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FRAMES FOR FRÉCHET SPACES

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A b s t r a c t. In this paper we study frame representations in projective and inductive limits of Banach spaces. We introduce the notion of a Fréchet pre-frame for a given Fréchet space with respect to a Fréchet sequence space. Main results of the paper include the use of density arguments and representations in the case of projective limits of isomorphic reflexive Banach spaces. Examples based on modulation spaces, Sobolev type spaces and Köthe type spaces are given.

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1. *Introduction*

Our goal is to get frame representations in projective and inductive limits of Banach spaces via families which are not necessarily (Schauder) bases. To that end we employ the existing theory of Banach frames, [2, 4, 7, 11, 14, 15, 17, 26] (see Section 3 for the definition of a Banach frame). We use general Banach sequence spaces for the analysis and provide necessary background for a general theory of frames for Fréchet spaces, which can be found in [23].

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In Section 2 we recall basic notions and list the main properties of sequence spaces under consideration. Although the spaces of sequences, which are usually used in the frame theory, are the spaces l^p $(1 \leq p < \infty)$ and their weighted versions, in the background of our approach lies a more general family the so called Köthe type sequence spaces.

It is known that, in the case of a Hilbert space, the frame conditions (2) and (3) can be extended from a dense subset. In Section 3 we prove analogous statement for Banach spaces with minimal conditions on the corresponding Banach sequence space, Theorem 3.3.

In Section 4 we study Fréchet spaces defined by a family of isomorphic Banach spaces, Theorem 4.2. Although the result is not surprising, it is applicable to important examples of Fréchet spaces such as the Schwartz spaces $\mathcal{S}$ and $\mathcal{D}_{L^2}$. Both spaces are projective limits of certain modulation spaces which are deeply connected to the theory of Weyl-Heisenberg frames, [8, 9, 16, 27]. We end the section with examples which illuminate applications of Theorem 4.2 and relate our results to the ones given in the above mentioned papers.

In order to develop the concept of a Fréchet frame for a Fréchet space in full generality, a family of sequence spaces and their projective limit should be used. This is explained in Section 5. It is well known that sequence spaces which are useful for the analysis of Banach spaces should contain the space of sequences of coefficients as a complemented subspace (see [4, 11, 17]). The analogous property holds for Fréchet spaces, see Theorem 5.3 which is proved in [23].

2. *Basic Notions and Notation*

We denote by $(X, \|\cdot\|_X,)$ a Banach space and by $(X^*, \|\cdot\|_{X^*})$ its dual space. Let X_1 and X_2 be Banach spaces and let $G : X_1 \to X_2$. If $g \in X_2^*$ and $f \in X_1$, then $G^* g(f) = g(Gf)$.

We are interested in the reconstruction of $f \in X$ via a family $\{g_i(f)\}_{i\in I}$ indexed by a countable set I, where $\{g_i\}_{i\in I}$ is a suitable sequence in X^*. We will use notation $\{c_i\} = \{c_i\}_{i=1}^\infty$, $\sum x_i = \sum_{i=1}^\infty x_i \ldots$ The canonical vector $(0, \ldots, 0, 1, 0, \ldots)$, where 1 is at the i-th position, is denoted by e_i. The sequence $\{e_i\}$ is a Schauder basis for l^p, $1 \leq p < \infty$. We denote by c the space of all convergent sequences equipped with the sup-norm, and c_0 is the subspace of l^∞ which consists of all zero-convergent sequences.

Let X_1 and X_2 be subspaces of X such that $X_1 + X_2 = X$ and $X_1 \cap X_2 = \{0\}$. The space X_1 is *complemented* in X if there exists a continuous projection of X on X_1 along X_2. The space c_0 is not complemented in l^∞, [21].

Recall, a complete locally convex space which has a countable fundamental system of seminorms is called a *Fréchet space*. A Fréchet space E with a fundamental system $(\|\cdot\|_k)_{k\in\mathbb{N}}$ of seminorms is said to have the property (DN) if there is $p \in \mathbb{N}$ such that for each $q \in \mathbb{N}$ there exist $n \in \mathbb{N}$ and $C > 0$ such that

$$\|x\|_q^2 \leq C\|x\|_p\|x\|_n, \quad \forall x \in E.$$

Then, $\|\cdot\|_p$ is a norm. A Fréchet space E with a fundamental system $(\|.\|_k)_{k\in\mathbb{N}}$ of seminorms has the property (Ω) if for each $p \in \mathbb{N}$ there exists $q \in \mathbb{N}$ so that for every $n \in \mathbb{N}$ there exist $\theta \in (0,1)$ and $C > 0$ with

$$\|y\|_q^* \leq C\|y\|_p^{*^{1-\theta}}\|y\|_n^{*^{\theta}}, \quad \forall y \in E^*,$$

where $\|y\|_k^* := \sup\{|y(x)| : \|x\|_k \leq 1\}$, $k \in \mathbb{N}$.

We denote by $(\Theta, |||\cdot|||_\Theta)$ a Banach sequence space. The space Θ is called *solid* if the conditions $\{c_i\} \in \Theta$ and $|d_i| \leq |c_i|$, for all $i \in \mathbb{N}$, imply that $\{d_i\} \in \Theta$ and $|||\{d_i\}|||_\Theta \leq |||\{c_i\}|||_\Theta$.

A Banach sequence space is a *BK-space* if the coordinate functionals are continuous or, equivalently, if the convergence implies the convergence of the corresponding coordinates. A BK-space, which contains all the canonical vectors e_i and for which there exists a constant $\lambda \geq 1$ such that

$$|||\{c_i\}_{i=1}^n|||_\Theta \leq \lambda|||\{c_i\}_{i=1}^\infty|||_\Theta,\ \forall n \in \mathbb{N},\ \forall\{c_i\}_{i=1}^\infty \in \Theta \tag{1}$$

$(\{c_i\}_{i=1}^n := \sum_{i=1}^n c_ie_i)$, will be called *$\lambda$–$BK$-space*.

A BK-space Θ is called a *CB-space* if the set of the canonical vectors $\{e_i\}$ forms a (Schauder) basis, which will be called *the canonical basis* for the CB-space Θ. When Θ is a CB-space, we have $1 \leq \sup_N \|S_N\| < \infty$ (see [18, 25]), where $S_N : \Theta \to \Theta$ are given by $S_N(\{c_i\}) = \sum_{i=1}^N c_ie_i$, $N \in \mathbb{N}$. The number $\sup_N \|S_N\|$ is called the canonical basis constant. Thus every CB-space is a λ–BK-space, where λ is the canonical basis constant. Solid CB-spaces, in particular, ℓ^p spaces, $1 \leq p < \infty$, and c_0 are examples of 1–BK-spaces. Note that a λ–BK–space need not be a CB-space – take for example the space c or the space ℓ^∞, which are 1–BK-spaces.

If Θ is a CB-space, then the space $BK\Theta^* := \{\{g(e_i)\} : g \in \Theta^*\}$ with the norm $|||\{g(e_i)\}|||_{BK\Theta^*} := \|g\|_{\Theta^*}$ is a BK-space isometrically isomorphic to Θ^* (see e.g. [19]). Moreover, if Θ is a reflexive CB-space, then the coefficient functionals $\{E_i\}$ associated to the canonical basis $\{e_i\}$ form a Schauder basis for Θ^* and thus $BK\Theta^*$ is a CB-space, since the canonical vectors $\{E_j(e_i)\}$ form a Schauder basis for $BK\Theta^*$. From now on when Θ is a CB-space, we will always identify Θ^* with $BK\Theta^*$.

Let $\mathcal{A} = (a_{i,k})_{i,k\in\mathbb{N}}$ be a matrix with positive elements such that $a_{i,k} \leq a_{i,k+1}$ for all $i, k \in \mathbb{N}$, the so called *Köthe matrix*. For any fixed $p \in [1, \infty)$ and $k \in \mathbb{N}$ we consider the *Köthe type spaces*

$$\lambda^{p,k}(\mathcal{A}) := \{\{x_i\}_{i=1}^{\infty} \ : \ |||\{x_i\}|||_{p,k} := \left(\sum |x_i a_{i,k}|^p\right)^{1/p} < \infty\}.$$

We refer to [21] for a detailed exposition on Köthe type spaces. We will consider the spaces $\lambda^{2,k}((i^k)_{i\in\mathbb{N}})$, $k \in \mathbb{N}$. Their projective limit as $k \to \infty$ is the space of rapidly decreasing sequences

$$\mathbf{s} = \left\{\{x_i\} \ : \ |||\{x_i\}|||_k := \left(\sum |x_i i^k|^2\right)^{1/2} < \infty, \ \forall k \in \mathbb{N}\right\},$$

which is a nuclear Fréchet space. A Fréchet space F is isomorphic to a complemented subspace of $\mathbf{s}$ if and only if it is nuclear and has the properties (DN) and (Ω).

3. *Density theorem*

Recall [4], if Θ is a BK-space, a countable family $\{g_i\}_{i\in I}$, $g_i \in X^*$, is a Θ*–frame for* X, if there exist constants $0 < A \leq B < \infty$ such that

$$\{g_i(f)\}_{i\in I} \in \Theta \text{ and} \tag{2}$$

$$A\|f\|_X \leq |||\{g_i(f)\}_{i\in I}|||_\Theta \leq B\|f\|_X \tag{3}$$

for all $f \in X$. The constants A and B are called Θ*-frame bounds*. When (2) and at least the upper inequality in (3) are satisfied for all $f \in X$, then $\{g_i\}_{i\in I}$ is called a Θ*-Bessel sequence for* X *with the bound* B.

If $\Theta = \ell^p$, $1 < p < \infty$, then an ℓ^p-frame is a p-frame introduced in [2]; p--frames in shift-invariant spaces of ℓ^p are considered in [2, 3], while p-frames in general Banach spaces are studied in [7].

The definition of a Θ-frame is part of the definition of the notion of a Banach frame, introduced by Gröchenig [15] and studied in [4, 6, 11, 12, 13, 14, 17]. A Banach frame for a Banach space X with respect to a BK-space Θ is a Θ-frame $\{g_i\}$ for X, for which there exists a bounded linear operator $S : \Theta \to X$ such that $S(\{g_i(f)\}) = f$ for all $f \in X$. In other words, a Θ-frame is a Banach frame when the operator $U : X \to \Theta$, $Uf = \{g_i(f)\}$, has a bounded left inverse $S : \Theta \to X$. The difference between Banach frames and Θ-frames is studied in [4]. We recall the main result from [4] which will be used later on.

Proposition 3.1. ([4, Proposition 3.4]) *Let Θ be a BK-space and $\{g_i\} \in (X^*)^{\mathbb{N}}$ be a Θ-frame for X. Then:*

(i) *The family $\{g_i\}$ is a Banach frame for X with respect to Θ if and only if the set $\{\{g_i(f)\} : f \in X\}$ is complemented in Θ,*

(ii) *If Θ is a CB-space, then there exists a Θ^*-Bessel sequence $\{f_i\}$ ($f_i \in X \subseteq X^{**}, i \in \mathbb{N}$) for X^* such that*

$$f = \sum g_i(f) f_i, \quad \forall f \in X, \tag{4}$$

if and only if the set $\{\{g_i(f)\} : f \in X\}$ is complemented in Θ.

(iii) *If both Θ and Θ^* are CB-spaces, there exists a Θ^*-Bessel sequence $\{f_i\}$ ($f_i \in X \subseteq X^{**}, i \in \mathbb{N}$) for X^* such that*

$$g = \sum g(f_i) g_i, \ \forall g \in X^*, \tag{5}$$

if and only if the set $\{\{g_i(f)\} : f \in X\}$ is complemented in Θ.

In each of the cases (ii) *and* (iii)*, $\{f_i\}$ is actually a Θ^*-frame for X^*.*

A Θ^*-frame $\{f_i\}$ for X^*, satisfying (4) and (5), is called a *dual frame* of the Θ-frame $\{g_i\}$.

Remark 3.2. *By the proof of the above Proposition 3.1* (iii)*, given in [4], we have that a Θ^*-Bessel sequence $\{f_i\}$ satisfies (4) if and only if it satisfies (5).*

Our aim is to extend Proposition 3.1 to Fréchet spaces and obtain series expansions by the use of a "Fréchet frame" and the corresponding "dual frame", see Section 5 for the definition. The first result in that direction is Theorem 3.3.

If X is a Hilbert space and $\Theta = \ell^2$, and if (2) and (3) hold on a dense subset of X then (2) and (3) hold for all $f \in X$, [5]. The same conclusion holds when X is a Banach space and $\Theta = \ell^p$, $p \in (1, \infty)$, see [26]. Analogous result for Banach frames can be found in [11]. Now we generalize these results for larger class of sequence spaces Θ and for Θ-frames.

Theorem 3.3. *Let $(\Theta, |||\cdot|||)$ be a λ–BK-space. Let W be a dense subset of a Banach space $(X, \|\cdot\|)$ and $\{g_i\} \in (X^*)^{\mathbb{N}}$.*

(i) *If (2) and the upper inequality in (3) hold for all $f \in W$, then $\{g_i\}$ is a Θ-Bessel sequence for X with a bound λB.*

(ii) *If (2) and (3) hold for all $f \in W$, then $\{g_i\}$ is a Θ-frame for X with bounds $A, \lambda B$.*

P r o o f. (i) Assume that $\{g_i\}$ satisfies (2) and the upper inequality in (3) for all $f \in W$. First we will prove that $\{g_i(f)\}$ belongs to Θ for all $f \in X$. Fix an arbitrary $f \in X \setminus W$ and assume that there exists $N \in \mathbb{N}$ such that $||| \sum_{i=1}^N g_i(f)e_i||| > \lambda B\|f\|$. Denote

$$\delta := \left|\left|\left| \sum_{i=1}^N g_i(f)e_i \right|\right|\right| - \lambda B\|f\| > 0.$$

By the density of W in X, there exists a family $\{f_n\} \in W^{\mathbb{N}}$ such that $f_n \to f$ when $n \to \infty$ and hence there exists $N_1 \in \mathbb{N}$ such that

$$\|f_n\| - \|f\| < \frac{\delta}{2\lambda B}, \ \forall n > N_1.$$

By the continuity of g_i, $(i = 1, 2, \dots, N)$, there exists $N_2 \in \mathbb{N}$ such that

$$\left|\left| \sum_{i=1}^N g_i(f)e_i \ |||-||| \sum_{i=1}^N g_i(f_n)e_i \right|\right| < \frac{\delta}{2}, \ \forall n > N_2.$$

Now for $n > max(N_1, N_2)$ one has

$$\left|\left|\left| \sum_{i=1}^N g_i(f_n)e_i \right|\right|\right| > \lambda B\|f\| + \frac{\delta}{2} \geq \lambda B(\|f_n\| - \frac{\delta}{2\lambda B}) + \frac{\delta}{2} = \lambda B\|f_n\|.$$

By (1),

$$|||\{g_i(f_n)\}_{i=1}^{\infty}||| \geq \frac{1}{\lambda} \left|\left|\left| \sum_{i=1}^N g_i(f_n)e_i \right|\right|\right| > B\|f_n\|,$$

which contradicts to the validity of (3) for $f_n \in W$. Therefore

$$\left|\left|\left| \sum_{i=1}^N g_i(f)e_i \right|\right|\right| \leq \lambda B\|f\|, \ \forall N \in \mathbb{N}.$$

Thus, for every $N \in \mathbb{N}$, the operator $S_N : X \to \Theta$, given by $S_N(f) = \sum_{i=1}^N g_i(f)e_i$, is bounded with $\|S_N\| \leq \lambda B$. Moreover, $S_N(f)$ converges when

$N \to \infty$ for all f in the dense subset W of X. Therefore $S_N(f)$ converges for all $f \in X$ (see, e.g., [20]). Since Θ is a BK-space, $\lim_{N\to\infty} S_N(f)$ is the sequence $\{g_i(f)\}_{i=1}^{\infty}$ and hence $\{g_i(f)\}_{i=1}^{\infty} \in \Theta$ for all $f \in X$. Now the Banach-Steinhaus theorem implies that

$$|||\{g_i(f)\}_{i=1}^{\infty}||| \leq \lambda B\|f\|, \text{ for all } f \in X,$$

and hence $\{g_i\}$ is a Θ-Bessel sequence for X with bound λB.

(ii) Assume now that (2) and (3) hold for all $f \in W$. It remains only to prove the validity of the lower Θ-frame inequality for all $f \in X \setminus W$. Take $f \in X \setminus W$ and a family $\{f_n\} \in W^{\mathbb{N}}$ such that $f_n \to f$ when $n \to \infty$. For any $n \in \mathbb{N}$ we have

$$\left|\ |||\{g_i(f_n)\}_{i=1}^{\infty}||| - |||\{g_i(f)\}_{i=1}^{\infty}|||\ \right| \leq \lambda B\|f_n - f\|$$

and therefore

$$\lim_{n\to\infty} |||\{g_i(f_n)\}_{i=1}^{\infty}||| = |||\{g_i(f)\}_{i=1}^{\infty}|||.$$

By the inequality $A\|f_n\| \leq |||\{g_i(f_n)\}_{i=1}^{\infty}|||$, which holds for all $n \in \mathbb{N}$, when $n \to \infty$ we get

$$A\|f\| \leq |||\{g_i(f)\}_{i=1}^{\infty}|||.$$

□

4. *Isomorphic spaces*

Let $\{Y_s, |\cdot|_s\}_{s\in\mathbb{N}_0}$ be a family of Banach spaces such that

$$\{\mathbf{0}\} \neq \cap_{s\in\mathbb{N}_0} Y_s \subset \ldots \subset Y_2 \subset Y_1 \subset Y_0 \tag{6}$$

$$|\cdot|_0 \leq |\cdot|_1 \leq |\cdot|_2 \leq \ldots \tag{7}$$

$$Y_F := \cap_{s\in\mathbb{N}_0} Y_s \text{ is dense in } Y_s, \quad \text{for every} \quad s \in \mathbb{N}_0. \tag{8}$$

Then Y_F is a Fréchet space with the sequence of norms $|\cdot|_s$, $s \in \mathbb{N}_0$. We will use the above sequences for general Banach spaces, $Y_s = X_s$ with norms $\|\cdot\|_s, s \in \mathbb{N}_0$, and for Banach sequence spaces $Y_s = \Theta_s$ with norms $|||\cdot|||_s, s \in \mathbb{N}_0$.

Note that if $\{\Theta_s, |||\cdot|||_s\}_{s\in\mathbb{N}_0}$ is a family of CB-spaces, satisfying (6) and (7), then finite sequences belong to Θ_F and form a dense subset of Θ_s; thus (8) is automatically satisfied. In this case Θ_F has the canonical basis in the sense that every $x \in \Theta_F$ can be written uniquely as $x = \sum x_i e_i$ with the convergence in Θ_s for every $s \in \mathbb{N}_0$.

For any given $p \in [1, \infty)$, the family $\{\lambda^{p,s}(\mathcal{A}), |||\cdot|||_{p,s}\}_{s\in\mathbb{N}_0}$ defined in Section 2 is a family of solid CB-spaces which satisfy (6)-(8).

It is well known that a (Hilbert) frame remains a frame via an isomorphism (see, e.g. [5, 18]). Let us show that the same holds in the more general case of Banach frames.

Lemma 4.1. *Let $(\Theta, |||\cdot|||)$ be a BK-space, X and Y be reflexive Banach spaces and let the operator G be an isomorphism of X^* onto Y^*. Let $\{g_i\} \in (X^*)^{\mathbb{N}}$.*

(i) *If $\{g_i\}$ is a Θ-Bessel sequence (resp. Θ-frame) for X, then $\{Gg_i\}$ is a Θ-Bessel sequence (resp. Θ-frame) for Y,*

(ii) *If $\{g_i\}$ is a Banach frame for X w.r.t. Θ, then $\{Gg_i\}$ is a Banach frame for Y w.r.t. Θ.*

P r o o f. (i) Let $\{g_i\}$ be a Θ-Bessel sequence for X. By the reflexivity of X, for all $y \in Y$ we have $\{Gg_i(y)\} = \{g_i(G^*y)\} \in \Theta$ and

$$|||\{Gg_i\,(y)\}||| = |||\{g_i(G^*y)\}||| \leq B\|G^*y\|_X \leq B\|G\|\|y\|_Y.$$

Let now $\{g_i\}$ be a Θ-frame for X. It remains to prove that $\{Gg_i\}$ satisfies the lower Θ-frame inequality. For every $y \in Y$ the following inequalities hold

$$|||\{Gg_i\,(y)\}||| = |||\{g_i(G^*y)\}||| \geq A\|G^*y\|_X \geq A\frac{1}{\|(G^*)^{-1}\|}\|y\|_Y.$$

(ii) We need only to prove that the operator $\tilde{U} : Y \to \Theta$, $\tilde{U}y = \{Gg_i(y)\}$, has a bounded left inverse. Since $\{g_i\}$ is a Banach frame for X w.r.t. Θ, the operator $U : X \to \Theta$, $Uf = \{g_i(f)\}$, has a bounded left inverse $S : \Theta \to X$. Since $\tilde{U} = UG^*$, we have $(G^*)^{-1}S\tilde{U} = Id|_Y$ and hence the operator $(G^*)^{-1}S : \Theta \to Y$ is a bounded left inverse of $\tilde{U}$. □

Theorem 4.2. *Let $\{X_s\}_{s\in\mathbb{N}_0}$ be a family of reflexive Banach spaces, satisfying (6)-(8), and assume that for every $s \in \mathbb{N}_0$ there exists an isomorphism G_s of X_0 onto X_s. Let Θ be a CB-space such that Θ^* is also a CB-space (in particular, reflexive CB-space). Assume that there exists $s_0 \in \mathbb{N}_0$ and a sequence $\{g_i\} \in (X^*_{s_0})^{\mathbb{N}}$, which is a Banach frame for X_{s_0} w.r.t. Θ. Then* *there exists a Banach frame* $\{f_i\} \in (X_{s_0})^{\mathbb{N}}$ *for* $X_{s_0}^*$ *w.r.t.* Θ^* *such that for every* $s \in \mathbb{N}_0$ *the following holds:*

$$f = \sum {G_s^*}^{-1} G_{s_0}^* g_i\,(f)\; G_s G_{s_0}^{-1} f_i, \quad \textit{for all } f \in X_s, \tag{9}$$

$$g = \sum g(G_s G_{s_0}^{-1} f_i)\; {G_s^*}^{-1} G_{s_0}^* g_i, \quad \textit{for all } g \in X_s^*, \tag{10}$$

$$\{{G_s^*}^{-1} G_{s_0}^* g_i\} \textit{ is a Banach frame for } X_s \textit{ w.r.t. } \Theta, \tag{11}$$

$$\{G_s G_{s_0}^{-1} f_i\} \textit{ is a Banach frame for } X_s^* \textit{ w.r.t. } \Theta^*. \tag{12}$$

P r o o f. By Proposition 3.1 and Remark 3.2, there exists a Banach frame $\{f_i\}$ for $X_{s_0}^*$ w.r.t. Θ^* such that

$$x = \sum g_i(x) f_i, \text{ for all } x \in X_{s_0}, \tag{13}$$

$$y = \sum y(f_i) g_i, \text{ for all } y \in X_{s_0}^*. \tag{14}$$

For any given $s \in \mathbb{N}_0$, the operator $G_s G_{s_0}^{-1}$ is an isomorphism of X_{s_0} onto X_s and ${G_s^*}^{-1} G_{s_0}^*$ is an isomorphism of $X_{s_0}^*$ onto X_s^*. Let $f \in X_s$ and thus $f = G_s G_{s_0}^{-1} x_f$ for some $x_f \in X_{s_0}$. By (13) we have

$$G_s G_{s_0}^{-1} x_f = \sum g_i(x_f) G_s G_{s_0}^{-1} f_i = \sum g_i(G_{s_0} G_s^{-1} f) G_s G_{s_0}^{-1} f_i$$

and finally

$$f = \sum \Big((G_{s_0} G_s^{-1})^* g_i\Big)\,(f)\; G_s G_{s_0}^{-1} f_i.$$

Let now $g \in X_s^*$ and thus $g = {G_s^*}^{-1} G_{s_0}^* y_g$ for some $y_g \in X_{s_0}^*$. By (14) we have

$$ {G_s^*}^{-1} G_{s_0}^* y_g = \sum y_g(f_i) {G_s^*}^{-1} G_{s_0}^* g_i = \sum {G_{s_0}^*}^{-1} G_s^* g(f_i) {G_s^*}^{-1} G_{s_0}^* g_i$$

and therefore

$$g = \sum g(G_s G_{s_0}^{-1} f_i)\, {G_s^*}^{-1} G_{s_0}^* g_i.$$

By Lemma 4.1 we conclude that $\{{G_s^*}^{-1} G_{s_0}^* g_i\}$ is a Banach frame for X_s w.r.t. Θ and $\{G_s G_{s_0}^{-1} f_i\}$ is a Banach frame for X_s^* w.r.t. Θ^*. □

In the same way as above, if there exists $s_0 \in \mathbb{N}_0$ and a sequence $\{f_i\} \in (X_{s_0})^{\mathbb{N}}$, which is a Banach frame for $X_{s_0}^*$ w.r.t. Θ, then there exists a Banach frame $\{g_i\} \in (X^*_{s_0})^{\mathbb{N}}$ for X_{s_0} w.r.t. Θ^* such that (9)-(12) hold for every $s \in \mathbb{N}_0$.

Remark 4.3. *Theorem 4.2 deals with families of reflexive Banach spaces together with their corresponding dual spaces,*

$$\{\mathbf{0}\} \neq \cap_{s\in\mathbb{N}} X_s \subset \ldots \subset X_2 \subset X_1 \subset X_0 \subset X_0^* \subset X_1^* \subset X_2^* \subset \ldots \cup_{s\in\mathbb{N}} X_s^*.$$

Thus, (10) gives expansions of distributions $g \in X_s^* \subset \sup_{s\in\mathbb{N}} X_s^*$, via distributions ${G_s^*}^{-1} G^*_{s_0} g_i$, $\{g_i\} \in (X^*_{s_0})^{\mathbb{N}}$, for some $s_0 \in \mathbb{N}$. This is different from the usual structural theorems for distributions where the synthesis is done by elements from $\cap_{s\in\mathbb{N}} X_s$. See, for example [30], for the expansion of tempered distributions by orthogonal polynomials. We also mention [9, 10, 24] for the representation of tempered (ultra)-distributions by Gabor frames and Wilson bases. More details on this remark will be given in a forthcoming paper.

4.1. *Examples*

As usual, $\mathcal{S}(\mathbb{R}^n)$ denotes the Schwartz class of rapidly decreasing functions whose dual is the space of tempered distributions, $\mathcal{S}'(\mathbb{R}^n)$. By $\mathcal{F}f = \hat{f}$ and $\mathcal{F}^{-1}f$ we denote the Fourier and the inverse Fourier transform. We use the notation $\langle\cdot\rangle = (1 + |\cdot|^2)^{1/2}$ and $\langle D\rangle = (1 - \triangle)^{1/2}$. The Bessel potential space of order $s \in \mathbb{R}$, H^2_s, is the Hilbert space of all tempered distributions f such that $f = \mathcal{F}g$, for some g which satisfies

$$\int |g(\xi)|^2 \langle\xi\rangle^{2s} d\xi < \infty.$$

We have $\mathcal{S} \subset H^2_s \subset H^2_{s'} \subset \mathcal{S}'$ for $s' < s$. The Bessel potential spaces satisfy (6)–(8), and therefore we may apply Theorem 4.2. We now generalize this example to families of Banach spaces.

4.1.1. *Modulation spaces*

Modulation spaces are recognized as the most important spaces of functions and distributions in time-frequency analysis [5, 8, 9, 11, 16, 27]. Frame decompositions of modulation spaces are based on Gabor (Weyl-Heisenberg) frames, [16]. We refer to [8, 16] for general theory of modulation spaces and list only those features which are necessary for the present work.

Let g be any nonzero function from the Schwartz class $\mathcal{S}(\mathbb{R}^d)$. The short-time Fourier transform of $f \in \mathcal{S}'(\mathbb{R}^d)$ with respect to the window g is given by

$$V_g f(x,\omega) = \int_{\mathbb{R}^d} f(t)\, \overline{g(t-x)}\, e^{-2\pi i\omega t}\, dt. \tag{15}$$

Modulation space $M^{p,q}_{s,t} = M^{p,q}_{s,t}(\mathbb{R}^d)$, $1 \le p, q \le \infty$, $s, t \in \mathbb{R}$, is a Banach space of $f \in \mathcal{S}'(\mathbb{R}^n)$ such that $\|f\|_{M^{p,q}_{s,t}} < \infty$, with the norm $\|\cdot\|_{M^{p,q}_{s,t}}$ defined by

$$\left(\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}|V_g f(x,\omega)\langle x\rangle^t|^p dx\right)^{q/p}\langle\omega\rangle^{qs}d\omega\right)^{1/q},$$

with obvious modifications when $p = \infty$ and/or $q = \infty$. We have $(M^{p,q}_{s,t})^* = M^{p',q'}_{1/s,1/t}$, $1 \le p, q \le \infty$, $s, t \in \mathbb{R}$, $1/p + 1/p' = 1$, $1/q + 1/q' = 1$.

Let there be given monotonically increasing sequences of positive numbers $\{s_n\}$ and $\{t_n\}$, $\lim_{n\to\infty} s_n = \infty$ and $\lim_{n\to\infty} t_n = \infty$, and let there be given modulation space $M^{p,q}_{s_0,t_0}$. Then we have $M^{p,q}_{s_n,t_m} \hookrightarrow M^{p,q}_{s_{n-1},t_m}$ and $M^{p,q}_{s_n,t_m} \hookrightarrow M^{p,q}_{s_n,t_{m-1}}$. Moreover, for any given $s, t, u, v \in \mathbb{R}$ the map $f \mapsto \langle\cdot\rangle^v f$ is an isomorphism between $M^{p,q}_{s,t+v}$ and $M^{p,q}_{s,t}$ and the map $f \mapsto \langle D\rangle^u f$ is an isomorphism between $M^{p,q}_{s+u,t}$ and $M^{p,q}_{s,t}$, see [8, 27].

Therefore, it is easy to observe a family of modulation spaces such that (6) - (8) holds. If $\{s_n\}$ is an increasing sequence such that $\lim_{n\to\infty} s_n = \infty$, then $\mathcal{S} = \cap_{n\to\infty} M^{p,q}_{s_n,s_n}$ and, consequently, $\mathcal{S}' = \cup_{n\to\infty} M^{p',q'}_{-s_n,-s_n}$. In particular, for $p = q = 2$, we obtain $\mathcal{S}$ as the projective limit of the sequence of Hilbert spaces $M^{2,2}_{s_n,s_n}$, known also as Shubin spaces.

We have $M^{2,2}_{s,0} = H^2_s$, and obtain the space $\mathcal{D}_{L^2}$ (see [29]) as projective limit of isomorphic (Hilbert) modulation spaces $M^{2,2}_{s,0}$, $s \in \mathbb{R}$.

Both $\mathcal{S}$ and $\mathcal{D}_{L^2}$ are Fréchet spaces which have properties (DN) and (Ω). However, $\mathcal{S}$ is nuclear while $\mathcal{D}_{L^2}$ is not a Montel space, and therefore it is not nuclear. Therefore $\mathcal{S}$ is isomorphic to (a complemented subspace of) the Köthe sequence space **s**, see Section 2.

Theorem 4.2 can be easily applied to modulation spaces. Actually, by [15, Chapter 12], there exist Gabor frames which are frames for all modulation spaces, see also [28]. The elements of the Gabor frames belong to $M^{1,1}_{s,t}$, for some $s, t \in \mathbb{R}$. On the other side, in Theorem 4.2, different modulation spaces are characterized by different frames.

In the above examples, one can choose Hilbert modulation spaces to obtain the Fréchet spaces $\mathcal{S}$ and $\mathcal{D}_{L^2}$. Let us now give an example with isomorphic Banach spaces which are not Hilbert spaces.

4.1.2. *Banach spaces which are not Hilbert spaces*

Let $p \in (1, \infty)$, $X_0 := \mathcal{F}^{-1}(L^p)$,

$$X_s := \{f \in \mathcal{S}' \mid f^{(\alpha)} \in \mathcal{F}^{-1}(L^p),\ \ \forall\alpha \in \mathbb{N}^d_0, |\alpha| \le s\},\ \ s \in \mathbb{N},$$

with the norm $\|f\|_{X_s} = (\int (1+|\xi|^2)^{ps/2} |\hat{f}(\xi)|^p d\xi)^{1/p}$, $s \in \mathbb{N}_0$. The derivatives are understood in distributional sense. If we restrict to $1 < p \leq 2$, then $X_0 \subset L^q$, where $q = \frac{p}{p-1}$. Similar constructions related to Sobolev spaces can be found in [1, 22].

The operator G_s given by

$$G_s f = \mathcal{F}^{-1}((1+|\xi|^2)^{-s/2} \hat{f}(\xi)),\ f \in X_0,$$

is an isomorphism of X_0 onto X_s. If $p = 2$, we have $X_s = H^2_s$, as above. Since

$$X_s = \{f \in \mathcal{S}' \mid ((1+|\xi|^2)^{s/2} \hat{f} \in L^p\} = \{f \in \mathcal{S}' \mid \hat{f} \in L^p_{(1+|\xi|^2)^{s/2}}\}$$

its dual is

$$X_s^* = \{g \in \mathcal{S}' \mid ((1+|\xi|^2)^{-s/2} \hat{g} \in L^q\} = \{g \in \mathcal{S}' \mid \hat{g} \in L^q_{(1+|\xi|^2)^{-s/2}}\},$$

where $q = \frac{p}{p-1}$, and the dual pairing is given by $\langle g, f\rangle = \int \hat{g}(\xi)\hat{f}(\xi) d\xi$. Therefore X_s, $s \in \mathbb{N}$, is a reflexive Banach space.

Let $\{g_i\} \subset L^q_{(1+|\xi|^2)^{-s/2}}$ be a $p-$frame for $L^p_{(1+|\xi|^2)^{s/2}}$. Then $\{\mathcal{F}^{-1} g_i\}$ is a frame for X_s with respect to ℓ^p and we can again apply Theorem 4.2.

5. *Fréchet frames*

We begin with the definition of a pre-frame for a Fréchet space.

Definition 5.1. *Let $\{X_s, \|\cdot\|_s\}_{s\in\mathbb{N}_0}$ be a family of Banach spaces, satisfying (6)-(8) and let $\{\Theta_s, |||\cdot|||_s\}_{s\in\mathbb{N}_0}$ be a family of BK-spaces, satisfying (6)-(8). A sequence $\{g_i\} \in (X_F^*)^{\mathbb{N}}$ is called a pre-Fréchet frame (a pre-F-frame) for X_F with respect to Θ_F if for every $s \in \mathbb{N}_0$ there exist constants $0 < A_s \leq B_s < \infty$ such that*

$$\{g_i(f)\} \in \Theta_F, \tag{16}$$

$$A_s \|f\|_s \leq |||\{g_i(f)\}|||_s \leq B_s \|f\|_s, \tag{17}$$

for all $f \in X_F$. When (16) and the upper inequality in (17) hold for all $f \in X_F$, $\{g_i\}$ is called a Fréchet-Bessel sequence (an F-Bessel sequence) for X_F with respect to Θ_F.

If $X = X_F = X_s$ and $\Theta = \Theta_F = \Theta_s$, $s \in \mathbb{N}_0$, the above definition of a pre-F-frame gives a Θ-frame for X. Having in mind the definition of a Banach frame, we define a Fréchet frame as follows.

A Fréchet frame (F-frame) for a Fréchet space X_F with respect to a Fréchet sequence space Θ_F is a pre-F-frame $\{g_i\}$ for X_F with respect to Θ_F, for which there exists a continuous linear operator $S : \Theta_F \to X_F$ such that $S(\{g_i(f)\}) = f$ for all $f \in X_F$. In other words, a pre-F-frame is an F-frame when the operator $U : X_F \to \Theta_F$, $Uf = \{g_i(f)\}$, has a continuous left inverse $S : \Theta_F \to X_F$.

Let $\Theta_s = \Theta$, $\forall s \in \mathbb{N}_0$, be a CB-space and assume that $\{X_s\}_{s\in\mathbb{N}_0}$ is a family of Banach spaces, satisfying (6)-(8). If $\{g_i\} \in (X_F^*)^{\mathbb{N}}$ is a pre-F-frame for X_F with respect to Θ and if there is a continuous projection U from Θ onto $R(U) = \{\{g_i(f)\} : f \in X_F\}$, then X_F should be isomorphic to $R(U)$, which is a closed subspace of Θ, see Theorem 5.3 below. Therefore X_F should be (isomorphic to) a Banach space. This explains that in case when X_F is a Fréchet space which is not a Banach space a sequence of different sequence spaces Θ_s, $s \in \mathbb{N}_0$, should be used.

Let $\{g_i\}_{i\in I} \in (X_F^*)^{\mathbb{N}}$ be a pre-F-frame for X_F with respect to Θ_F. For any given $s \in \mathbb{N}_0$ and $i \in I$, the unique continuous extension of g_i on X_s will be denoted by g_i^s.

Lemma 5.2. *Let $\{X_s, \|\cdot\|_s\}_{s\in\mathbb{N}_0}$ be a family of Banach spaces, satisfying (6)–(8) and and let $\{\Theta_s, |||\cdot|||_s\}_{s\in\mathbb{N}_0}$ be a family of λ_s–BK-spaces, satisfying (6)–(8). If $\{g_i\}_{i\in I} \in (X_F^*)^{\mathbb{N}}$ is an F-Bessel sequence (resp. pre-F-frame) for X_F with respect to Θ_F with bounds B_s (resp. A_s, B_s), then for any given $s \in \mathbb{N}_0$ the family $\{g_i^s\}_{i\in I}$ is a Θ_s-Bessel sequence (resp. Θ_s-frame) for X_s with a bound $\lambda_s B_s$ (resp. $A_s, \lambda_s B_s$).*

P r o o f. The result follows from Theorem 3.3, because of the density of X_F in each X_s, $s \in \mathbb{N}_0$. □

Theorem 5.3. *Let $\{X_s, \|\cdot\|_s\}_{s\in\mathbb{N}_0}$ be a family of Banach spaces, satisfying (6)–(8) and let $\{\Theta_s, |||\cdot|||_s\}_{s\in\mathbb{N}_0}$ be a family of CB-spaces, satisfying (6)-(8) and we assume that Θ_s^* is a CB-space for every $s \in \mathbb{N}_0$. Let $\{g_i\}_{i\in I} \in (X_F^*)^{\mathbb{N}}$ be a pre-F-frame for X_F with respect to Θ_F. There exists a family $\{f_i\} \in (X_F)^{\mathbb{N}}$ such that*

(a) $f = \sum g_i(f) f_i$, $\forall f \in X_F$, *and* $g = \sum g(f_i) g_i$, $\forall g \in X_F^*$;

(b) $f = \sum g_i^s(f) f_i$, $\forall f \in X_s$, *and* $g = \sum g(f_i) g_i^s$, $\forall g \in X_s^*$, $\forall s \in \mathbb{N}_0$;

(c) *for every $s \in \mathbb{N}_0$, $\{f_i\}$ is a Θ_s^*-frame for X_s^*.*

if, and only if, there exists a continuous projection U from Θ_F onto its *subspace* $\{\{g_i(f)\} : f \in X_F\}$.

A family $\{f_i\} \in (X_F)^{\mathbb{N}}$, which satisfies the conditions (a)–(c) in the above theorem, is called a *dual pre-F-frame* of the given pre-F-frame $\{g_i\}$.

We refer to [23] for the proof of Theorem 5.3, the existence of frames for Fréchet spaces, and more details on frames for Fréchet spaces. In the case of a single Banach space, Theorem 5.3 follows from Proposition 3.1 and Remark 3.2.

Very general concept of "localization of frames" is proposed by Gröchenig, [17]. Families of Banach spaces are associated to a Riesz basis of a Hilbert space. Note that our approach is different since Theorem 5.3 does not refer to any Hilbert space.

We end with an example when the sequence spaces Θ_s, $s \in \mathbb{N}_0$, are Hilbert spaces. If $\{g_i\} \in (X_F^*)^{\mathbb{N}}$ is a pre-F-frame for X_F with respect to Θ_F, then, by Lemma 5.2, the family $\{g_i^s\}$ is a Θ_s-frame for X_s for any $s \in \mathbb{N}_0$. Therefore the space X_s is isomorphic to a closed subspace of Θ_s and thus X_s is a Hilbert space. Theorem 5.3 points toward a complementedness condition, which is necessary and sufficient for series expansions via given pre-F-frame and the corresponding dual pre-F-frame. In particular, when $\Theta_s = \lambda^{2,s}((i^s)_{i\in\mathbb{N}})$ (see Section 2, by Theorem 5.3 we have the following.

Corollary 5.4. *Let $\{X_s\}_{s\in\mathbb{N}}$ be a family of Hilbert spaces, satisfying (6)–(8). Let $\{g_i\} \in (X_F^*)^{\mathbb{N}}$ and let for every $s \in \mathbb{N}_0$ there exist constants $0 < A_s \le B_s < \infty$ such that*

$$A_s\|f\|_s \le \left(\sum_i |i^s g_i(f)|^2\right)^{1/2} \le B_s\|f\|_s, \quad \forall f \in X_F.$$

Then the following statements are equivalent:

(i) *There exists a family $\{f_i\} \in X_F^{\mathbb{N}}$ such that for every $s \in \mathbb{N}_0$ one has:*

$$\exists\, 0 < \tilde{A}_s \le \tilde{B}_s < \infty \;:\; \tilde{A}_s\|g\|_{X_s^*} \le \left(\sum |i^{-s} g(f_i)|^2\right)^{1/2} \le \tilde{B}_s\|g\|_{X_s^*},\ \forall g \in X_s^*;$$

$$f = \sum g_i(f) f_i,\ \forall f \in X_F; \quad g = \sum g(f_i) g_i,\ \forall g \in X_F^*;$$

$$f = \sum g_i^s(f) f_i,\ \forall f \in X_s; \quad g = \sum g(f_i) g_i^s,\ \forall g \in X_s^*.$$

(ii) *The set $\{\{g_i(f)\} : f \in X_F\}$ is nuclear and has the properties (DN) and (Ω).*

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