)$.*
(b) $\mathcal{G}^0(\Omega) \cap \mathcal{D}'(\Omega) = \mathcal{G}^\infty(\Omega) \cap \mathcal{D}'(\Omega) = \mathcal{C}^\infty(\Omega)$.

P r o o f. Let $w \in \mathcal{D}'(\Omega)$. Using the fact that the imbedding ι is a sheaf morphism, it suffices to assume that w is compactly supported. According to (3), $\iota(w)$ is given by convolution with a mollifier φ_ε as in (4). Assertion (a) is now obvious. As for (b), it is well-known (see [21, Thm. 5.2]) that $\mathcal{G}^\infty(\Omega) \cap \mathcal{D}'(\Omega) = \mathcal{C}^\infty(\Omega)$; this is all the more so true of its subalgebra $\mathcal{G}^0(\Omega)$. □

We note that $\mathcal{D}'(\Omega) \subset \mathcal{G}^{\mathcal{L}}(\Omega)$ so that this space carries a kind of regularity, if any, which cannot be expressed by means of subspaces of the space of distributions.

4. *Linear equations*

This section serves to test the notions introduced in Section 3 in the case of linear partial differential operators with (classical) constant coefficients. We begin by investigating local regularity. A linear partial differential operator

$$P(\partial) = \sum_{|\alpha|\leq m} a_\alpha \partial^\alpha \tag{6}$$

with coefficients $a_\alpha \in \mathbb{C}$ is called *hypoelliptic in the classical sense* if for every open subset Ω of $\mathbb{R}^n$ the following regularity property holds:

$$\left.\begin{array}{l} w \in \mathcal{D}'(\Omega) \\ P(\partial)w \in \mathcal{C}^\infty(\Omega) \end{array}\right\} \Rightarrow \; w \in \mathcal{C}^\infty(\Omega).$$

Let $\Omega \to \mathcal{A}(\Omega)$ be a subsheaf (of complex vector spaces) of the sheaf $\Omega \to \mathcal{G}(\Omega)$. A linear partial differential operator (6) is called $\mathcal{A}$*-hypoelliptic (in the sense of Colombeau algebras)*, if for every open set $\Omega \subset \mathbb{R}^n$,

$$\left.\begin{array}{l} u \in \mathcal{G}(\Omega) \\ P(\partial)u \in \mathcal{A}(\Omega) \end{array}\right\} \Rightarrow \; u \in \mathcal{A}(\Omega).$$

Remark 6. *No linear partial operator of order $m > 0$ is $\mathcal{C}^\infty$-hypoelliptic (in the sense of Colombeau algebras). Indeed, let $\xi \in \mathbb{C}^n$ be a zero of $P(\mathrm{i}\xi)$ and $c \in \widetilde{\mathbb{C}} \setminus \mathbb{C}$ be a generalized constant. Then the family $u_\varepsilon(x) = c_\varepsilon \exp(\mathrm{i}x\xi)$ defines an element $u \in \mathcal{G}(\Omega)$ which does not belong to $\mathcal{C}^\infty(\Omega)$ and solves the homogeneous equation $P(\partial)u = 0$ in $\mathcal{G}(\Omega)$. The same example shows that no such operator is $\mathcal{G}^0$-hypoelliptic either.*

The first part of the following result was already proved in [21]. The proof is briefly displayed below in order to shed light on what can or cannot be done with the other algebras from Section 3.

Proposition 7. *(a) A linear partial differential operator (6) is hypoelliptic in the classical sense if and only if it is $\mathcal{G}^\infty$-hypoelliptic.*

(b) Every linear partial differential operator (6) which is hypoelliptic in the classical sense is also $\mathcal{G}^{\mathcal{L}}$-hypoelliptic.

P r o o f. (a) Let $P(\partial)$ be hypoelliptic in the classical sense, $u \in \mathcal{G}(\Omega)$ and $P(\partial)u = f \in \mathcal{G}^\infty(\Omega)$. The operator $P(\partial)$ has a fundamental solution Q whose $\mathcal{C}^\infty$-singular support consists of the origin. Let χ be a smooth cut-off function with support in a small neighborhood of the origin. Further, take a relatively compact open subset $\omega \subset \Omega$ and let ψ be a smooth cut-off function with compact support L in Ω which is identically equal to one in a neighborhood of ω. Then

$$\psi u = Q * P(\partial)(\psi u) = Q * (\psi f) + (Q - \chi Q) * g + (\chi Q) * g \tag{7}$$

where $g \in \mathcal{G}(\Omega)$ vanishes on ω and has its (compact) support contained in L. Convolution with the fixed distribution Q is a continuous map from $\mathcal{D}_L(\Omega)$

into $\mathcal{C}^\infty(\Omega)$. Therefore, if K is any compact subset of Ω, there is ℓ and $C > 0$ such that

$$\sup_{x\in K} |\partial^\alpha Q * (\psi f_\varepsilon)(x)| \le C \sup_{|\beta|\le\ell} \sup_{x\in L} |\partial^{\alpha+\beta}(\psi f_\varepsilon)(x)| \tag{8}$$

holds for all $\alpha \in \mathbb{N}_0^n$. This shows that $Q * (\psi f)$ belongs to $\mathcal{G}^\infty(\Omega)$, because ψf does. Next, we use the fact that $Q - \chi Q$ is a smooth function and that $\partial^\alpha(Q - \chi Q)$ maps $\mathcal{C}_L(\Omega)$ into $\mathcal{C}(\Omega)$ continuously via convolution, for every $\alpha \in \mathbb{N}_0^n$. This observation yields the inequality

$$\sup_{x\in K} |\partial^\alpha((Q - \chi Q) * g_\varepsilon)(x)| \le C_\alpha \sup_{x\in L} |g_\varepsilon(x)|, \tag{9}$$

so all derivatives of $(Q - \chi Q) * g_\varepsilon$ inherit the asymptotic growth of g_ε itself, so $(Q - \chi Q) * g$ belongs to $\mathcal{G}^\infty(\Omega)$ as well. Finally, the term $(\chi Q) * g$ possibly carries singularities, but it vanishes in an open subset ω' of ω which is only slightly smaller than ω, depending on the diameter of the support of χ. Thus ψu is $\mathcal{G}^\infty$-regular on ω'. As the choice of ω and χ is free, this proves the $\mathcal{G}^\infty$-regularity of u on Ω.

Conversely, assume that $P(\partial)$ is $\mathcal{G}^\infty$-hypoelliptic and let $w \in \mathcal{D}'(\Omega)$ be a distributional solution of $P(\partial)w = h$ with $h \in \mathcal{C}^\infty(\Omega)$. Then $P(\partial)\iota(w) = \iota(P(\partial)w) = \iota(h)$, because the imbedding ι commutes with linear constant coefficient operators. Thus $\iota(w) \in \mathcal{G}^\infty(\Omega)$, implying $w \in \mathcal{C}^\infty(\Omega)$ as noted in Prop. 5.

(b) We proceed as in (a). If $f \in \mathcal{G}^{\mathcal{L}}(\Omega)$ then $\|\partial^\alpha f_\varepsilon\|_{\mathrm{L}^\infty(L)} = \mathcal{O}(\varepsilon^{-a|\alpha|-b})$ for some $a, b \ge 0$ and all $\alpha \in \mathbb{N}_0^n$. Also, $g_\varepsilon = \mathcal{O}(\varepsilon^{-p})$ for some $p \ge 0$. Therefore, inequalities (8) and (9) show that $\|\partial^\alpha(\psi u_\varepsilon)\|_{\mathrm{L}^\infty(\omega')} = \mathcal{O}(\varepsilon^{-a|\alpha|-b'})$ with $b' = \max(\ell + b, p)$. □

The second test for the notions under scrutiny is (global) propagation of singularities for the linear wave equation. Consider the Cauchy problem for the wave equation in n space dimensions

$$\begin{aligned} &\partial_t^2 u(x,t) - \textstyle\sum_{j=1}^n \partial_{x_j}^2 u(x,t) = 0, \quad x \in \mathbb{R}^n, t > 0, \\ &u(x,0) = u_0(x),\ \partial_t u(x,0) = u_1(x), \quad x \in \mathbb{R}^n. \end{aligned} \tag{10}$$

Denote by $\Gamma = \{(x,t) : t \ge 0, |x| = t\}$ the forward light-cone issuing from the origin. For distributional solutions - e.g., solutions $u \in \mathcal{C}^\infty([0,\infty) : \mathcal{D}'(\mathbb{R}^n))$ - the following result on propagation of singularities from the initial data is well-known. Assume that u_0, u_1 belong to $\mathcal{C}^\infty(\mathbb{R}^n \setminus \{0\})$. Then the singular support of the solution is contained in Γ, that is, $u \in \mathcal{C}^\infty(\mathbb{R}^n \setminus \Gamma)$. Thus regularity of the initial data outside the origin not only entails regularity outside the light-cone (which is clear from finite propagation speed), but is transported inside the light-cone. Again, this transport of $\mathcal{C}^\infty$-regularity does not hold for Colombeau solutions:

Example 8. *Consider the wave equation (10) in one space dimension* $(n = 1)$ *with initial data* $u(\cdot, 0) = 0,\ \partial_t u(\cdot, 0) = \iota(\delta)^2$, *the square of the Dirac measure in* $\mathcal{G}(\mathbb{R})$. *By d'Alembert's formula, the solution* $u \in \mathcal{G}(\mathbb{R} \times [0, \infty))$ *has the constant value* $\frac{1}{2}\int_{-\infty}^{\infty} \iota(\delta)^2(x)\,\mathrm{d}x$ *inside the light-cone, which is a generalized constant in* $\widetilde{\mathbb{C}} \setminus \mathbb{C}$.

Proposition 9. *Let* $u \in \mathcal{G}(\mathbb{R}^n \times [0, \infty))$ *be a solution to the liner wave equation (10) with initial data* $u_0, u_1 \in \mathcal{G}(\mathbb{R}^n)$.

(a) If $u_0, u_1 \in \mathcal{G}^\infty(\mathbb{R}^n \setminus \{0\})$ *then* $u \in \mathcal{G}^\infty((\mathbb{R}^n \times [0, \infty)) \setminus \Gamma)$.

(b) If $u_0, u_1 \in \mathcal{G}^{\mathcal{L}}(\mathbb{R}^n \setminus \{0\})$ *then* $u \in \mathcal{G}^{\mathcal{L}}((\mathbb{R}^n \times [0, \infty)) \setminus \Gamma)$.

P r o o f. The proof relies on the fact that the Cauchy problem (10) admits a fundamental solution with singular support equal to Γ and vanishing outside Γ. Cut-off arguments similar to those employed in the proof of Prop. 7 yield the result (details in the $\mathcal{G}^\infty$-case have been elaborated in [21]). $\square$

Example 8 shows that an assertion analogous to Prop. 9 does not hold for $\mathcal{G}^0$-regularity.

5. *Regularity results for semilinear waves*

In this section, we shall put the notions introduced in Section 3 to test in a simple model problem, namely the initial value problem for a semilinear wave equation in one space dimension,

$$\begin{aligned} &\partial_t^2 u(x,t) - \partial_x^2 u(x,t) = F(u(x,t)), \quad x \in \mathbb{R}, t > 0, \\ &u(x,0) = u_0(x),\ \partial_t u(x,0) = u_1(x), \quad x \in \mathbb{R}. \end{aligned} \tag{11}$$

We assume here that the function $u \to F(u)$ is smooth with all derivatives of at most polynomial growth as $|u| \to \infty$, and that it satisfies a global Lipschitz estimate, i.e., has a globally bounded derivative. Existence and uniqueness of a solution in the Colombeau algebra on the upper half plane can be inferred from various much more general theorems on hyperbolic equations. The result relevant for our purpose is the following; one simple proof can be found in [22]:

Remark 10 Assume that the function F is as described above. Let $u_0, u_1 \in \mathcal{G}(\mathbb{R})$. Then problem (11) has a unique solution $u \in \mathcal{G}(\mathbb{R} \times [0,\infty))$. One of the representatives of the solution u is the family $(u_\varepsilon)_{\varepsilon\in(0,1]}$ of classical smooth solutions with initial data given by representatives of u_0 and u_1.

The question of global regularity asks whether a certain regularity of the initial data on $\mathbb{R}$ is preserved in the solution on $\mathbb{R} \times [0,\infty)$. Such a *global* result is true of $\mathcal{C}^\infty$-regularity, as follows from the fact that $\iota(f) = \sigma(f)$ for smooth functions f. Thus if u_0 and u_1 are elements of $\mathcal{C}^\infty(\mathbb{R}) \subset \mathcal{G}(\mathbb{R})$, the solution $u \in \mathcal{G}(\mathbb{R} \times [0,\infty))$ belongs to $\mathcal{C}^\infty(\mathbb{R} \times [0,\infty))$ as well, because the classical $\mathcal{C}^\infty$-solution is one of its representatives. Before turning to answering the same question for the other algebras presented in Section 3, we need to recall estimates for classical solutions to the non-homogeneous linear wave equation. Thus let z be a continuous function that solves of the linear wave equation

$$\begin{aligned} &\partial_t^2 z(x,t) - \partial_x^2 z(x,t) = h(x,t), \quad x \in \mathbb{R}, t > 0,\\ &z(x,0) = z_0(x),\ \partial_t z(x,0) = z_1(x), \quad x \in \mathbb{R} \end{aligned} \tag{12}$$

in the sense of distributions. Let $K_0 = [-\kappa, \kappa]$ be a compact interval. For $0 \le t \le \kappa$, the trapezoidal region K_t is defined by

$$K_t = \{(x,s) \in \mathbb{R} \times [0,\infty) : 0 \le s \le t, |x| \le \kappa - s\}. \tag{13}$$

The following estimate is easily deduced ($0 \le t \le T \le \kappa$):

$$\|z\|_{\mathrm{L}^\infty(K_T)} \le \|z_0\|_{\mathrm{L}^\infty(K_0)} + T\|Iz_1\|_{\mathrm{L}^\infty(K_0)} + T\int_0^T \|h\|_{\mathrm{L}^\infty(K_t)}\,\mathrm{d}t, \tag{14}$$

where

$$Iz_1(x) = \int_0^x z_1(y)\,\mathrm{d}y.$$

The same estimate holds with L^1 in place of L^∞.

Proposition 11. *Assume that the function F is smooth, globally Lipschitz, and all its derivatives are polynomially bounded. Let $u \in \mathcal{G}(\mathbb{R} \times [0,\infty))$ be the solution to the semilinear wave equation (11) with initial data $u_0, u_1 \in \mathcal{G}(\mathbb{R})$. Then:*

(a) If $u_0, u_1 \in \mathcal{G}^0(\mathbb{R})$, then $u \in \mathcal{G}^0(\mathbb{R} \times [0,\infty))$.

(b) If $u_0, u_1 \in \mathcal{G}^\infty(\mathbb{R})$ and F is linear, then $u \in \mathcal{G}^\infty(\mathbb{R} \times [0,\infty))$.

(c) If $u_0, u_1 \in \mathcal{G}^{\mathcal{L}}(\mathbb{R})$ and F as well as all its derivatives are bounded, then $u \in \mathcal{G}^{\mathcal{L}}(\mathbb{R} \times [0,\infty))$.

(d) If $u_0, u_1 \in \mathcal{L}^1_{\mathcal{G}}(\mathbb{R})$, then $u \in \mathcal{L}^1_{\mathcal{G}}(\mathbb{R} \times [0,\infty))$.

P r o o f. Let K_T be a region as described above; let $(u_\varepsilon)_{\varepsilon\in(0,1]}$ be the representative of the solution described in Remark 10. Then u_ε and its first and second derivatives with respect to x satisfy

$$\begin{aligned} \left(\partial_t^2 - \partial_x^2\right) u_\varepsilon &= F'(\theta_\varepsilon)u_\varepsilon + F(0), \quad (|\theta_\varepsilon| \le |u_\varepsilon|) \\ \left(\partial_t^2 - \partial_x^2\right) \partial_x u_\varepsilon &= F'(u_\varepsilon)\partial_x u_\varepsilon, \\ \left(\partial_t^2 - \partial_x^2\right) \partial_x^2 u_\varepsilon &= F'(u_\varepsilon)\partial_x^2 u_\varepsilon + F''(u_\varepsilon)(\partial_x u_\varepsilon)^2, \end{aligned}$$

while the first derivative with respect to t satisfies

$$\left(\partial_t^2 - \partial_x^2\right) \partial_t u_\varepsilon = F'(u_\varepsilon)\partial_t u_\varepsilon.$$

The initial data for the derivatives are

$$\begin{array}{ll} \partial_x u_\varepsilon(\cdot,0) = \partial_x u_{0\varepsilon}, & \partial_t\partial_x u_\varepsilon(\cdot,0) = \partial_x u_{1\varepsilon}, \\ \partial_x^2 u_\varepsilon(\cdot,0) = \partial_x^2 u_{0\varepsilon}, & \partial_t\partial_x^2 u_\varepsilon(\cdot,0) = \partial_x^2 u_{1\varepsilon}, \\ \partial_t u_\varepsilon(\cdot,0) = u_{1\varepsilon}, & \partial_t\partial_t u_\varepsilon(\cdot,0) = \partial_x^2 u_{0\varepsilon} + F(u_{0\varepsilon}), \end{array}$$

respectively. Using (14) and Gronwall's inequality, one can estimate the L^∞-norm of u_ε and its derivatives on K_T in terms of the L^∞-norms of the initial data as well as the derivatives of lower order. This yields the estimates required to prove (a). Case (c) follows by the same argument, applied with additional care for the t-derivatives: the loss of one order of derivative in the second slot for the initial data is made up for by the antiderivative I appearing in estimate (13). Point (b) is a result on propagation of $\mathcal{G}^\infty$-regularity for solutions to linear hyperbolic equations which can be proved similar to Prop. 9, while (d) is just a reformulation of continuous dependence of $\mathcal{L}^1_{\mathrm{loc}}$-solutions of semilinear wave equations with Lipschitz nonlinearity on the initial data. □

We now address propagation of singularities for the semilinear wave equation. Thus we study initial data that are regular outside the origin. Following the classical theory of propagation of jump discontinuities one may hope to prove that, in the one-dimensional case, the solution will be regular (in the appropriate sense) inside the light-cone Γ. That this behavior occurs in the $\mathcal{C}^\infty$-category for the semilinear wave equation in one space dimension was shown by [27]; it does not hold in higher space dimensions or for higher order operators [25], for which *anomalous* singularities may occur. The question we ask is whether this transport of regularity into the interior of the light-cone happens in the categories introduced in Section 3. In fact, this question remains open here. What we will be able to prove is transport of regularity under conditions which resemble the situation leading to *delta waves*. The existence of delta waves for semilinear hyperbolic equations has been discovered in [20, 23, 26]: the weak limits of regularized solutions have been found to split in a sum of two terms, a singular term satisfying a linear equation and a regular term satisfying a nonlinear equation. A similar phenomenon will now be described pertaining to regularity for Colombeau solutions.

We consider the semilinear wave equation (11) with a nonlinear function F as described at the beginning of this section. We take initial data of the form

$$u_i = r_i + s_i \in \mathcal{G}(\mathbb{R}), \quad i = 0, 1$$

where

$$r_i \in \mathcal{L}^1_{\mathcal{G}}(\mathbb{R}) \text{ and } \operatorname{supp}(s_i) = \{0\}, \quad i = 0, 1.$$

Define the generalized complex number $M \in \widetilde{\mathbb{C}}$ as the class of

$$M_\varepsilon = \frac{1}{2}\int_{-\infty}^{\infty} s_{1\varepsilon}(x)\,\mathrm{d}x$$

where $(s_{1\varepsilon})_{\varepsilon\in(0,1]}$ is a representative of s_1. We shall write $M \approx m$ if M_ε converges to a complex number $m \in \mathbb{C}$ and $|M| \approx \infty$ if $|M_\varepsilon| \to \infty$ as $\varepsilon \to 0$.

Proposition 12. *Assume that the function F is smooth, globally Lipschitz, and all its derivatives are polynomially bounded. Let $u \in \mathcal{G}(\mathbb{R} \times [0,\infty))$ be the solution to the semilinear wave equation (11) with initial data u_0, u_1 as described above. If either*
(a) F is globally bounded and $M \approx m$ for some $m \in \mathbb{C}$, or
(b) F is globally bounded, $\lim_{|y|\to\infty} F(y)$ exists and $|M| \approx \infty$,
then $u \in \mathcal{G}^\infty((\mathbb{R} \times [0,\infty)) \setminus \Gamma) + \mathcal{L}^1_{\mathcal{G}}(\mathbb{R} \times [0,\infty))$.

P r o o f. Denote by $\mathbf{1}_\Sigma$ the characteristic function of the solid light-cone $\Sigma = \{(x,t) : t \geq 0, |x| \leq t\}$. Let $(r_{i\varepsilon})_{\varepsilon\in(0,1]}$, $(s_{i\varepsilon})_{\varepsilon\in(0,1]}$, $i = 0, 1$, be representatives of the initial data, where we may assume that $\operatorname{supp}(s_{i\varepsilon}) \subset [-\eta, \eta]$ with η as small as we wish. Let $u_\varepsilon, v_\varepsilon \in \mathcal{C}^\infty(\mathbb{R} \times [0,\infty))$, $w_\varepsilon \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R} \times [0,\infty))$ be the solutions to

$$\begin{aligned}
&(\partial_t^2 - \partial_x^2)\, u_\varepsilon = F(u_\varepsilon), && u_\varepsilon(\cdot,0) = u_{0\varepsilon}, \quad \partial_t u_\varepsilon(\cdot,0) = u_{1\varepsilon},\\
&(\partial_t^2 - \partial_x^2)\, v_\varepsilon = 0, && v_\varepsilon(\cdot,0) = s_{0\varepsilon}, \quad \partial_t v_\varepsilon(\cdot,0) = s_{1\varepsilon},\\
&(\partial_t^2 - \partial_x^2)\, w_\varepsilon = F(M_\varepsilon \mathbf{1}_\Sigma + w_\varepsilon), && w_\varepsilon(\cdot,0) = r_{0\varepsilon}, \quad \partial_t w_\varepsilon(\cdot,0) = r_{1\varepsilon}.
\end{aligned}$$

Then

$$\left(\partial_t^2 - \partial_x^2\right)(u_\varepsilon - v_\varepsilon - w_\varepsilon) \\ = F(u_\varepsilon) - F(v_\varepsilon + w_\varepsilon) + F(v_\varepsilon + w_\varepsilon) - F(M_\varepsilon \mathbf{1}_\Sigma + w_\varepsilon)$$

with zero initial data. The difference of the first two terms on the right hand side can be estimated by the L^∞-norm of F' times $|u_\varepsilon - v_\varepsilon - w_\varepsilon|$, while the difference of the last two terms vanishes off an η-neighborhood of the light-cone Γ, because $v_\varepsilon(x,t) = M_\varepsilon$ for $|x| < t - \eta$. Taking a trapezoidal region K_T as in (13), the boundedness of F, inequality (14) and Gronwall's lemma give an estimate of the form

$$\|u_\varepsilon - v_\varepsilon - w_\varepsilon\|_{\mathrm{L}^1(K_T)} \le C\eta \tag{15}$$

for some constant $C > 0$ and all $\varepsilon \in (0,1]$. In the case (a), let $w \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R} \times [0,\infty))$ be the solution to

$$\left(\partial_t^2 - \partial_x^2\right) w = F(m\mathbf{1}_\Sigma + w), \qquad w(\cdot,0) = r_0, \quad \partial_t w(\cdot,0) = r_1$$

where r_i is the limit in $\mathrm{L}^1_{\mathrm{loc}}(\mathbb{R})$ of $r_{i\varepsilon}$ as $\varepsilon \to 0$, $i = 0,1$. Now

$$\left(\partial_t^2 - \partial_x^2\right)(w_\varepsilon - w) \\ = F(M_\varepsilon \mathbf{1}_\Sigma + w_\varepsilon) - F(M_\varepsilon \mathbf{1}_\Sigma + w) + F(M_\varepsilon \mathbf{1}_\Sigma + w) - F(m\mathbf{1}_\Sigma + w)$$

with initial data given by $r_{i\varepsilon} - r_i$, $i = 0,1$. The difference of the first two terms on the right hand side is bounded by the L^∞-norm of F' times $|w_\varepsilon - w|$. By assumption, the difference of the last two terms converges to zero almost everywhere. Using inequality (14) with L^1-norms in place of the L^∞-norms and Gronwall's lemma as above shows that

$$\|w_\varepsilon - w\|_{\mathrm{L}^1(K_T)} \to 0 \quad \text{as } \varepsilon \to 0. \tag{16}$$

In case (b), defining $w \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R} \times [0,\infty))$ as the solution to

$$\left(\partial_t^2 - \partial_x^2\right) w = L\mathbf{1}_\Sigma, \qquad w(\cdot,0) = r_0, \quad \partial_t w(\cdot,0) = r_1,$$

the same argument as above leads to the convergence result (16) in this case as well. Combining (15) which holds for arbitrarily chosen $\eta > 0$ with (16) shows that $u_\varepsilon - v_\varepsilon$ converges to w in $\mathrm{L}^1(K_T)$ as $\varepsilon \to 0$. By Prop. 9, v_ε enjoys the $\mathcal{G}^\infty$-estimate (5) off the light cone Γ. Thus $u_\varepsilon = v_\varepsilon + (u_\varepsilon - v_\varepsilon)$ defines an element of $\mathcal{G}^\infty((\mathbb{R} \times [0,\infty)) \setminus \Gamma) + \mathcal{L}^1_{\mathcal{G}}(\mathbb{R} \times [0,\infty))$. □

Remark 13. *The hypotheses on the generalized constant M in Prop. 12 are satisfied when the term s_1 in the initial data is a polynomial in the Dirac measure and its derivatives. In fact, when $s_1 = \partial^\alpha \iota(\delta)$, we have $M \approx \frac{1}{2}$ for $\alpha = 0$ and $M \approx 0$ for $\alpha > 0$. When $s_1 = \pi(\iota(\delta))$ for some polynomial function π, only the cases $M \approx m$ for some $m \in \mathbb{C}$ or $|M| \approx \infty$ can occur.*

Prop. 12 shows that regularity of the type $\mathcal{G}^\infty + \mathcal{L}^1_{\mathcal{G}}$ is propagated into the region inside the light-cone. It is clear that it can be generalized in various ways: for example, the support of the singular part s_0, s_1 of the data could consist of a discrete set rather than just a point, other norms in place of the L^1-norm in $\mathcal{L}^1_{\mathcal{G}}$ could be used to define spaces measuring the regularity. However, it should be noted that continuous dependence of the regularized solutions on the data in terms of this norm enters into the argument, and such a property depends decisively on the particular equation, the space dimension and the nonlinearity F. Prop. 12 exploits such special properties and thus falls short of providing a prototypical description of nonlinear propagation of regularity for Colombeau solutions, which remains a challenging open question.

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