Bulletin T.CXXXIV de l'Académie serbe des sciences et des arts − 2007
Classe des Sciences mathématiques et naturelles
Sciences mathématiques, № 32

NORDHAUS-GADDUM-TYPE RELATIONS FOR THE ENERGY AND LAPLACIAN ENERGY OF GRAPHS

B. ZHOU, I. GUTMAN

(Presented at the 8th Meeting, held on November 24, 2006)

A b s t r a c t. *Let $\overline{G}$ denote the complement of the graph G. If $I(G)$ is some invariant of G, then relations (identities, bounds, and similar) pertaining to $I(G) + I(\overline{G})$ are said to be of Nordhaus-Gaddum type. A number of lower and upper bounds of Nordhaus-Gaddum type are obtained for the energy and Laplacian energy of graphs. Also some new relations for the Laplacian graph energy are established.*

AMS Mathematics Subject Classification (2000): 05C50
Key Words: spectrum (of graph), Laplacian spectrum (of graph), energy (of graph), Laplacian energy (of graph), Nordhaus-Gaddum-type relation

1. *Introduction*

In this paper we are concerned with simple graphs. Let G be such a graph, and let n and m denote, respectively, the number of its vertices and edges. Then G is said to be an (n, m)-graph.

The (ordinary) spectrum of G is the spectrum of its adjacency matrix [6], and consists of the numbers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. The Laplacian spectrum of G is the spectrum of its Laplacian matrix [10, 11, 21, 22], and consists of the numbers $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n = 0$.

The *energy* of a graph G, denoted by $E(G)$, is defined as

$$E(G) = \sum_{i=1}^{n} |\lambda_i| \ .$$

This graph-spectrum-based invariant has its origin in theoretical chemistry (for details see [13, 14]), but has recently attracted the interest of mathematicians. The basic mathematical properties of graph energy can be found in the review [12], whereas some most recent mathematical studies in the papers [3, 4, 25–30, 32, 33, 35].

The *Laplacian energy* of a graph G, denoted by $LE(G)$, has been recently defined as [15]

$$LE(G) = \sum_{i=1}^{n} \left| \mu_i - \frac{2m}{n} \right|$$

and was aimed at being the Laplacian-spectral analog of graph energy. Until now, only two papers [15, 37] are devoted to the study of Laplacian graph energy.

As usual, $\overline{G}$ will symbolize the complement of the graph G. The number of vertices and edges of the complement of an (n,m)-graph will be denoted by $\overline{n}$ and $\overline{m}$, respectively.

Nordhaus and Gaddum [23] reported bounds for the sum of the chromatic numbers of a graph and its complement. Eventually, Norhhaus-Gaddum-type relations were established for many other graph invariants [1, 2, 5, 8, 9, 16, 17, 20, 31, 34, 36]. In this paper we obtain bounds of this kind for the graph energy and Laplacian graph energy.

2. *Nordhaus-Gaddum-Type Bounds for Graph Energy*

Let $\overline{\lambda_1}$ be the largest eigenvalue of $\overline{G}$. Nosal [24] demonstrated that for a graph G with n vertices,

$$n - 1 \leq \lambda_1 + \overline{\lambda_1} < \sqrt{2}\, n \tag{1}$$

which itself is a Nordhaus-Gaddum-type relation. In connection with the right-hand side inequality in (1), it was shown in [17] that

$$\lambda_1 + \overline{\lambda_1} \leq \sqrt{\left(2 - \frac{1}{\omega} - \frac{1}{\overline{\omega}}\right) n(n-1)}\ , \tag{2}$$

where ω and $\overline{\omega}$ denote the clique numbers of G and $\overline{G}$, respectively.

Theorem 2.1. *Let G be a graph with n vertices. Then*

$$E(G) + E(\overline{G}) \geq 2(n-1) \tag{3}$$

with equality if and only if G is the complete graph K_n or its complement, the empty graph (the n-vertex graph without edges).

P r o o f. We first observe that $E(G) \geq 2\,\lambda_1$ with equality if and only if G has at most one positive eigenvalue, i.e., if G is the empty graph or a complete multipartite graph [6]. Therefore,

$$E(G) + E(\overline{G}) \geq 2(\lambda_1 + \overline{\lambda_1}) \geq 2(n-1)\ .$$

If equality holds in (3), then both G and $\overline{G}$ are empty or complete multipartite graphs, and so G must be the complete graph or the empty graph. Conversely, knowing the spectrum of K_n and $\overline{K_n}$, see [6], it is easily shown that (3) is an equality if $G \cong K_n$ or $G \cong \overline{K_n}$. □

In [19] it was shown that for an (n,m)-graph G,

$$E(G) \leq \lambda_1 + \sqrt{(n-1)\,(2m - \lambda_1^2)}\ . \tag{4}$$

From this upper bound it could be deduced that [18]

$$E(G) \leq \frac{n}{2}\ (\sqrt{n} + 1)$$

which immediately implies

$$E(G) + E(\overline{G}) \leq n\ (\sqrt{n} + 1)\ .$$

In what follows we improve the latter upper bound.

Theorem 2.2. *Let G be a graph with n vertices. Then*

$$E(G) + E(\overline{G}) < \sqrt{2}\,n + (n-1)\sqrt{n-1}\ . \tag{5}$$

P r o o f. Let m and $\overline{m}$ denote, respectively, the number of edges of G and $\overline{G}$. By (4) and (1), we have

$$E(G) + E(\overline{G}) \leq \lambda_1 + \overline{\lambda_1} + \sqrt{(n-1)\ (2m - \lambda_1^2)} + \sqrt{(n-1)\ \left(2\overline{m} - \overline{\lambda_1}^2\right)}$$

$$
\begin{aligned}
&\le \lambda_1 + \overline{\lambda_1} + \sqrt{2(n-1)\,\left[2m + 2\overline{m} - \left(\lambda_1^2 + \overline{\lambda_1}^2\right)\right]} \\
&\le \lambda_1 + \overline{\lambda_1} + \sqrt{2(n-1)\,\left[n(n-1) - \frac{1}{2}\,\left(\lambda_1 + \overline{\lambda_1}\right)^2\right]} \\
&< \sqrt{2}\,n + \sqrt{2(n-1)\,\left[n(n-1) - \frac{1}{2}\,(n-1)^2\right]} \\
&= \sqrt{2}\,n + (n-1)\,\sqrt{n-1}\;.
\end{aligned}
$$

This completes the proof. □

Remark 2.3. Let G be an n-vertex regular graph of degree r. Then (4) becomes $E(G) \le r + \sqrt{(n-1)r(n-r)}$ and we have

$$
\begin{aligned}
E(G) + E(\overline{G}) &\le n - 1 + \sqrt{(n-1)}\,\left[\sqrt{r(n-r)} + \sqrt{(r+1)(n-r-1)}\right] \\
&\le (n-1)\,\left(\sqrt{n+1} + 1\right)
\end{aligned}
$$

which for $n \ge 6$ is better than (5).

Remark 2.4. A strongly regular graph G with parameters (n, r, ρ, σ) is an r-regular graph on n vertices, in which each pair of adjacent vertices has ρ common neighbors and each pair of non-adjacent vertices has σ common neighbors. If $\sigma \ge 1$ and G is non-complete, then the eigenvalues of G are [6] r, s, and t, with multiplicities 1, m_s, and m_t, where s and t are the solutions of the equation $x^2 + (\sigma - \rho)x + (\sigma - r) = 0$, and m_s and m_t are determined by $m_s + m_t = n - 1$ and $r + m_s\,s + m_t\,t = 0$. If G is a strongly regular graph with parameters $(n, (n + \sqrt{n})/2, (n + 2\sqrt{n})/4, (n + 2\sqrt{n})/4)$ (for some conveniently chosen value of n), then

$$
E(G) + E(\overline{G}) = \frac{n}{2}\,(\sqrt{n}+1) + \frac{n}{2}\,(\sqrt{n}+1) - \sqrt{n} - 2 = (n-1)\,(\sqrt{n}+1) - 1\;.
$$

If we consider a Paley graph H, which is a strongly regular graph with parameters $(n, (n-1)/2, (n-5)/4, (n-1)/4)$, then

$$
E(H) + E(\overline{H}) = (n-1)(\sqrt{n}+1)\;.
$$

The results stated in Remark 2.4 show that the bound (5) is asymptotically tight.

Remark 2.5. Using (2), from the proof of Theorem 2.2, we have

$$E(G)+E(\overline{G}) \leq \sqrt{\left(2-\frac{1}{\omega}-\frac{1}{\overline{\omega}}\right) n(n-1)+(n-1)\sqrt{n-1}}\ .$$

3. *Some Properties of the Laplacian Graph Energy*

Details of the theory of Laplacian graph spectra are found in the reviews [10, 11, 21, 22]. For the following consideration we need the properties: $\mu_n = 0$ for all graphs, and $\mu_{n-1} > 0$ if and only if G is connected.

Let $G_1 * G_2$ denote the join of the graphs G_1 and G_2, i.e., the graph obtained from the disjoint union of G_1 and G_2 by adding all possible edges between vertices of G_1 and vertices of G_2.

Theorem 3.1. *Let G_1 and G_2 be (n,m)-graphs. Then*

$$LE(G_1 * G_2) = LE(G_1) + LE(G_2) + 2n - \frac{4m}{n}\ .$$

P r o o f. Let $\mu_1', \mu_2', \ldots, \mu_n'$ be the Laplacian eigenvalues of G_1 and $\mu_1'', \mu_2'', \ldots, \mu_n''$ the Laplacian eigenvalues of G_2. Then the Laplacian eigenvalues of $G_1 * G_2$ are [22]

$$2n\ ,\ n+\mu_1'\ ,\ n+\mu_1''\ ,\ n+\mu_2'\ ,\ n+\mu_2''\ ,\ldots,\ n+\mu_{n-1}'\ ,\ n+\mu_{n-1}''\ ,\ 0\ .$$

Note that $G_1 * G_2$ is a $(2n, 2m+n^2)$-graph with average vertex degree $(2m+n^2)/n$. Therefore,

$$\begin{aligned} LE(G_1 * G_2) &= 2n + \sum_{i=1}^{n-1}\left|n+\mu_i' - \frac{2m+n^2}{n}\right| + \sum_{i=1}^{n-1}\left|n+\mu_i'' - \frac{2m+n^2}{n}\right| \\ &= 2n + \sum_{i=1}^{n-1}\left|\mu_i' - \frac{2m}{n}\right| + \sum_{i=1}^{n-1}\left|\mu_i'' - \frac{2m}{n}\right| \\ &= 2n + LE(G_1) - \frac{2m}{n} + LE(G_2) - \frac{2m}{n}\ . \end{aligned}$$

The result follows. □

Remark 3.2. Let G_1 and G_2 be regular graphs of degrees r' and r'', respectively, with n' and n'' vertices, respectively. Then

$$E(G_1 * G_2) = E(G_1) + E(G_2) + \sqrt{(r' - r'')^2 + 4\,n'\,n''} - r' - r'' \ .$$

Let $G_1 \times G_2$ denote the Cartesian product of graphs G_1 and G_2. Then $V(G_1 \times G_2) = V(G_1) \times V(G_2)$ and (u_1, u_2) is adjacent to (v_1, v_2) if and only if $u_1 = v_1$ and $(u_2, v_2) \in E(G_2)$, or $u_2 = v_2$ and $(u_1, v_1) \in E(G_1)$.

Theorem 3.3. *Let G_1 and G_2 be, respectively, (n, m_1)– and (n, m_2)-graphs. Then*

$$LE(G_1 \times G_2) \leq n\,LE(G_1) + n\,LE(G_2) \ .$$

P r o o f. Let the notation be the same as in the proof of Theorem 3.1. Then the Laplacian eigenvalues of $G_1 \times G_2$ are [22]

$$\mu_i' + \mu_j'' \ , \ i, j = 1, 2, \ldots, n \ .$$

Note that $G_1 \times G_2$ is an $(n^2, n(m_1 + m_2))$-graph with average vertex degree $(2m_1 + 2m_2)/n$. Therefore,

$$\begin{aligned} LE(G_1 \times G_2) &= \sum_{i=1}^{n} \sum_{j=1}^{n} \left| \mu_i' + \mu_j'' - \frac{2\,m_1 + 2\,m_2}{n} \right| \\ &\leq \sum_{i=1}^{n} \sum_{j=1}^{n} \left(\left| \mu_i' - \frac{2\,m_1}{n} \right| + \left| \mu_j'' - \frac{2\,m_2}{n} \right| \right) \\ &= n\,LE(G_1) + n\,LE(G_2) \ . \end{aligned}$$

The result follows. □

Let G be an (n, m)-graph. Note that $\mu_1 \geq 2m/n$. Then

$$LE(G) = \mu_1 + \sum_{i=2}^{n-1} \left| \mu_i - \frac{2m}{n} \right| \ .$$

If G is not a complete graph, then $\mu_{n-1} \leq 2m/n$ [7], and therefore

$$LE(G) = \mu_1 - \mu_{n-1} + \frac{2m}{n} + \sum_{i=2}^{n-2} \left| \mu_i - \frac{2m}{n} \right| \ .$$

Theorem 3.4. *Let G be an (n,m)-graph with $n \geq 2$ and $m \geq 1$. Then $LE(G) \geq \mu_1$, with equality if and only if $G \cong K_{n/2,n/2}$, in which case, of course, n must be even.*

P r o o f. It is easy to see that $LE(G) \geq \mu_1$, with equality if and only if $n = 2$ or for $n \geq 3$, if $\mu_2 = \cdots = \mu_{n-1} = \frac{2m}{n}$. Suppose that $n \geq 3$ and $LE(G) = \mu_1$. Then by a result from [37], G is a regular complete k-partite graph with $1 < k \leq n$. Then

$$n - \frac{n}{k} + (k-1)\,\frac{n}{k} = n\,,$$

implying $k = 2$. Thus, $G \cong K_{n/2,n/2}$. Conversely, if $G \cong K_{n/2,n/2}$, then it is easy to verify that $LE(G) = \mu_1$. □

In a similar manner we arrive at

Theorem 3.5. *Let G be an (n,m)-graph, such that $n \geq 3$ and $m \geq 1$. Then*

$$LE(G) \geq \mu_1 - \mu_{n-1} + \frac{2m}{n}$$

with equality if and only if $n = 3$ or for $n \geq 4$, if $\mu_2 = \cdots = \mu_{n-2} = 2m/n$.

4. *Nordhaus-Gaddum-Type Bounds for Laplacian Graph Energy*

Lemma 4.1. *If G is not the complete graph, and has at least one edge, then $\mu_1 - \mu_{n-1} > 1$.*

P r o o f. Since G has at least one edge, $\mu_1 \geq \Delta + 1$, where Δ is the maximum vertex degree of G [10, 21]. If G is connected, then equality holds if and only if $\Delta = n - 1$.

Suppose that G is connected. Then $\mu_1 - \mu_{n-1} \geq \Delta - 2m/n + 1 \geq 1$. If $\mu_1 - \mu_{n-1} = 1$, then $2m/n = \Delta = n - 1$ and then it would be $G \cong K_n$, a contradiction.

If G is not connected, then $\mu_1 - \mu_{n-1} = \mu_1 \geq \Delta + 1 > 1$. □

Theorem 4.2. *Let G be a graph with n vertices. Then*

$$LE(G) + LE(\overline{G}) \geq 2n - 2$$

with equality if and only if G is isomorphic to K_n or $\overline{K_n}$.

P r o o f. If G is isomorphic to K_n or $\overline{K_n}$, then it is easy to show that $LE(G) + LE(\overline{G}) = 2n - 2$. Suppose that $n \geq 3$ and that G is different from

K_n and $\overline{K_n}$. Then

$$
\begin{aligned}
LE(G)+LE(\overline{G}) &= \mu_1-\mu_{n-1}+\frac{2m}{n}+\sum_{i=2}^{n-2}\left|\mu_i-\frac{2m}{n}\right| \\
&\quad + \mu_1-\mu_{n-1}+\frac{2\,\overline{m}}{n}+\sum_{i=2}^{n-2}\left|n-\mu_i-\frac{2\,\overline{m}}{n}\right| \\
&\geq 2\,(\mu_1-\mu_{n-1})+n-1+\sum_{i=2}^{n-2}1 = 2\,(\mu_1-\mu_{n-1})+2n-4\ .
\end{aligned}
$$

By Lemma 4.1, $LE(G)+LE(\overline{G})>2n-2$. □

Theorem 4.3. *Let G be a graph with n vertices. Then*

$$LE(G)+LE(\overline{G})<n\sqrt{n^2-1}\ .$$

P r o o f. Denote by $d_1,d_2,\ldots,d_n$ the vertex degrees of G . Assume that $n\geq 2$. Let the auxiliary quantity M be defined as [15]

$$M=M(G)=m+\frac{1}{2}\sum_{i=1}^{n}\left(d_i-\frac{2m}{n}\right)^2\ .$$

Then

$$M(\overline{G})=\overline{m}+\frac{1}{2}\sum_{i=1}^{n}\left(d_i-\frac{2m}{n}\right)^2\ .$$

Using the fact

$$\sum_{i=1}^{n}(d_i)^2\leq 2(n-1)m$$

with equality if and only if G is the empty graph or the complete graph, we have

$$
\begin{aligned}
M(G)+M(\overline{G}) &= \frac{1}{2}\,n(n-1)+\sum_{i=1}^{n}\left(d_i-\frac{2m}{n}\right)^2 \\
&= \frac{1}{2}\,n(n-1)+\sum_{i=1}^{n}(d_i)^2-\frac{4m^2}{n} \\
&\leq \frac{1}{2}\,n(n-1)+2(n-1)m-\frac{4m^2}{n} \\
&\leq \frac{1}{2}\,n(n-1)+\frac{1}{4}\,n(n-1)^2=\frac{1}{4}\,(n-1)n(n+1)\ .
\end{aligned}
$$

Now, because for $n \geq 2$ the number of edges of K_n and $\overline{K_n}$ differs from $n(n-1)/4$, we have

$$M(G) + M(\overline{G}) < \frac{1}{4}\,(n-1)n(n+1)\ . \tag{6}$$

In [15] it has been shown that $LE(G) \leq \sqrt{2nM}$, which combined with (6) implies

$$LE(G) + LE(\overline{G}) \leq \sqrt{4n\left[M(G) + M(\overline{G})\right]} < n\,\sqrt{n^2-1}\ . \qquad \square$$

Example 4.4. Let $G \cong K_{n/2} \cup \overline{K_{n/2}}$. Then the Laplacian eigenvalues of G are

$$\frac{n}{2}\,\left(\frac{n}{2} - 1 \text{ times}\right) \text{ and } 0\,\left(\frac{n}{2} + 1 \text{ times}\right)$$

and therefore

$$LE(G) = \left(\frac{n}{2} - 1\right)\frac{n+2}{4} + \left(\frac{n}{2} + 1\right)\frac{n-2}{4} = \frac{1}{4}\,(n^2-4)\ .$$

The Laplacian eigenvalues of $\overline{G}$ are

$$n\,\left(\frac{n}{2} \text{ times}\right),\ \frac{n}{2}\,\left(\frac{n}{2} - 1 \text{ times}\right) \text{ and } 0 \text{ (1 time)}$$

and therefore

$$LE(\overline{G}) = \frac{n}{2}\,\frac{n+2}{4} + \left(\frac{n}{2} - 1\right)\frac{n-2}{4} + \frac{3n-2}{4} = \frac{1}{4}\,(n^2+2n)\ .$$

This implies

$$LE(G) + LE(\overline{G}) = \frac{1}{2}\,(n^2+n-2)\ .$$

Acknowledgement. This work was supported by the National Natural Science Foundation of China through Grant no. 10671076, and by the Serbian Ministry of Science and Environmental Protection, through Grant no. 144015G.

REFERENCES

[1] N. Achuthan, N. R. Achuthan, L. Caccetta, *On the Nordhaus-Gaddum class problem*, Australas. J. Combin. **2** (1990) 5–27.

[2] Y. Alavi, M. Behzard, *Complementary graphs and edge chromatic numbers*, SIAM J. Appl. Math. **20** (1971) 161–163.

[3] R. Balakrishnan, *The energy of a graph*, Lin. Algebra Appl. **387** (2004) 287–295.

[4] R. B. Bapat, S. Pati, *Energy of a graph is never an odd integer*, Bull. Kerala Math. Assoc. **1** (2004) 129–132.

[5] G. Chartrand, S. Schuster, *On the independence numbers of complementary graphs*, Trans. New York Acad. Sci. Ser. II **36** (1974) 247–251.

[6] D. Cvetković, M. Doob, H. Sachs, *Spectra of Graphs – Theory and Application*, Academic Press, New York, 1980; 2nd revised ed.: Barth, Heidelberg, 1995.

[7] M. Fiedler, *Algebraic connectivity of graphs*, Czechoslovak Math. J. **25** (1975) 607–618.

[8] W. Goddard, M. A. Henning, *Nordhaus–Gaddum bounds for independent domination*, Discr. Math. **268** (2003) 299–302.

[9] W. Goddard, M. A. Henning, H. C. Swart, *Some Nordhaus–Gaddum–type results*, J. Graph Theory **16** (1992) 221–231.

[10] R. Grone, R. Merris, *The Laplacian spectrum of a graph II*, SIAM J. Discr. Math. **7** (1994) 221–229.

[11] R. Grone, R. Merris, V. S. Sunder, *The Laplacian spectrum of a graph*, SIAM J. Matrix Anal. Appl. **11** (1990) 218–238.

[12] I. Gutman, *The energy of a graph: old and new results*, in: A. Betten, A. Kohnert, R. Laue, A. Wassermann (Eds.), *Algebraic Combinatorics and Applications*, Springer–Verlag, Berlin, 2001, pp. 196–211.

[13] I. Gutman, *Topology and stability of conjugated hydrocarbons. The dependence of total π-electron energy on molecular topology*, J. Serb. Chem. Soc. **70** (2005) 441–456.

[14] I. Gutman, O. E. Polansky, *Mathematical Concepts in Organic Chemistry*, Springer–Verlag, Berlin, 1986.

[15] I. Gutman, B. Zhou, *Laplacian energy of a graph*, Lin. Algebra Appl. **414** (2006) 29–37.

[16] F. Harary, T. W. Haynes, *Nordhaus–Gaddum inequalities for domination in graphs*, Discr. Math. **155** (1996) 99–105.

[17] Y. Hong, J. Shu, *A sharp upper bound for the spectral radius of the Nordhas–Gaddum type*, Discr. Math. **211** (2000) 229–232.

[18] J. H. Koolen, V. Moulton, *Maximal energy graphs*, Adv. Appl. Math. **26** (2001) 47–52.

[19] J. H. Koolen, V. Moulton, I. Gutman, *Improving the McCelland inequality for total π-electron energy*, Chem. Phys. Lett. **320** (2000) 213–216.

[20] H. Liu, M. Lu, F. Tian, *On the ordering of trees with the general Randić index of the Nordhaus–Gaddum type*, MATCH Commun. Math. Comput. Chem. **55** (2006) 419–426.

[21] R. M e r r i s, *Laplacian matrices of graphs: A survey*, Lin. Algebra Appl. **197–198** (1994) 143–176.

[22] B. M o h a r, *The Laplacian spectrum of graphs*, in: Y. Alavi, G. Chartrand, O. R. Oellermann, A. J. Schwenk (Eds.), *Graph Theory, Combinatorics, and Applications*, Wiley, New York, 1991, pp. 871–898.

[23] E. A. N o r d h a u s, J. W. G a d d u m, *On complementary graphs*, Amer. Math. Monthly **63** (1956) 175–177.

[24] E. N o s a l, *Eigenvalues of Graphs*, M. Sc. thesis, Univ. Calgary, 1970.

[25] J. R a d a, *Energy ordering of catacondensed hexagonal systems*, Discr. Appl. Math. **145** (2005) 437–443.

[26] J. R a d a, A. T i n e o, *Upper and lower bounds for energy of bipartite graphs*, J. Math. Anal. Appl. **289** (2004) 446–455.

[27] H. S. R a m a n e, I. G u t m a n, H. B. W a l i k a r, S. B. H a l k a r n i, *Another class of equienergetic graphs*, Kragujevac J. Math. **26** (2004) 15–18.

[28] H. S. R a m a n e, H. B. W a l i k a r, S. B. R a o, B. D. A c h a r y a, P. R. H a m p i h o l i, S. R. J o g, I. G u t m a n, *Equienergetic graphs*, Kragujevac J. Math. **26** (2004) 5–13.

[29] H. S. R a m a n e, H. B. W a l i k a r, S. B. R a o, B. D. A c h a r y a, P. R. H a m p i h o l i, S. R. J o g, I. G u t m a n, *Spectra and energies of iterated line graphs of regular graphs*, Appl. Math. Lett. **18** (2005) 679–682.

[30] I. S h p a r l i n s k i, *On the energy of some circulant graphs*, Lin. Algebra Appl. **414** (2006) 378–382.

[31] M. S t e n c e l, *Theorems of Nordhaus–Gaddum type for some numbers*, Discuss. Math. **7** (1985) 107–112.

[32] D. S t e v a n o v i ć, *Energy and NEPS of graphs*, Lin. Multilin. Algebra **53** (2005) 67–74.

[33] D. S t e v a n o v i ć, I. S t a n k o v i ć, *Remarks on hyperenergetic circulant graphs*, Lin. Algebra Appl. **400** (2005) 345–348.

[34] M. S t i e b i t z, *On Hadwiger's number – a problem of the Nordhaus–Gaddum type*, Discr. Math. **101** (1992) 307–317.

[35] W. Y a n, L. Y e, *On the minimal energy of trees with a given diameter*, Appl. Math. Lett. **18** (2005) 1046–1052.

[36] L. Z h a n g, B. W u, *The Nordhaus–Gaddum–type inequalities for some chemical indices*, MATCH Commun. Math. Comput. Chem. **54** (2005) 189–194.

[37] B. Z h o u, I. G u t m a n, *On Laplacian energy of graphs*, MATCH Commun. Math. Comput. Chem. **57** (2007) 211–220.

Department of Mathematics
South China Normal University
Guangzhou 510631
P. R. China

Faculty of Science
University of Kragujevac
P. O. Box 60
34000 Kragujevac
Serbia