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INHOMOGENEOUS GEVREY ULTRADISTRIBUTIONS AND CAUCHY PROBLEM[1]

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A b s t r a c t. After a short survey on Gevrey functions and ultradistributions, we present the inhomogeneous Gevrey ultradistributions introduced recently by the authors in collaboration with A. Morando, cf. [7]. Their definition depends on a given weight function λ *, satisfying suitable hypotheses, according to Liess-Rodino [16]. As an application, we define* (s, λ)*-hyperbolic partial differential operators with constant coefficients (for* $s > 1$*), and prove for them the well-posedness of the Cauchy problem in the frame of the corresponding inhomogeneous ultradistributions. This sets in the dual spaces a similar result of Calvo [4] in the inhomogeneous Gevrey classes, that in turn extends a previous result of Larsson [14] for weakly hyperbolic operators in standard homogeneous Gevrey classes.*

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Introduction

Let us begin this note by a short survey on Gevrey classes and related ultradistributions. Starting from the investigation of the solutions of the heat equation, Gevrey [9] considered subclasses of $C^\infty(\Omega)$, set of the infinitely differentiable functions in an open set $\Omega \subset \mathbb{R}^n$. Namely, with the notation we shall use in the following:

f belongs to $G^s(\Omega)$, $1 \leq s < \infty$, if for every compact subset $K \subset\subset \Omega$ it is satisfied:

$$\sup_{x\in K} |D^\alpha f(x)| \leq RC^{|\alpha|}\alpha!^s \tag{1}$$

for suitable positive constants R and C independent of the multi-index α.

In particular, for $s = 1$ we recapture the class of the analytic functions of the real variable $x \in \Omega$. Somewhat different classes are defined if we assume that for every $C > 0$ there exists R such that (1) is satisfied (Beurling-type Gevrey functions, we say now with reference to Beurling [1]). Concerning the applications to the heat equation, we have that all the solutions f, also in the weak sense, of

$$Hf = \partial_t f - \Delta f = 0$$

are in $G^2(\mathbb{R}^{n+1})$. Moreover, assuming space dimension $n = 1$ and denoting y for the space variable, the Cauchy problem at $y = 0$:

$$\begin{cases} Hf = 0 \\ f(t,0) = f_0 \\ \partial_y f(t,0) = f_1 \end{cases} \tag{2}$$

admits solution $f(t, y)$, $y \in \mathbb{R}$, if and only if $f_0, f_1 \in G^s(\mathbb{R})$, $s < 2$ (note that, with respect to the standard parabolic problem with datum at $t = 0$, one is rather treating here H as a weakly hyperbolic operator, with initial data at $y = 0$). Starting from the original contribution of Gevrey, researches proceeded then along different lines. Let us distinguish two main streams. From one side, it is natural to generalize (1) as

$$\sup_{x\in K} |D^\alpha f(x)| \leq RC^{|\alpha|}M_{|\alpha|}, \tag{3}$$

where $M = (M_j)$, $j = 0, 1, 2, \ldots$, is a sequence of positive numbers satisfying suitable properties, cf. f.i. [13, 17]. The corresponding classes are usually denoted by $C^M(\Omega)$. It is easy to see that when $M_j = (j!)^s$ we recapture from (3) the Gevrey classes defined by (1). Other interesting examples are given by the classes for which $\sum_{j=0}^{\infty} \frac{M_{j-1}}{M_j} = \infty$, as for the case $M_j = j!$; they are called quasi-analytic classes, because of their properties, which are similar to those of the analytic class. See for example Mandelbrojt [18], Beurling [1], Rudin [24] and the references there, for details. In the non-quasi-analytic case, functions in $C^M(\Omega)$ with compact support in Ω exist, and the corresponding set is denoted by $C_0^M(\Omega)$. Giving the natural topology to $C_0^M(\Omega)$ and taking the dual, one obtains the space of the generalized functions $D'_M(\Omega)$, see for example Roumieu [22], Komatsu [13], Lions-Magenes [17] and Matsumoto [20], containing as a subset the space of the Schwartz distributions $\mathcal{D}'(\Omega)$. Researches in this area are still very active, see for example the alternative presentation of Braun-Meise-Taylor [2]. A further generalization is given by Roumieu [23], who considers sequences M_α depending on all components of α.

In strict connection with such a study, we emphasize a second point of view, where the main objective is given by the analysis of partial differential equations, and the previous spaces play the role of natural functional frames to obtain results of existence, uniqueness and regularity. In this line, the attention is often limited to the Gevrey spaces defined by (1) and to the corresponding ultradistribution spaces $\mathcal{D}'_s(\Omega)$.As an example, let us mention the famous result of Hörmander (see [12]), concerning the regularity of the solutions of the partial differential equations with constant coefficients $P(D)u = 0$. Namely, all the solutions $u \in D'(\Omega)$ are actually in $G^s(\Omega)$ if and only if the symbol $P(\xi)$ satisfies for large $|\xi|$ the estimates $|D^\alpha P(\xi)|/|P(\xi)| \leq C|\xi|^{-\frac{|\alpha|}{s}}$, i.e. $P(D)$ is s-hypoelliptic. One can easily deduce from this the above-mentioned result of regularity for the solutions of the heat equation.

Concerning the Cauchy problem (2), this also can be extended to general operators with constant coefficients. Namely, the Cauchy problem with respect to the time variable is well posed for initial data in the class G^r, $1 \leq r < s$, if $P(D)$ is s-hyperbolic (according to the definition of Larsson [14])in the following sense:

$$P(\tau,\xi) = \tau^m + \sum_{|\nu|+j\leq m, j\neq m} a_{\nu j}\xi^\nu \tau^j = 0 \quad \text{for } (\tau,\xi) \in \mathbb{C}\times\mathbb{R}^n \Longrightarrow |\Im\tau| \leq C\langle\xi\rangle^{\frac{1}{s}}, \tag{4}$$

where τ and ξ denote the duals of the time and space variables, respec-

tively, and we write $\langle\xi\rangle = \sqrt{1+|\xi|^2}$. For the proof we refer for instance to [12], Theorem 12.7.5, [21], Theorem 2.5.2 and [14]. Generalizing such contributions, several other papers appeared, addressed to linear partial differential equations with analytic-Gevrey coefficients, or non-linear equations with analytic-Gevrey nonlinearity, dealing with the problems of the local existence and regularity of the solutions, the well-posedness of the Cauchy problem and related questions. In particular, in these last thirty years, new techniques of proof and terminology were given by Microlocal Analysis (Gevrey pseudo-differential operators, Gevrey wave front sets, etc). We address to the references in Rodino [21] and Mascarello-Rodino [19] for some recent results in this connection.
Finally, to introduce the contents of the present paper, we recall that the Fourier transform $\hat{f}$ of $f \in G^s_0(\mathbb{R}^n) = G^s(\mathbb{R}^n) \cap C^\infty_0(\mathbb{R}^n)$ satisfies for positive constants C and ε the estimates

$$|\hat{f}(\xi)| \leq C \exp(-\varepsilon\langle\xi\rangle^{\frac{1}{s}}). \tag{5}$$

Starting from (5), we extend the definition of Gevrey classes by replacing $\langle\xi\rangle$ by a weight function $\lambda(\xi)$, satisfying suitable properties. When $\lambda(\xi) = \tilde{\lambda}(|\xi|)$, one can establish precise connections with generalizations in terms of (3), see for instance Braun-Meise-Taylor [2]. Generic inhomogeneous weight functions $\lambda(\xi)$ were considered by Liess-Rodino [16] and Rodino [21], Section 1.8; the corresponding classes cannot in general be identified in terms of the estimates (3), however there are natural applications to hypoelliptic partial differential equations, motivating the inhomogeneous approach, cf. [16].
In the next section we shall recall shortly the definition of weight function from Liess-Rodino [16] and present also the corresponding Gevrey classes and ultradistributions, as treated recently by Calvo-Morando-Rodino [7]. We observe that a different approach is presented in Björk [3], under more restrictive hypotheses on the weight function; our setting allows us to include also the multi-anisotropic classes as particular case (cf. [6]). In the last section we shall give a new application, concerning the Cauchy problem for operators with constant coefficients in the frame of inhomogeneous ultradistributions. This extends to the dual spaces the above-mentioned result of Larsson [14] for the standard Gevrey classes, and preceding contributions of Calvo [4, 5] concerning the inhomogeneous and multi-anisotropic Gevrey classes.

1. *Inhomogeneous Gevrey ultradistributions*

We introduce the weight functions, following Liess-Rodino [16].

Definition 1.1. *We say that a function $\lambda : \mathbb{R}^n \to \mathbb{R}_+$ is a weight function if there are constants $C, C', \delta > 0$ such that*

$$\begin{array}{lll} (i) & |\lambda(\xi) - \lambda(\eta)| \leq C|\xi - \eta|, & \forall \xi, \eta \in \mathbb{R}^n, \\ (ii) & \langle \xi \rangle^{\delta} \leq C' \lambda(\xi), & \forall \xi \in \mathbb{R}^n. \end{array}$$

Observe that *(i)* implies $\lambda(\xi) \leq C\langle \xi \rangle$ for a constant $C > 0$, so that δ must be smaller than 1.

From now on λ is a weight function according to Definition 1.1 and $s > 1$. For short, in the sequel we refer to classes of functions and distributions in $\mathbb{R}^n$; classes in an open subset Ω of $\mathbb{R}^n$ can be defined similarly.

Definition 1.2. *A distribution $f \in \mathcal{D}'(\mathbb{R}^n)$ belongs to the inhomogeneous Gevrey class $G^{s,\lambda}(\mathbb{R}^n)$ if for any Gevrey function $\chi \in G_0^s(\mathbb{R}^n)$ of the same order s, χf satisfies for suitable $C, \varepsilon > 0$ the condition*

$$|\widehat{\chi f}(\xi)| \leq C \exp(-\varepsilon \lambda(\xi)^{\frac{1}{s}}), \quad \forall \xi \in \mathbb{R}^n. \tag{6}$$

Definition 1.3. *The inhomogeneous Gevrey classes with compact support are*

$$G_0^{s,\lambda}(\mathbb{R}^n) = G^{s,\lambda}(\mathbb{R}^n) \cap C_0^{\infty}(\mathbb{R}^n).$$

It follows easily that the space $G_0^{s,\lambda}(\mathbb{R}^n)$ is defined by the following inhomogeneous version of (5) (cf. [7], Theorem 2.1).
A compactly supported distribution $f \in \mathcal{E}'(\mathbb{R}^n)$ belongs to $G_0^{s,\lambda}(\mathbb{R}^n)$ if and only if its Fourier transform satisfies for suitable $C, \varepsilon > 0$ the condition

$$|\hat{f}(\xi)| \leq C \exp(-\varepsilon \lambda(\xi)^{\frac{1}{s}}), \quad \forall \xi \in \mathbb{R}^n.$$

In order to endow the spaces $G_0^{s,\lambda}(\mathbb{R}^n)$ and $G^{s,\lambda}(\mathbb{R}^n)$ with a topology, we then define the following Banach spaces.

Definition 1.4. *Let K be a compact subset of $\mathbb{R}^n$ and ε a positive number. Then*

$$G_0^{s,\lambda}(K, \varepsilon) := \{f \in \mathcal{E}'(\mathbb{R}^n) : \mathrm{supp} f \subset K, \sup_{\xi \in \mathbb{R}^n} \exp(\varepsilon \lambda(\xi)^{\frac{1}{s}})|\hat{f}(\xi)| < \infty\}.$$

Therefore, if $\{K_j\}_{j\in\mathbb{N}}$ is an exhaustive sequence of compact sets in $\mathbb{R}^n$ and $\{\varepsilon_j\}_{j\in\mathbb{N}}$ a decreasing sequence of positive numbers converging to 0, we define the topology in $G_0^{s,\lambda}(\mathbb{R}^n)$ as

$$G_0^{s,\lambda}(\mathbb{R}^n) = \mathrm{indlim}_{K_j\to\mathbb{R}^n,\varepsilon_j\to 0} G_0^{s,\lambda}(K_j,\varepsilon_j).$$

Hence, $G_0^{s,\lambda}(\mathbb{R}^n)$ is a (DFS)- space, cf. [11].
We now define the topology of the classes $G^{s,\lambda}(\mathbb{R}^n)$ with arbitrary support. Firstly let $\{K_j\}_{j=1}^{\infty}$ be an exhaustive sequence of compact sets of $\mathbb{R}^n$ (we set $K_0 := \emptyset$) and, for every $j = 1, 2, \ldots$, take a function $\chi_j \in G_0^s(K_j)$ such that $\chi_j \equiv 1$ on K_{j-1}. For any pair of positive integers $l > j$ we define the continuous map

$$\rho_{l,j} : G_0^{s,\lambda}(K_l) \longrightarrow G_0^{s,\lambda}(K_j), \quad f \longmapsto \chi_j f.$$

We can then define $G^{s,\lambda}(\mathbb{R}^n) = \mathrm{projlim}_{l>j}\left(G_0^{s,\lambda}(K_j), \rho_{l,j}\right)$.
The space $G^{s,\lambda}(\mathbb{R}^n)$ is a complete Schwartz space, as it is the projective limit of complete Schwartz spaces (cf. [11]).
For the properties of the inhomogeneous Gevrey functions and the algebraic operations on them, we refer to Calvo-Morando-Rodino [7]; we just point out that, for δ as in Definition 1.1, the following inclusions involving the inhomogeneous and the standard Gevrey classes hold:

$$G^s(\mathbb{R}^n) \subset G^{s,\lambda}(\mathbb{R}^n) \subset G^{\frac{s}{\delta}}(\mathbb{R}^n). \tag{7}$$

We now pass to define the inhomogeneous Gevrey ultradistributions associated to the inhomogeneous Gevrey classes previously defined.

Definition 1.5. *We denote by $\mathcal{D}'_{s,\lambda}(\mathbb{R}^n) := (G_0^{s,\lambda}(\mathbb{R}^n))'$ the topological dual space of $G_0^{s,\lambda}(\mathbb{R}^n)$. We also set $\mathcal{E}'_{s,\lambda}(\mathbb{R}^n) := (G^{s,\lambda}(\mathbb{R}^n))'$ for the topological dual of $G^{s,\lambda}(\mathbb{R}^n)$.*

The space $\mathcal{E}'_{s,\lambda}(\mathbb{R}^n)$ coincides with the subspace of the ultradistributions in $\mathcal{D}'_{s,\lambda}(\mathbb{R}^n)$ with compact support. As a consequence of (7), the following inclusions of the standard and inhomogeneous ultradistributions hold:

$$\mathcal{D}'_{\frac{s}{\delta}}(\mathbb{R}^n) \subset \mathcal{D}'_{s,\lambda}(\mathbb{R}^n) \subset \mathcal{D}'_s(\mathbb{R}^n).$$

We point out the following property of the Fourier transform on $\mathcal{E}'_{s,\lambda}(\mathbb{R}^n)$, that we will use to prove the well-posedness of the Cauchy problem in the next section.

Proposition 1.6. *If u belongs to $\mathcal{E}'_{s,\lambda}(\mathbb{R}^n)$, then for every $\varepsilon > 0$ there exists a constant $C_\varepsilon > 0$ such that the Fourier transform $\hat{u}$ of u satisfies*

$$|\hat{u}(\xi)| \leq C_\varepsilon \exp\left(\varepsilon\lambda(-\xi)^{\frac{1}{s}}\right), \quad \forall \xi \in \mathbb{R}^n. \tag{8}$$

To antitransform a generic function $\hat{u}(\xi)$ satisfying (8), we need some spaces of inhomogeneous ultradistributions in which the Fourier transform is an automorphism. Namely, we first introduce the space $\mathcal{S}^{s,\lambda}(\mathbb{R}^n)$, in analogy with [3], [22], [23].

Definition 1.7. *We say that a function $f \in L^1(\mathbb{R}^n)$ belongs to $\mathcal{S}^{s,\lambda}(\mathbb{R}^n)$ if $f, \hat{f} \in C^\infty(\mathbb{R}^n)$ and there is a constant $\varepsilon > 0$ such that for all $\alpha \in \mathbb{N}^n$ it holds:*

$$\begin{aligned} p_{\alpha,\varepsilon}(f) &= \sup_{x\in\mathbb{R}^n} \exp(\varepsilon\lambda(x)^{\frac{1}{s}})|D^\alpha f(x)| < \infty \\ \pi_{\alpha,\varepsilon}(f) &= \sup_{\xi\in\mathbb{R}^n} \exp(\varepsilon\lambda(\xi)^{\frac{1}{s}})|D^\alpha \hat{f}(\xi)| < \infty. \end{aligned} \tag{9}$$

Then we can naturally define a locally convex topology in $\mathcal{S}^{s,\lambda}(\mathbb{R}^n)$ given by the semi-norms $p_{\alpha,\varepsilon}, \pi_{\alpha,\varepsilon}$. We easily see that $G_0^{s,\lambda}(\mathbb{R}^n) \subset \mathcal{S}^{s,\lambda}(\mathbb{R}^n) \subset G^{s,\lambda}(\mathbb{R}^n)$ and $\mathcal{S}^{s,\lambda}(\mathbb{R}^n)$ is included in the Schwartz space $\mathcal{S}(\mathbb{R}^n)$. In analogy with [3], from (9) we have that the Fourier transform is an automorphism in $\mathcal{S}^{s,\lambda}(\mathbb{R}^n)$. Therefore, we define the dual spaces of $\mathcal{S}^{s,\lambda}(\mathbb{R}^n)$.

Definition 1.8. *The space of the inhomogeneous ultradistribution $\mathcal{S}'_{s,\lambda}(\mathbb{R}^n)$ is the topological dual of $\mathcal{S}^{s,\lambda}(\mathbb{R}^n)$.*

We have obviously the inclusions $\mathcal{E}'_{s,\lambda}(\mathbb{R}^n) \subset \mathcal{S}'_{s,\lambda}(\mathbb{R}^n) \subset \mathcal{D}'_{s,\lambda}(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n) \subset \mathcal{S}'_{s,\lambda}(\mathbb{R}^n)$, where $\mathcal{S}'(\mathbb{R}^n)$ is the space of tempered distributions. We can define the Fourier transform for $u \in \mathcal{S}'_{s,\lambda}(\mathbb{R}^n)$ using Parseval' formula: $\hat{u}(f) = u(\hat{f})$, for all $f \in \mathcal{S}^{s,\lambda}(\mathbb{R}^n)$ (as also $\hat{f} \in \mathcal{S}^{s,\lambda}(\mathbb{R}^n)$). The Fourier transform is an automorphism in $\mathcal{S}'_{s,\lambda}(\mathbb{R}^n)$. Note finally that any function $\hat{u}(\xi)$ satisfying for all $\varepsilon > 0$ the condition (8) can be regarded as an element of $\mathcal{S}'_{s,\lambda}(\mathbb{R}^n)$, and therefore is the Fourier transform of an inhomogeneous ultradistribution in $\mathcal{S}'_{s,\lambda}(\mathbb{R}^n)$. It follows that if $\hat{u}$ satisfies (8), then u belongs to $\mathcal{D}'_{s,\lambda}(\mathbb{R}^n)$ from the inclusion $\mathcal{S}'_{s,\lambda}(\mathbb{R}^n) \subset \mathcal{D}'_{s,\lambda}(\mathbb{R}^n)$.
For other properties of the inhomogeneous ultradistributions we refer to [7]. An important example of the inhomogeneous Gevrey functions is represented by the multi-anisotropic case, that is widely studied in literature and presents many applications to the theory of partial differential equations, cf. f.i. [5], [10], and the bibliography there; and see [6] for the dual spaces. Here

the weight function is defined in terms of a completely regular polyhedron, according to

Definition 1.9. *A convex polyhedron $\mathcal{P} \subset \mathbb{R}^n$ is complete if it satisfies the following conditions:*

i) all vertices (we denote their set by $\mathcal{V}(\mathcal{P})$) have rational non-negative coordinates;

ii) the origin $(0,\dots,0)$ belongs to $\mathcal{P}$;

iii) $\mathcal{P}$ has one vertex in each coordinate axis (different from the origin);

iv) the outer normals (we denote by $\mathcal{N}_1(\mathcal{P})$ their set) to the non-coordinate faces of $\mathcal{P}$ have strictly positive components.

Then we define $\mu = \max_{j=1,\dots,n} \max_{\nu\in\mathcal{N}_1(\mathcal{P})} \nu_j^{-1}$ the formal order of $\mathcal{P}$. Therefore, the function $|\xi|_{\mathcal{P}} = \sum_{v\in\mathcal{V}(\mathcal{P})} |\xi^v|^{\frac{1}{\mu}}$ is a weight function according to Definition 1.1. The multi-anisotropic Gevrey classes obtained by fixing $\lambda(\xi) = |\xi|_{\mathcal{P}}$ in (6), can be defined also in terms of estimates on the derivatives, cf. the previous references (with respect to (3), bounds are given in terms of a suitable multi-sequence M_α, depending on each component of α).

2. *Cauchy problem in inhomogeneous ultradistributions*

As an application of the inhomogeneous Gevrey ultradistributions, we study in this setting the well-posedness of the Cauchy problem, by considering a class of weakly hyperbolic operators, called (s,λ)-hyperbolic: these operators were introduced in [4], modeled to have well-posedness in the inhomogeneous Gevrey classes. They are an extension of the s-hyperbolic operators of Larsson [14] (see formula (4) and of the multi-quasi-hyperbolic operators in [5]). For simplicity, we suppose that $\lambda(\xi) = \lambda(-\xi)$, for all $\xi \in \mathbb{R}^n$, as satisfied in the standard homogeneous and multi-anisotropic cases.

Definition 2.1. *We say that a differential operator with constant coefficients in $\mathbb{R}_t \times \mathbb{R}^n_x$:*

$$P(D) = P(D_t, D_x) = D_t^m + \sum_{|\nu|+j\le m, j\ne m} a_{\nu j} D_x^\nu D_t^j \tag{10}$$

is (s,λ)-hyperbolic if there exists a constant $C > 0$ such that for any $(\tau,\xi) \in \mathbb{C}\times\mathbb{R}^n$ the symbol of $P(D)$ satisfies the condition:

$$P(\tau,\xi) = \tau^m + \sum_{|\nu|+j\le m, j\ne m} a_{\nu j}\xi^\nu\tau^j = 0 \quad \textit{for } (\tau,\xi)\in\mathbb{C}\times\mathbb{R}^n \Longrightarrow |\Im\tau| \le C\lambda(\xi)^{\frac{1}{s}}.$$

Obviously, if $P(D)$ is (s,λ)-hyperbolic, then $P(D)$ is also (r,λ)-hyperbolic for $1<r<s$ and s-hyperbolic (cf. [14]). The (s,λ)-hyperbolic operators are obviously weakly hyperbolic, i.e. all the roots τ of the characteristic equation $P_m(\tau,\xi)=\tau^m+\sum_{|\nu|+j=m, j\neq m}\xi^\nu\tau^j=0$ are real. In the opposite direction, if $P_m(\tau,\xi)$ satisfies the weakly hyperbolic assumption, in order to have the (s,λ)-hyperbolicity, we need to ask some Levi-type conditions on the lower order terms, modeled on the weight function λ, as shown for instance by the following result. We omit the proof, detailed in [4] together with many other examples.

Proposition 2.2. *Let $P(D)$ in (10) be weakly hyperbolic. Assume that the multiplicity of the roots of its principal symbol $P_m(\tau,\xi)$ is equal to $M\leq m$ and the lower order terms satisfy for some $k<M$ and a constant $C>0$:*

$$|a_{\nu j}\xi^\nu|\leq C\lambda(\xi)^k\langle\xi\rangle^{m-M-j} \qquad \text{for } \ |\nu|+j\leq m-1.$$

Then $P(D)$ is $(\frac{M}{k},\lambda)$-hyperbolic.

Finally, we can state the announced new result, of which we give a short proof.

Theorem 2.3. *Let $P(D)$ be an (s,λ)-hyperbolic differential operator in $\mathbb{R}_t\times\mathbb{R}^n_x$. Let $1<r<s$ and $u_k\in\mathcal{D}'_{r,\lambda}(\mathbb{R}^n_x)$ $(k=0,1,\ldots,m-1)$. Then the Cauchy problem:*

$$\begin{cases} P(D)u=D_t^m u+\sum_{|\nu|+j\leq m,\ j\neq m} a_{\nu j}D_x^\nu D_t^j u=0\\ D_t^k u(0,x)=u_k(x), \qquad k=0,1,\ldots,m-1\end{cases} \tag{11}$$

admits a unique solution

$$u\in C^\infty([-T,T],\mathcal{D}'_{r,\lambda}(\mathbb{R}^n_x))$$

for any $T>0$. If $r=s$, the solution is only local in time.

P r o o f. As $P(D)$ is a weakly hyperbolic operator with constant coefficients, then the phenomena related to $P(D)$ propagate with a finite speed, and therefore it is not restrictive to prove the theorem for data in the set of compactly supported inhomogeneous ultradistributions $\mathcal{E}'_{r,\lambda}(\mathbb{R}^n)$. Analogously to [4], [12] and [14], after performing the partial Fourier transform with respect to the space variable in the Cauchy problem (11), the unique solution $u(t,x)$ is obtained by anti-transforming

$$\hat{u}(t,\xi)=\sum_{j=0}^{m-1}\hat{u}_j(\xi)F_j(t,\xi), \tag{12}$$

where $\hat{u}_j$ $(j = 0, \dots, m-1)$ are the Fourier transforms of the data of (11) and F_j are the unique solutions of the Cauchy problems

$$\begin{cases} P(D_t, \xi)F_j = 0 \\ D_t^k F_j(0, \xi) = \delta_{jk}, \quad k = 0, \dots, m-1, \end{cases}$$

for $\delta_{jk} = 1$ if $j = k$ and 0 otherwise. As u_j belong to $\mathcal{E}'_{r,\lambda}(\mathbb{R}^n)$, then from (8) it follows that for all $\varepsilon > 0$ there is a constant $C_\varepsilon > 0$ such that $|\hat{u}_j(\xi)| \leq C_\varepsilon \exp(\varepsilon \lambda(\xi)^{\frac{1}{r}})$. To estimate F_j we use the following result (cf. [12], Lemma 12.7.7):

Lemma 2.4. *Let $L(D) = D^m + \sum_{j=0}^{m-1} a_j D^j$ be an ordinary differential operator with constant coefficients $a_j \in \mathbb{C}$. Write $\Lambda = \{\tau \in \mathbb{C} : L(\tau) = 0\}$ and assume: $\max_{\tau \in \Lambda} |\tau| \leq A$ and $\max_{\tau \in \Lambda} |\Im \tau| \leq B$. Then the solutions $v_j(t)$, $j = 0, 1, \dots, m-1$, of the Cauchy problems:*

$$\begin{cases} L(D)v_j = 0 \\ (D^k v_j)(0) = \delta_{jk}, \qquad k = 0, \dots, m-1 \end{cases}$$

satisfy the estimates:

$$|D^N v_j(t)| \leq 2^m (A+1)^{N+m+1} e^{(B+1)|t|}, \qquad N = 0, 1, \dots, \ t \in \mathbb{R}. \tag{13}$$

We now determine the constants A, B in order to apply the estimate (13), for $N = 0$, to the functions $F_j(t, \xi)$, $j = 0, 1, \dots, m-1$ (with ξ as parameter). As $P(D)$ is (s, λ)-hyperbolic, then we can take $B = C'\lambda(\xi)^{\frac{1}{s}}$, while it is easy to see that $A = C_1 \langle \xi \rangle$. This leads to have for new constants:

$$|F_j(t, \xi)| \leq (C_1 \langle \xi \rangle)^{m+1} c_1 \exp(C'|t|\lambda(\xi)^{\frac{1}{s}}) \leq c_2 \exp(C_2(1+|t|)\lambda(\xi)^{\frac{1}{s}}).$$

Then we can estimate $\hat{u}(t, \xi)$ given by (12) as follows: for all $\varepsilon > 0$ there is a constant $C_\varepsilon > 0$ such that

$$|\hat{u}(t, \xi)| \leq \sum_{j=0}^{m-1} c_2 C_\varepsilon \exp(\varepsilon \lambda(\xi)^{\frac{1}{r}}) \exp(C_2(1+|t|)\lambda(\xi)^{\frac{1}{s}}).$$

As $r < s$, for $t \in [-T, T]$ (for $T > 0$ fixed), for all $\varepsilon' > 0$, taking ε sufficiently small (depending on ε', r, s, T), then there is a constant $C_{\varepsilon'} > 0$ such that holds $|\hat{u}(t, \xi)| \leq C_{\varepsilon'} \exp(\varepsilon' \lambda(\xi)^{\frac{1}{r}})$, $\forall \xi \in \mathbb{R}^n$, implying that u belongs to

$C^\infty([-T,T], \mathcal{D}'_{r,\lambda}(\mathbb{R}^n))$. We observe that for $r = s$ the solution is only local in time. □

To conclude our treatment, let us show the application of Theorem 2.3 to some operators. Let $P(D_x) = \sum c_\alpha D_x^\alpha$ be a hypoelliptic operator with constant coefficients in $\mathbb{R}^n$ and let $\mathcal{P}$ be its Newton polyhedron, i.e. the convex hull of the points $\{0\} \cup \{\alpha : c_\alpha \neq 0\}$, it is a complete polyhedron, according to the classical result of Friberg [8]. If μ is the formal order of $\mathcal{P}$ and $m > \mu$, then the operator

$$D_t^m + P(D_x)$$

is $(\frac{m}{\mu}, |\cdot|_{\mathcal{P}})$-hyperbolic, and therefore the Cauchy problem (11) admits a unique solution $u \in C^\infty([-T,T], \mathcal{D}'_{r,\mathcal{P}}(\mathbb{R}^n))$ (for all $T > 0$) for any data $u_k \in \mathcal{D}'_{r,\mathcal{P}}(\mathbb{R}^n)$ $(k = 0, \dots, m-1)$, if $r < s = \frac{m}{\mu}$.

REFERENCES

[1] A. Beurling, *Quasi-analyticity and general distributions*, Amer. Math. Soc. Summer Inst., Stanford 1962.

[2] R. W. Braun - R. Meise - B. A. Taylor, *Ultradifferentiable functions and Fourier analysis*, Results in Mathematics 17 (1990), 207–237.

[3] G. Björck, *Linear partial differential operators and generalized distributions*, Ark. Mat. 6 (1966), 351–407.

[4] D. Calvo, *Cauchy problem in inhomogeneous Gevrey classes for operators with constant coefficients*, Indian Journal of Mathematics 47, 1 (2005), 59–76.

[5] D. Calvo, *Generalized Gevrey classes and multi-quasi-hyperbolic operators*, Rend. Sem. Mat. Univ. Pol. Torino 60 (2) (2002), 73–100.

[6] D. Calvo - A. Morando, *Multianisotropic Gevrey classes and ultradistributions*, to appear in J. Funct. Anal. (2005).

[7] D. Calvo - A. Morando - L. Rodino, *Inhomogeneous Gevrey classes and ultradistributions*,J. Math. Anal. Appl. 297 (2004), 720–739.

[8] J. Friberg, *Multi quasi-elliptic polynomials*, Ann. Sc. Norm. Supl Pisa, Cl. di Sc. 21 (1967), 239–260.

[9] M. Gevrey, *Sur la nature analytique des solutions des équations aux dérivées partielles*, Ann. Ec. Norm. Sup. Paris, 35 (1918), 129–190.

[10] G. Gindikin - L. R. Volevich, *The method of Newton's polyhedron in the theory of partial differential equations*, Mathematics and its applications, Soviet Series, 86, 1992.

[11] A. Grothendieck, *Sur les espaces (F) and (DF)*, Summa Brasil. Math. 3 (1954), 57–123.

[12] L. H ö r m a n d e r, *The analysis of linear partial differential Operators*, Vol. 2, Springer-Velag, Berlin, 1983.

[13] H. K o m a t s u, *Ultradistributions I: structure theorem and a characterization*, J. Fac. Sci. Univ. Tokyo, Sect. IA, 20 (1973), 25-105.

[14] E. L a r s s o n, *Generalized hyperbolicity*, Ark. Mat. 7 (1967), 11–32.

[15] O. L i e s s, *Intersection properties of weak analytically uniform classes of functions*, Ark. Mat., 14, (1976), 93–111.

[16] O. L i e s s - L. R o d i n o, *Inhomogeneous Gevrey classes and related pseudo-differential operators*, Suppl. Boll. Un. Mat. It., Anal. Funz. Appl., Vol. 3 1-C (1984), 233–323.

[17] J. L. L i o n s - E. M a g e n e s, *Problèmes aux limites non homogènes et applications*, Vol.3, Dunod 1970, Paris.

[18] S. M a n d e l b r o j t, *Séries adhérentes, régularisation des suites, Applications*, Gauthier-Villars, Paris, 1952.

[19] M. M a s c a r e l l o - L. R o d i n o, *Partial differential equations with multiple characteristics*, Wiley-VCH, Berlin, 1997.

[20] W. M a t s u m o t o, *Theory of pseudo-differential operators of ultradifferentiable classes*, J. Math. Kyoto Univ. 27 (1987), 453–500.

[21] L. R o d i n o, *Linear partial differential operators in Gevrey spaces*, World Scientific, Singapore, 1993.

[22] C. R o u m i e u, *Sur quelques extensions de la notion de distribution*, Ann. Sci. École Norm. Sup. (3) 77, (1960), 41–121.

[23] C. R o u m i e u, *Ultra-distributions définies sur* $\mathbb{R}^n$ *et sur certaines classes des variétés différentiables*, J. Analyse Math. 10 (1962-1963), 153–192.

[24] W. R u d i n, *Division in algebra of infinitely differentiable functions*, J. Math. Mech., 11, (1962), 673–711.

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