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MINIMAL ANTI-KÄHLER HOLOMORPHIC HYPERSURFACES

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A b s t r a c t. M. Matsumoto examined in [10] the intrinsic properties of minimal hypersurfaces in a flat space and showed that for many of them the second fundamental form can be expressed in terms of the curvature and Ricci tensors.

The aim of this paper is to generalize the investigation of Matsumoto to holomorphic hypersurfaces of an anti-Kähler manifold of constant totally real sectional curvatures.

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The aim of this paper is to generalize the investigation of Matsumoto to the holomorphic hypersurface of the anti-Kähler manifold of constant totally real sectional curvatures. This is done in the Section 2. The Section 3 is devoted to special properties of HC-flat minimal holomorphic hypersurface. In the Section 4 we examine holomorphically Einstein manifolds. Finally, in the Section 5 we give some remarks concerning complex hypersurfaces of the Kähler manifold.

1. *Anti-Kähler manifold and its holomorphic hypersurface*

By an anti-Kähler manifold we mean a triple $(\widetilde{M}, G, F)$, where $\widetilde{M}$ is a connected differentiable manifold of dimension $2m$, $F = (F_B^A)$ is a $(1,1)$ tensor field and $G = (G_{AB})$ is a pseudo-Riemannian metric on $\widetilde{M}$ satisfying

$$F^2 = -\mathrm{Id.}, \quad \mathrm{tr} F = 0, \quad F_A^C F_B^D G_{CD} = -G_{AB}, \quad \widetilde{\nabla} F = 0, \tag{1.1}$$

where $\widetilde{\nabla}$ is the Levi-Civita connection with respect to G and $A, B, C, D, \ldots \in \{1, 2, \ldots, 2m\}$.

The manifold $(\widetilde{M}, G, F)$ is orientable and evendimensional. The metric G is indefinite and the signature is (m, m).

The anti-Kähler manifolds are investigated by many authors (for exam. [1]-[9], [11]-[13], [15]).

Let $T_P(M)$ be the tangent vector space of $\widetilde{M}$ at the point $P \in \widetilde{M}$. We denote by $\widetilde{R}(X, Y, Z, W)$, $X, Y, Z, W \in T_P(\widetilde{M})$, the Riemannian curvature tensor of $\widetilde{M}$. Because of $\widetilde{\nabla} F = 0$, it satisfies the condition

$$F_A^P F_B^Q \widetilde{R}_{PQCD} = -\widetilde{R}_{ABCD}. \tag{1.2}$$

The anti-Kähler manifold is of constant totally real sectional curvatures if ([5],[6])

$$\begin{aligned}\widetilde{R}_{ABCD} &= k_1(G_{AD}G_{BC} - G_{AC}G_{BD} - F_A^P G_{PD} F_B^Q G_{QC} + F_A^P G_{PC} F_B^Q G_{QD}) \\ &\quad + k_2(G_{AD} F_B^P G_{PC} + G_{BC} F_A^P G_{PD} - G_{AC} F_B^P G_{PD} - G_{BD} F_A^P G_{PC}).\end{aligned} \tag{1.3}$$

If $m \geq 3$, both functions k_1 and k_2 are constants.

Now we consider a differentiable submanifold M of $\widetilde{M}$, dim $M = 2n$, $n = m - 1$. Let the equation

$$x^A = x^A(u^a)$$

be the local parametric expression of M in $(\widetilde{M}, G)$, where u^a are the local coordinates in M, $a, b, c, \ldots, i, j, k, \ldots \in \{1, 2, \ldots, 2n\}$. A submanifold M is said to be holomorphic hypersurface of $\widetilde{M}$ if the restriction of G on M has the maximal rank and $FT_p(M) = T_p(M)$, $p \in M$. We denote the restriction of G and F on M by g and f. Then it can be proved [9] that (M, g, f) is itself an anti-Kähler manifold, i.e.

$$f^2 = -\mathrm{Id.}, \quad \mathrm{tr}\ f = 0, \quad f_i^a f_j^b g_{ab} = -g_{ij}, \quad \nabla f = 0 \tag{1.4}$$

$$f_i^a f_j^b R_{abkl} = -R_{ijkl}, \tag{1.5}$$

where ∇ is the Levi-Civita connection and R_{ijkl} are the local components of the Riemannian curvature tensor with respect to g. It follows from (1.5) that the Ricci tensor ρ_{ij} satisfies

$$f_i^a f_j^b \rho_{ab} = -\rho_{ij}, \tag{1.6}$$

and therefore

$$f_i^a \rho_{aj} = f_j^a \rho_{ia}.$$

An anti-Kähler manifold (M, g, f) is holomorphically Einstein if its Ricci tensor has the form

$$\rho_{ij} = \frac{\kappa}{2n} g_{ij} - \frac{\overset{*}{\kappa}}{2n} f_{ij} \tag{1.7}$$

(see, for ex. [15]), where κ and $\overset{*}{\kappa}$ are the first and the second scalar curvatures, and $f_{ij} = f_i^a g_{aj} = f_{ji}$.

Because F leaves invariant the tangent space of M, it leaves invariant the normal space, too. There exist locally vector fields, $N_{1|}$ and $N_{2|}$ normal to M such that [9]

$$G_{AB} N_{1|}^A N_{1|}^B = -G_{AB} N_{2|}^A N_{2|}^B = 1, \quad G_{AB} N_{1|}^A N_{2|}^B = 0,$$
$$F_B^A N_{1|}^B = -N_{2|}^A, \quad F_B^A N_{2|}^B = N_{1|}^A.$$

Let h_{ij} be the components of the second fundamental form corresponding to $N_{1|}$. Then $-f_i^a h_{aj}$ are those corresponding to $N_{2|}$, and

$$f_i^a f_j^b h_{ab} = -h_{ij}. \tag{1.8}$$

The relation (1.8) implies

$$f_j^a h_{ai} = f_i^a h_{aj} \quad \text{and} \quad f_a^i h_j^a = f_j^a h_a^i.$$

Now, using the induction, it is easy to see that

$$(h^r)_{ij} = (h^r)_{ji}, \quad f_i^a f_j^b (h^r)_{ab} = -(h^r)_{ij}, \tag{1.9}$$

where (h^r) is the fundamental form of order r and is defined as follows [14]

$$(h^r)_{ij} = (h^{r-1})_{ia}h^a_j, \quad (h^1)_{ij} = h_{ij}, \quad r = 2, 3, \dots \ .$$

Let v^i represent a principal direction of the holomorphic hypersurface M at $P \in M$ with respect to the normal $N_{1|}$, i.e., an eigenvector of the matrix (h_{ij}) so that

$$h_{ij}v^j = \lambda g_{ij}v^j \ , \tag{1.10}$$

where λ is the corresponding eigenvalue. Then

$$(h^r)_{ij}v^j = \lambda^r g_{ij}v^j \ ,$$

so that v^i is also an eigenvector of (h^r), but the corresponding eigenvalue is λ^r. We associate to (1.10) the equation

$$\det\,(h_{ij} - \lambda g_{ij}) = 0 \ ,$$

and denote its $2n$ roots by $\lambda_1, \lambda_2, \dots, \lambda_{2n}$.

On the other hand, if, at a fixed point $P \in M$, we choose the parameters u^i such that the tangents to the curves $u^i =$ const. at P coincide with the principal directions of M at P, the components $(h^r)_{ij}$ are given by

$$(h^r)_{ij} = \begin{vmatrix} \lambda_1^r & 0 & \dots & 0 \\ 0 & \lambda_2^r & \dots & 0 \\ & & \ddots & \\ 0 & 0 & \dots & \lambda_{2n}^r \end{vmatrix} . \tag{1.11}$$

We denote by H_p the p-th elementary symmetric function of $\lambda_1, \dots, \lambda_{2n}$, i.e.,

$$\begin{aligned} H_1 \ &= \lambda_1 + \lambda_2 + \dots + \lambda_{2n} \\ H_2 \ &= \lambda_1\lambda_2 + \lambda_1\lambda_3 + \dots + \lambda_{2n-1}\lambda_{2n} \\ &\dots \\ H_{2n} &= \lambda_1\lambda_2 \cdots \lambda_{2n}, \end{aligned}$$

and put

$$P_p = \sum_{a=1}^{2n} (\lambda_a)^p.$$

According to the theory of the symmetric polynomials, by means of the Newton formula, we have

$$P_p + \sum_{q=1}^{p-1} (-1)^q H_q P_{p-q} + (-1)^p p H_p = 0,$$

i.e.,

$$\begin{aligned} &P_1 - H_1 = 0 \\ &P_3 - H_1 P_2 + H_2 P_1 - 3H_3 = 0, \\ &P_5 - H_1 P_4 + H_2 P_3 - H_3 P_2 + H_4 P_1 - 5H_5 = 0, \end{aligned}$$

and so on. This means that

If $P_1 = 0$, then $H_1 = 0$,

if $P_1 = P_3 = 0$, then $H_3 = 0$,

....

if $P_1 = P_3 = \cdots = P_{2p+1} = 0$, then $H_{2p+1} = 0$.

But, in view of (1.11), $P_r = \mathrm{tr}\ (h^r)$. On the other hand $H_r = 0$ for $r > 2n$. Thus

$$\text{if} \quad \mathrm{tr}\ (h^{2p+1}) \neq 0, \quad \text{then} \quad 2p+1 < 2n. \tag{1.12}$$

In view of (1.10), we have

$$f_i^a h_{aj} v^j = \lambda f_{ij} v^j = \lambda f_j^a v^j g_{ai},$$

and taking into account that $-f_i^a h_{aj}$ is the second fundamental form with respect to the normal $N_{2|}$, we see that if v^i represents a principal direction of M with respect to $N_{1|}$ then $f_a^i v^a$ represents the principal direction with respect to $N_{2|}$, and the corresponding eigenvalue is $-\lambda$. Putting

$$(fh^{2p+1})_{ij} = f_i^a (h^{2p+1})_{aj},$$

we conclude, in the similar way as above, that

$$\text{if} \quad \mathrm{tr}\ (fh^{2p+1}) \neq 0, \quad \text{then} \quad 2p+1 < 2n. \tag{1.13}$$

2. *Minimal holomorphic hypersurface of an anti-Kähler manifold of constant totally real sectional curvatures*

The Gauss equation for the holomorphic hypersurface (M, g, f) is

$$\widetilde{R}_{ABCD}\frac{\partial x^A}{\partial u^i}\frac{\partial x^B}{\partial u^j}\frac{\partial x^C}{\partial u^k}\frac{\partial x^D}{\partial u^l} =$$

$$R_{ijkl} - (h_{il}h_{jk} - h_{ik}h_{jl}) + (f_i^a h_{al} f_j^b h_{bk} - f_i^a h_{ak} f_j^b h_{bl}).$$

Let us suppose that the ambient manifold $(\widetilde{M}, G, F)$ is a manifold of constant totally real sectional curvatures. Then, substituting (1.3) into above Gauss equation, we get

$$R_{ijkl} = k_1 G_{ijkl} + k_2 f_i^a G_{ajkl} + h_{il}h_{jk} - h_{ik}h_{jl} - f_i^a h_{al} f_j^b h_{bk} + f_i^a h_{ak} f_j^b h_{bl},$$

where

$$G_{ijkl} = g_{il}g_{jk} - g_{ik}g_{jl} - f_{il}f_{jk} + f_{ik}f_{jl}.$$

If we put

$$T_{ijkl} = R_{ijkl} - k_1 G_{ijkl} - k_2 f_i^a G_{ajkl}, \tag{2.1}$$

the Gauss equation can be written in the form

$$T_{ijkl} = h_{il}h_{jk} - h_{ik}h_{jl} - f_i^a h_{al} f_j^b h_{bk} + f_i^a h_{ak} f_j^b h_{bl}. \tag{2.2}$$

Then, for

$$\tau_{ij} = T_{iabj}g^{ab},$$

from (2.1), we have

$$\tau_{il} = \rho_{il} - 2(n-1)(k_1 g_{il} + k_2 f_{il}), \tag{2.3}$$

while from (2.2), it follows

$$\tau_{il} = \operatorname{tr} h\, h_{il} - \operatorname{tr}(fh)(fh)_{il} - 2(h^2)_{il}. \tag{2.4}$$

We note that the tensor T has all algebraic properties as the curvature tensor of the anti-Kähler manifold. In particular

$$f_i^a f_j^b T_{abkl} = -T_{ijkl},$$

and therefore

$$f_i^a T_{ajkl} = f_j^b T_{ibkl}. \tag{2.5}$$

Now we suppose that M is the minimal holomorphic hypersurface, i.e., we suppose

$$\text{tr}\ h = \text{tr}\ (fh) = 0. \tag{2.6}$$

Then (2.4) reduces to

$$\tau_{ij} = -2(h^2)_{ij}$$

because of which we have

$$(\tau^p)_{ij} = (-2)^p (h^{2p})_{ij}, \tag{2.7}$$

where

$$(\tau^p)_{ij} = (\tau^{p-1})_{ia}\tau^a_j, \quad p = 1, 2, \ldots, \qquad (\tau^0)_{ij} = g_{ij}.$$

Definition. *Let (M, g, f) be a holomorphic hypersurface of an anti-Kähler manifold of constant totally real sectional curvatures, satisfying (2.6). (M, g, f) is said to be* **the minimal holomorphic hypersurface of type** *p if*

$$\text{tr}(h^{2p+1}) \neq 0, \quad \text{tr}(fh^{2p+1}) \neq 0, \tag{2.8}$$

while

$$\text{tr}(h^{2q+1}) = 0, \quad \text{tr}(fh^{2q+1}) = 0, \tag{2.9}$$

for all $q < p$, $p = 1, 2, \ldots$.

According to (1.12) and (1.13), (M, g, f) can be of type $1, 2, \ldots, p$ such that $2p + 1 < 2n$.

For (M, g, f) of type p, we can determine the second fundamental forms h_{ij} and $-(fh)_{ij}$. To do this, we use (2.2), (2.5) and (2.7), to get

$$\begin{aligned} T_{rsia}(\tau^p)^a{}_j + T_{rsja}(\tau^p)^a{}_i = \\ &= (-2)^p \left[h_{ra}h_{si} - h_{ri}h_{sa} - f^c_r h_{ca} f^d_s h_{di} + f^c_r h_{ci} f^d_s h_{da}\right] (h^{2p})^a{}_j \\ &+ (-2)^p \left[h_{ra}h_{sj} - h_{rj}h_{sa} - f^c_r h_{ca} f^d_s h_{dj} + f^c_r h_{cj} f^d_s h_{da}\right] (h^{2p})^a{}_i, \end{aligned}$$

from which, contracting with g^{sj} and using (1.9) and (2.6), we find

$$\text{tr}\ (h^{2p+1})h_{ri} - \text{tr}\ (fh^{2p+1})(fh)_{ri} = \frac{1}{(-2)^p}\left[T_{rabi}(\tau^p)^{ab} - (\tau^{p+1})_{ri}\right]. \tag{2.10}$$

This relation, together with

$$\text{tr}\ (fh^{2p+1})h_{ri} + \text{tr}\ (h^{2p+1})(fh)_{ri} = \frac{1}{(-2)^p} f^t_r \left[T_{tabi}(\tau^p)^{ab} - (\tau^{p+1})_{ti}\right], \tag{2.11}$$

implies

$$\left\{\left[\mathrm{tr}\ (h^{2p+1})\right]^2+\left[\mathrm{tr}\ (fh^{2p+1})\right]^2\right\}h_{ij}=\frac{1}{(-2)^p}\mathrm{tr}\ (h^{2p+1})\left[T_{iabj}(\tau^p)^{ab}-(\tau^{p+1})_{ij}\right]$$

$$+\frac{1}{(-2)^p}\mathrm{tr}\ (fh^{2p+1})f_i^t\left[T_{tabj}(\tau^p)^{ab}-(\tau^{p+1})_{tj}\right].$$

Therefore

$$\begin{aligned}h_{ij}=\frac{1}{\theta_p}&\left\{\mathrm{tr}\ (h^{2p+1})\left[T_{iabj}(\tau^p)^{ab}-(\tau^{p+1})_{ij}\right]\right.\\&\left.+\mathrm{tr}\ (fh^{2p+1})f_i^t\left[T_{tabj}(\tau^p)^{ab}-(\tau^{p+1})_{tj}\right]\right\},\end{aligned}\tag{2.12}$$

where

$$\theta_p=(-2)^p\left[(\mathrm{tr}\ (h^{2p+1}))^2+(\mathrm{tr}\ (fh^{2p+1}))^2\right].$$

For $q<p$, we have the equations similar to (2.11) and (2.12), but, in view of (2.9), they now yield

$$T_{iabj}(\tau^q)^{ab}-(\tau^{q+1})_{ij}=0\ ,\tag{2.13}$$

for all $q<p$.

Conversely, if (2.13) holds, then the corresponding equations (2.10) and (2.11) imply

$$\begin{aligned}&\mathrm{tr}\,(h^{2q+1})h_{ij}-\mathrm{tr}\,(fh^{2q+1})f_i^a h_{aj}=0,\\&\mathrm{tr}\,(fh^{2q+1})h_{ij}+\mathrm{tr}\,(h^{2q+1})f_i^a h_{aj}=0,\end{aligned}$$

from which these follows

$$(\mathrm{tr}\ (h^{2q+1}))^2+(\mathrm{tr}\ (fh^{2q+1}))^2=0,$$

and therefore

$$\mathrm{tr}\,(h^{2q+1})=\mathrm{tr}\,(fh^{2q+1})=0\quad\text{for all}\quad q<p.$$

Thus, for the minimal holomorphic hypersurface of type p, the conditions (2.9) and (2.13) are equivalent. This means that (M,g,f) is minimal of type p if and only if (2.13) holds for all $q<p$, and

$$T_{iabj}(\tau^p)^{ab}-(\tau^{p+1})_{ij}\neq 0.\tag{2.14}$$

If the ambient manifold $(\widetilde{M}, G, F)$ is flat, i.e. if $k_1 = k_2 = 0$, (2.1) and (2.3) reduce to $T_{ijkl} = R_{ijkl}$ and $\tau_{ij} = \rho_{ij}$, while (2.14) and (2.13) became

$$R_{iabj}(\rho^p)^{ab} - (\rho^{p+1})_{ij} \neq 0$$
$$R_{iabj}(\rho^q)^{ab} - (\rho^{q+1})_{ij} = 0 \qquad \text{for all } q < p,$$

respectively. But these relations are the intrinsic conditions of (M, g, f). Thus, if (M, g, f) is holomorphic hypersurface of a flat anti-Kähler manifold, the property of (M, g, f) to be minimal of type p is its intrinsic characteristic. Also, $\mathrm{tr}\,(h^{2p+1})$ and $\mathrm{tr}\,(fh^{2p+1})$ are the object of the inner geometry of (M, g, f). To prove this, we note that now, (2.7) becomes

$$(\rho^p)_{ij} = (-2)^p (h^{2p})_{ij}, \tag{2.15}$$

because of which

$$(-2)^p (h^{2p+1})_{ij} = (-2)^p (h^{2p})_{ia} h^a_j = (\rho^p)_{ia} h^a_j,$$
$$(-2)^p (fh^{2p+1})_{ij} = (\rho^p)_{ia} (fh)^a_j.$$

Therefore

$$\begin{aligned} (-2)^p \mathrm{tr}\,(h^{2p+1}) &= h_{ij} (\rho^p)^{ij} \\ (-2)^p \mathrm{tr}\,(fh^{2p+1}) &= (fh)_{ij} (\rho^p)^{ij} \end{aligned} \tag{2.16}$$

On the other hand, (2.10) and (2.11) reduce to

$$\mathrm{tr}\,(h^{2p+1}) h_{ij} - \mathrm{tr}\,(fh^{2p+1})(fh)_{ij} = \frac{1}{(-2)^p} \left[R_{iabj}(\rho^p)^{ab} - (\rho^{p+1})_{ij} \right],$$
$$\mathrm{tr}\,(fh^{2p+1}) h_{ij} + \mathrm{tr}\,(h^{2p+1})(fh)_{ij} = \frac{1}{(-2)^p} f^t_i \left[R_{tabj}(\rho^p)^{ab} - (\rho^{p+1})_{tj} \right],$$

from which, transverting with $(\rho^p)^{ij}$ and using (2.6) we get

$$\left[\mathrm{tr}\,(h^{2p+1})\right]^2 - \left[\mathrm{tr}\,(fh^{2p+1})\right]^2 = \gamma,$$
$$\mathrm{tr}\,(h^{2p+1})\ \mathrm{tr}\,(fh^{2p+1}) = \delta,$$

where γ and δ are some functions of the inner geometry of (M, g, f). This system of equations shows that $\mathrm{tr}\,(h^{2p+1})$ and $\mathrm{tr}\,(fh^{2p+1})$, and therefore θ_p are the intrinsic properties of (M, g, f).

As for (2.12), it reduces to

$$\begin{aligned} h_{ij} = \ & \tfrac{1}{\theta_p} \Big\{ \mathrm{tr}\,(h^{2p+1}) \left[R_{iabj}(\rho^p)^{ab} - (\rho^{p+1})_{ij} \right] \\ & + \mathrm{tr}\,(fh^{2p+1}) f^t_i \left[R_{tabj}(\rho^p)^{ab} - (\rho^{p+1})_{tj} \right] \Big\}. \end{aligned} \tag{2.17}$$

From the above exposed, the following theorem holds.

Theorem 1. *Let (M, g, f) be a holomorphic hypersurface of an anti-Kähler manifold of constant totally real sectional curvatures. If it is the minimal hypersurface of type p, then:*

1) *Type p can be $p = 1, 2, \dots,$ such that $2p + 1 < 2n$, $2n = \dim M$.*

2) *The conditions (2.8) and (2.14) are equivalent, as well as the conditions (2.9) and (2.13).*

3) *The second fundamental form is given by (2.12).*

In particular, if (M, g, f) is a holomorphic hypersurface of the flat anti-Kähler manifold, then

4) *The property of (M, g, f) to be minimal of type p is its intrinsic characteristic.*

5) *The second fundamental form is given by (2.17); it is the intrinsic characteristic of (M, g, f), too.*

6) *Any fundamental form of even order satisfies (2.15).*

3. *Minimal HC-flat holomorphic hypersurface*

We consider in [4] HC-flat (holomorphically conformally flat) hypersurface, (M, g, f), $n > 3$, dim $M = 2n$, of an anti-Kähler manifold of constant totally real sectional curvatures and proved that for such (M, g, f) the following hold:

1. (M, g, f) is quasi-umbilical, i.e.

$$h_{ij} = \varphi g_{ij} + \psi f_{ij} + \tau V_{ij} + \sigma \bar{V}_{ij}, \tag{3.1}$$

where φ, ψ, τ and σ are some scalar functions,

$$V_{ij} = V_i V_j - \bar{V}_i \bar{V}_j, \quad \bar{V}_{ij} = f_i^a V_{aj} = \bar{V}_i V_j + V_i \bar{V}_j,$$

V is a vector field and $V = fV$;

2. (M, g, f) is of quasi-constant totally real sectional curvatures, i.e., its curvature tensor has the form

$$\begin{aligned} R_{ijlm} &= \lambda G_{ijlm} + \mu f_i^a G_{ajlm} \\ &+\xi(g_{im}V_{jl} + g_{jl}V_{im} - g_{il}V_{jm} - g_{jm}V_{il} - f_{im}\bar{V}_{jl} - f_{jl}\bar{V}_{im} + f_{il}\bar{V}_{jm} + f_{jm}\bar{V}_{il}) \\ &+\theta(g_{im}\bar{V}_{jl} + g_{jl}\bar{V}_{im} - g_{il}\bar{V}_{jm} - g_{jm}\bar{V}_{il} + f_{im}V_{jl} + f_{jl}V_{im} - f_{il}V_{jm} - f_{jm}V_{il}), \end{aligned} \tag{3.2}$$

where

$$\begin{aligned} &\lambda = \frac{\widetilde{\kappa}}{4n(n+1)} + \varphi^2 - \psi^2, \qquad \mu = -\frac{\widetilde{\overset{*}{\kappa}}}{4n(n+1)} + 2\varphi\psi, \\ &\xi = \tau\varphi - \sigma\psi, \qquad \theta = \sigma\varphi + \tau\psi \end{aligned} \tag{3.3}$$

while $\widetilde{\kappa}$ and $\widetilde{\overset{*}{\kappa}}$ are the scalar curvatures of the ambient manifold.

We get from (3.1)

$$\begin{aligned} \mathrm{tr}\, h &= 2n\varphi + 2\tau V_a V^a + 2\sigma V_a \bar{V}^a \ , \\ \mathrm{tr}\,(fh) &= -2n\psi + 2\tau V_a \bar{V}^a - 2\sigma V_a V^a \ . \end{aligned}$$

Thus, if (M, g, f) is minimal, we have

$$\begin{aligned} n\varphi + \tau V_a V^a + \sigma V_a \bar{V}^a &= 0 \ , \\ n\psi - \tau V_a \bar{V}^a + \sigma V_a V^a &= 0 \ . \end{aligned} \tag{3.4}$$

The relation (3.1) implies

$$\begin{aligned} (h^2)_{ij} = \ &(\varphi^2 - \psi^2) g_{ij} + 2\varphi\psi f_{ij} \\ &+ V_{ij}\left[2(\tau\varphi - \sigma\psi) + (\tau^2 - \sigma^2) V_a V^a + 2\tau\sigma V_a \bar{V}^a\right] \\ &+ \bar{V}_{ij}\left[2(\sigma\varphi + \tau\psi) - (\tau^2 - \sigma^2) V_a \bar{V}^a + 2\tau\sigma V_a V^a\right] \ . \end{aligned} \tag{3.5}$$

On the other hand, (3.4) yields

$$\begin{aligned} n(\tau\varphi - \sigma\psi) + (\tau^2 - \sigma^2) V_a V^a + 2\tau\sigma V_a \bar{V}^a &= 0, \\ n(\sigma\varphi + \tau\psi) - (\tau^2 - \sigma^2) V_a \bar{V}^a + 2\tau\sigma V_a V^a &= 0. \end{aligned} \tag{3.6}$$

Substituting this into (3.5), we get

$$\begin{aligned}(h^2)_{ij} = \;& (\varphi^2-\psi^2)g_{ij}+2\varphi\psi f_{ij}\\ &-(n-2)(\tau\varphi-\sigma\psi)V_{ij}-(n-2)(\sigma\varphi+\tau\psi)\bar{V}_{ij}.\end{aligned} \tag{3.7}$$

Next, we calculate $(h^3)_{ij}$ and, using (3.6), we find

$$\begin{aligned}(h^3)_{ij} = \;& \varphi(\varphi^2-3\psi^2)g_{ij}-\psi(\psi^2-3\varphi^2)f_{ij}\\ &+(n^2-3n+3)[\tau(\varphi^2-\psi^2)-2\sigma\varphi\psi]V_{ij}\\ &+(n^2-3n+3)[\sigma(\varphi^2-\psi^2)+2\tau\varphi\psi]\bar{V}_{ij}.\end{aligned} \tag{3.8}$$

This relation implies

$$\begin{aligned}(fh^3)_{ij} = \;& \psi(\psi^2-3\varphi^2)g_{ij}+\varphi(\varphi^2-3\psi^2)f_{ij}\\ &-(n^2-3n+3)\left[\sigma(\varphi^2-\psi^2)+2\tau\varphi\psi\right]V_{ij}\\ &+(n^2-3n+3)\left[\tau(\varphi^2-\psi^2)-2\sigma\varphi\psi\right]\bar{V}_{ij}.\end{aligned}$$

Now, we have

$$\begin{aligned}\operatorname{tr}(h^3) &= -2n(n-1)(n-2)\varphi(\varphi^2-3\psi^2),\\ \operatorname{tr}(fh^3) &= -2n(n-1)(n-2)\psi(\psi^2-3\varphi^2).\end{aligned} \tag{3.9}$$

If $\operatorname{tr}(h^3)\neq 0$ and $\operatorname{tr}(fh^3)\neq 0$, $(M,g,f)$is the minimal hypersurface of type 1.

Let us examine the case

$$\operatorname{tr}(h^3)=\operatorname{tr}(fh^3)=0.$$

Then, if $n>2$, (3.9) gives

$$\varphi(\varphi^2-3\psi^2)=0, \qquad \psi(\psi^2-3\varphi^2)=0,$$

which holds if and only if $\varphi=\psi=0$. But then (3.3) reduces to

$$\lambda=\frac{\widetilde{\kappa}}{4n(n+1)}, \quad \mu=-\frac{\widetilde{\overset{*}{\kappa}}}{4n(n+1)}, \quad \xi=0, \quad \theta=0,$$

while (3.2) becomes

$$R_{ijlm}=\frac{\widetilde{\kappa}}{4n(n+1)}G_{ijlm}-\frac{\widetilde{\overset{*}{\kappa}}}{4n(n+1)}f_i^aG_{ajlm}.$$

This mean that (M, g, f) is of constant totally real sectional curvatures too, and its scalar curvatures are related to the scalar curvatures of the ambient manifold in the following way:

$$\kappa = \frac{n-1}{n+1}\widetilde{\kappa}, \qquad \overset{*}{\kappa} = \frac{n-1}{n+1}\,\widetilde{\overset{*}{\kappa}}. \tag{3.10}$$

Also, in view of (3.7), we have

$$(h^2)_{ij} = (fh^2)_{ij} = 0 \, .$$

Therefore

$$(h^r)_{ij} = (fh^r)_{ij} = 0 \qquad \text{for all } r \geq 2$$

and

$$\mathrm{tr}\,(h^r) = \mathrm{tr}\,(fh^r) = 0 \qquad \text{for all } r.$$

But then, according to the discussion in the section 2, (2.13) holds for all q.

Finally, (3.4) reduce to

$$\tau V_a V^a + \sigma V_a \bar{V}^a = 0 \, ,$$
$$\tau V_a \bar{V}^a - \sigma V_a V^a = 0 \, .$$

These equations imply $V_a V^a = V_a \bar{V}^a = 0$ or $\tau = \sigma = 0$. The first case means that V is a null vector and is orthogonal to fV. In the second case (3.1) reduces to $h_{ij} = 0$, that is (M, g, f) is totally geodesic.

Thus, we can state

Theorem 2. *Let (M, g, f) be the holomorphically conformally flat holomorphic hypersurface of an anti-Kähler manifold of constant totally real sectional curvatures, and $n > 3$, $2n = \dim M$.*

If (M, g, f) is minimal, then

it is minimal of type 1,

or

it is itself a manifold of constant totally real sectional curvatures. In this case its scalar curvatures are related to the scalar curvatures of the ambient manifold according to (3.10), and (3.13) holds for all $q = 1, 2 \ldots$. The vector V is a null vector and is orthogonal to fV, or (M, g, f) is totally geodesic.

4. *Holomorphically Einstein hypersurface*

First we shall prove that for any anti-Kähler holomorphically Einstein hypersurface, (2.13) is satisfied for all $q = 1, 2, \dots$. Namely, substituting (1.7) into (2.3), we find

$$\tau_{ij} = \alpha g_{ij} - \beta f_{ij}, \tag{4.1}$$

where

$$\alpha = \frac{\kappa}{2n} - 2(n-1)k_1, \quad \beta = \frac{\overset{*}{\kappa}}{2n} + 2(n-1)k_2 \, . \tag{4.2}$$

Then

$$(\tau^p)_{ij} = \alpha_1 g_{ij} - \beta_1 f_{ij} \tag{4.3}$$

where α_1 and β_1 are some scalar functions. Therefore

$$(\tau^{p+1})_{ij} = (\alpha\alpha_1 - \beta\beta_1) g_{ij} - (\alpha\beta_1 + \alpha_1\beta) f_{ij}.$$

Using (1.5) and (4.3), we obtain

$$\begin{aligned} R_{iabj}(\tau^p)^{ab} &= \alpha_1 \rho_{ij} - \beta_1 f_i^a \rho_{aj} \\ &= \left[\alpha_1 \frac{\kappa}{2n} - \beta_1 \frac{\overset{*}{\kappa}}{2n}\right] g_{ij} - \left[\alpha_1 \frac{\overset{*}{\kappa}}{2n} + \beta_1 \frac{\kappa}{2n}\right] f_{ij}, \end{aligned}$$

and therefore

$$\begin{aligned} &T_{iabj}(\tau^p)^{ab} - (\tau^{p+1})_{ij} = \\ &\quad (R_{iabj} - k_1 G_{iabj} - k_2 f_i^t G_{tabj})(\tau^p)^{ab} - (\tau^{p+1})_{ij} = \\ &\quad = \left\{\alpha_1 \left[\frac{\kappa}{2n} - 2(n-1)k_1 - \alpha\right] - \beta_1 \left[\frac{\overset{*}{\kappa}}{2n} + 2(n-1)k_2 - \beta\right]\right\} g_{ij} \\ &\quad\quad - \left\{\alpha_1 \left[\frac{\overset{*}{\kappa}}{2n} + 2(n-1)k_2 - \beta\right] + \beta_1 \left[\frac{\kappa}{2n} - 2(n-1)k_1 - \alpha\right]\right\} f_{ij} = 0 \end{aligned}$$

because of (4.2).

We note that for any holomorphically Einstein anti-Kähler manifold, being it holomorphic hypersurface or not, we have

$$R_{iabj}(\rho^p)^{ab} - (\rho^{p+1})_{ij} = 0 \, .$$

So, if (M, g, f) is minimal holomorphic hypersurface and, at the same time, holomorphically Einstein, besides (2.6) the conditions (2.13) are satisfied for all $q = 1, 2, \dots$, and we can not determine the second fundamental form using the method described in the Section 2. But, holomorphically Einstein hypersurface may not be minimal and in that case the second fundamental form can be determined.

From now on in this section we suppose that (M, g, f) is a holomorphically Einstein holomorphic non-minimal hypersurface of an anti-Kähler manifold of constant totally real sectional curvatures. Thus (2.2) and (2.4) hold, as well as (4.1) and (4.2).

Using (2.2) we can calculate

$$\begin{aligned}(T_{abhj}T_{stik} - 2T_{haib}T_{jskt})g^{at}g^{bs} ={}& -4(h^2)_{ih}(h^2)_{jk} + 4h_{ih}(h^3)_{jk} + 4(h^3)_{ih}h_{jk} \\ & +4(fh^2)_{ih}(fh^2)_{jk} - 4(fh)_{ih}(fh^3)_{jk} - 4(fh^3)_{ih}(fh)_{jk} \\ & -2\operatorname{tr}(h^2)\left[h_{ih}h_{jk} - f_i^a h_{ah} f_j^b h_{bk}\right] + 2\operatorname{tr}(fh^2)\left[h_{ih}f_j^a h_{ak} + f_i^a h_{ah}h_{jk}\right].\end{aligned} \tag{4.4}$$

In view of (4.1), (2.4) becomes

$$(h^2)_{ij} = \frac{1}{2}\left[-\alpha g_{ij} + \beta f_{ij} + \operatorname{tr} h\, h_{ij} - \operatorname{tr}(fh) f_i^a h_{aj}\right], \tag{4.5}$$

from which we obtain

$$(fh^2)_{ij} = \frac{1}{2}\left[-\beta g_{ij} - \alpha f_{ij} + \operatorname{tr}(fh)h_{ij} + \operatorname{tr} h\, f_i^a h_{aj}\right], \tag{4.6}$$

$$\begin{aligned}(h^3)_{ij} ={}& \frac{1}{4}\left[-\alpha \operatorname{tr} h + \beta \operatorname{tr}(fh)\right] g_{ij} + \frac{1}{4}\left[\alpha \operatorname{tr}(fh) + \beta \operatorname{tr} h\right] f_{ij} \\ & + \frac{1}{4}\left[-2\alpha + (\operatorname{tr} h)^2 - (\operatorname{tr}(fh))^2\right] h_{ij} + \frac{1}{2}\left[\beta - \operatorname{tr} h \operatorname{tr}(fh)\right] f_i^a h_{aj},\end{aligned}$$

$$\begin{aligned}(fh^3)_{ij} ={}& -\frac{1}{4}\left[\alpha \operatorname{tr}(fh) + \beta \operatorname{tr} h\right] g_{ij} + \frac{1}{4}\left[-\alpha \operatorname{tr} h + \beta \operatorname{tr}(fh)\right] f_{ij} \\ & -\frac{1}{2}\left[\beta - (\operatorname{tr} h)(\operatorname{tr}(fh))\right] h_{ij} + \frac{1}{4}\left[-2\alpha + (\operatorname{tr} h)^2 - (\operatorname{tr}(fh))^2\right] f_i^a h_{aj}.\end{aligned}$$

Substituting this into (4.4), we get

$$\begin{aligned}& (T_{abhj}T_{stik} - 2T_{haib}T_{jskt})g^{at}g^{bs} + \\ & +(\alpha^2 - \beta^2)(g_{ih}g_{jk} - f_{ih}f_{jk}) - 2\alpha\beta(g_{ih}f_{jk} + g_{jk}f_{ih}) = \\ & = \left[-4\alpha + (\operatorname{tr} h)^2 - (\operatorname{tr} fh)^2 - 2\operatorname{tr}(h^2)\right]\left[h_{ih}h_{jk} - f_i^a h_{ah} f_j^b h_{bk}\right] \\ & \quad + \left[4\beta - 2\operatorname{tr} h \operatorname{tr}(fh) + 2\operatorname{tr}(fh^2)\right]\left[f_i^a h_{ah}h_{jk} + h_{ih}f_j^b h_{bk}\right]\end{aligned} \tag{4.7}$$

According to (4.5) and (4.6)

$$(\mathrm{tr}\,h)^2 - (\mathrm{tr}\,fh)^2 - 2\mathrm{tr}\,(h^2) = 2n\alpha$$

$$-2\mathrm{tr}\,h\,\mathrm{tr}\,(fh) + 2\mathrm{tr}\,(fh^2) = -2n\beta,$$

because of which (4.7) reduces to

$$\begin{aligned}&2(n-2)\alpha\left[h_{ih}h_{jk} - f_i^a h_{ah} f_j^b h_{bk}\right] - 2(n-2)\beta\left[f_i^a h_{ah} h_{jk} + h_{ih} f_j^b h_{bk}\right]\\ &= (T_{abhj}T_{stik} - 2T_{haib}T_{jskt})g^{at}g^{bs} + (\alpha^2-\beta^2)(g_{ih}g_{jk} - f_{ih}f_{jk})\\ &\quad -2\alpha\beta(g_{ih}f_{jk} + g_{jk}f_{ih}).\end{aligned}$$

Transecting this relation with g^{ih}, we get

$$\begin{aligned}&2(n-2)\left[\alpha\,\mathrm{tr}\,h - \beta\,\mathrm{tr}\,(fh)\right]h_{jk} - 2(n-2)\left[\alpha\,\mathrm{tr}\,(fh) + \beta\,\mathrm{tr}\,h\right]f_j^a h_{ak} =\\ &= -T^{abc}{}_{j}T_{abck} - 2T_{jabk}\tau^{ab} + 2n\left[(\alpha^2-\beta^2)g_{jk} - 2\alpha\beta f_{jk}\right].\end{aligned} \tag{4.8}$$

Putting

$$2(n-2)\left[\alpha\,\mathrm{tr}\,h - \beta\,\mathrm{tr}\,(fh)\right] = p\,,$$

$$2(n-2)\left[\alpha\,\mathrm{tr}\,(fh) + \beta\,\mathrm{tr}\,h\right] = q\,,$$

and using the condition (2.13), we rewrite (4.8) in the form

$$\begin{aligned}ph_{ij} - qf_i^a h_{aj} = \;& -\left[T^{abc}{}_{i}T_{abcj} + 2(\tau^2)_{ij}\right]\\ & +2n\left[(\alpha^2-\beta^2)g_{ij} - 2\alpha\beta f_{ij}\right]\,.\end{aligned}$$

This relation, together with

$$\begin{aligned}qh_{ij} + pf_i^a h_{aj} = \;& -f_i^t\left[T^{abc}{}_{t}T_{abcj} + 2(\tau^2)_{tj}\right]\\ & +2n\left[2\alpha\beta g_{ij} + (\alpha^2-\beta^2)f_{ij}\right]\end{aligned}$$

implies

$$\begin{aligned}(p^2+q^2)h_{ij} = \;& p\left\{-\left[T^{abc}{}_{i}T_{abcj} + 2(\tau^2)_{ij}\right] + 2n\left[(\alpha^2-\beta^2)g_{ij} - 2\alpha\beta f_{ij}\right]\right\}\\ & +q\left\{-f_i^t\left[T^{abc}{}_{t}T_{abcj} + 2(\tau^2)_{tj}\right] + 2n\left[2\alpha\beta g_{ij} + (\alpha^2-\beta^2)f_{ij}\right]\right\}.\end{aligned} \tag{4.9}$$

But if $\mathrm{tr}\,h = \mathrm{tr}\,(fh) = 0$, (4.8) yields

$$T^{abc}{}_{i}T_{abcj} + 2(\tau^2)_{ij} = 2n\left[(\alpha^2-\beta^2)g_{ij} - 2\alpha\beta f_{ij}\right]\,. \tag{4.10}$$

Thus, we can state

Theorem 3. *Let (M, g, f) be the holomorphically Einstein hypersurface of an anti-Kähler manifold of constant totally real sectional curvatures. Then, if it is not minimal, the second fundamental form is given by (4.9). But if* $\operatorname{tr} h = \operatorname{tr}(fh) = 0$, *then (4.10) holds.*

5. *Remarks on complex hypersurfaces of Kähler manifold*

A differentiable manifold $\widetilde{M}$, $\dim \widetilde{M} = 2m$ is a Kähler manifold if it is endowed with metric G and complex structure $\widetilde{J}$ such that

$$\widetilde{J}^2 = -\mathrm{Id.}, \quad \widetilde{J}_A^C \widetilde{J}_B^D G_{CD} = G_{AB}, \quad \widetilde{\nabla} \widetilde{J} = 0 \,.$$

A differentiable submanifold M of $\widetilde{M}$, $\dim M = 2n$, $m = n+1$, is said to be complex hypersurface of $\widetilde{M}$, if the complex structure $\widetilde{J}$ of $\widetilde{M}$ leaves invariant the tangent space of M at each point $P \in M$. In this case, G and $\widetilde{J}$ induce on M the metric g and the complex structure J such that (M, g, J) is itself a Kählar manifold [16], i.e.

$$J^2 = -\mathrm{Id.}, \quad J_i^a J_j^b g_{ab} = g_{ij}, \quad \nabla J = 0 \,.$$

If we put

$$F_{ij} = J_i^a g_{aj}, \quad \text{then} \quad F_{ij} = -F_{ji} \,.$$

Also, the complex structure $\widetilde{J}$ leaves invariant the normal plane to M at each point $P \in M$. Thus, there exist, in each neighborhood U of $P \in M$, two local unit vector fields, N and $\widetilde{J}N$, mutually orthogonal and normal to M. If h and k are second fundamental forms corresponding to N and JN, and h_{ij} and k_{ij} are their local components, than [16]

$$\begin{aligned}
&h_{ij} = h_{ji}, \qquad k_{ij} = k_{ji} \,, \\
&h_{ij} = J_i^a k_{aj}, \qquad k_{ij} = -J_i^a h_{aj} \,, \\
&J_i^a J_j^b h_{ab} = -h_{ij}, \qquad J_i^a J_j^b k_{ab} = -k_{ij}
\end{aligned}$$

Thus

$$\operatorname{tr} h = \operatorname{tr} k = 0 \,,$$

and therefore <u>any complex hypersurface of any Kähler manifold is minimal.</u>

If (h^r) is fundamental form of order r, then

$$J_i^a J_j^b (h^{2p})_{ab} = (h^{2p})_{ij}, \quad J_i^a J_j^b (h^{2p+1})_{ab} = -(h^{2p+1})_{ij} \,. \tag{5.1}$$

The second relation (5.1) shows that

$$\mathrm{tr}\,(h^{2p+1}) = 0 \quad \text{for all } p = 1, 2, \dots \ . \tag{5.2}$$

Similarly,

$$\mathrm{tr}\,(Jh^{2p+1}) = 0 \quad \text{for all } p = 0, 1, 2, \dots \ . \tag{5.3}$$

Now, let us suppose that the ambient $(\widetilde{M}, G, \widetilde{J})$ is a manifold of constant holomorphic sectional curvature c. Then the Gauss equation for (M, g, J) is

$$\begin{aligned} R_{ijlm} - \frac{c}{4}(g_{im}g_{jl} - g_{il}g_{jm} + F_{im}F_{jl} - F_{il}F_{jm} - 2F_{ij}F_{lm}) = \\ = h_{im}h_{jl} - h_{il}h_{jm} + k_{im}k_{jl} - k_{il}k_{jm} \ , \end{aligned}$$

where R_{ijlm} is the Riemannian curvature tensor of M.

If we put

$$T_{ijlm} = R_{ijlm} - \frac{c}{4}(g_{im}g_{jl} - g_{il}g_{jm} + F_{im}F_{jl} - F_{il}F_{jm} - 2F_{ij}F_{lm}) \ , \tag{5.4}$$

the Gauss equation becomes

$$T_{ijlm} = h_{im}h_{jl} - h_{il}h_{jm} + k_{im}k_{jl} - k_{il}k_{jm} \ . \tag{5.5}$$

We remark that the tensor (5.4) has all algebraic properties as the Riemannian curvature tensor of a Kähler manifold. In particular

$$J_i^a J_j^b T_{abkl} = T_{ijkl} \ . \tag{5.6}$$

If

$$\tau_{im} = T_{ijlm}g^{jl} \ ,$$

and if we denote by ρ_{ij} the Ricci tensor of M, from (5.4) we have

$$\tau_{ij} = \rho_{ij} - \frac{n+1}{2}cg_{ij} \tag{5.7}$$

On the other hand, (5.5) yields

$$\tau_{ij} = \mathrm{tr}\,h\ h_{ij} + \mathrm{tr}\,k\ k_{ij} - (h)^2_{ij} - (k^2)_{ij} \ .$$

But $\mathrm{tr}\,h = \mathrm{tr}\,k = 0$, $(k^2)_{ij} = (h^2)_{ij}$. Therefore

$$\tau_{ij} = -2(h^2)_{ij} \ ,$$

from which there follows

$$(\tau^p)_{ij} = (-2)^p (h^{2p})_{ij} \ . \tag{5.8}$$

Now, using (5.5) and (5.8), we find

$$\begin{aligned} &T_{rsia}(\tau^p)^a{}_j + T_{rsja}(\tau^p)^a{}_i = \\ &= (-2)^p \Big[(h^{2p+1})_{rj} h_{si} - h_{ri}(h^{2p+1})_{sj} + (h^{2p+1})_{tj} J^t_r h_{qi} J^q_s - h_{ti} J^t_r (h^{2p+1})_{qj} J^q_s \\ &\quad + (h^{2p+1})_{ri} h_{sj} - h_{rj}(h^{2p+1})_{si} + (h^{2p+1})_{ti} J^t_r h_{qj} J^q_s - h_{tj} J^t_r (h^{2p+1})_{qi} J^q_s \Big] \ , \end{aligned}$$

from which, transvectin with g^{sj} and using (5.2) and (5.3), we get

$$\begin{aligned} &-T_{rabi}(\tau^p)^{ab} + (\tau^{p+1})_{ri} = \\ &\qquad = (-2)^p \Big[(h^{2p+1})_{tj} J^t_r h_{qi} J^q_s g^{sj} - h_{tj} J^t_r (h^{2p+1})_{qi} J^q_s g^{sj} \Big] \ . \end{aligned}$$

But, according to the second relation (5.1)

$$J^a_i (h^{2p+1})_{ak} = J^a_k (h^{2p+1})_{ia} \ ,$$

because of which

$$(h^{2p+1})_{tj} J^t_r h_{qi} J^q_s g^{sj} = (h^{2p+1})_{tr} J^t_j h_{qi} J^q_s q^{sj} = (h^{2p+1})_{tr} h_{qi} g^{tq} = (h^{2p+2})_{ri} \ ,$$

and

$$-h_{tj} J^t_r (h^{2p+1})_{qi} J^q_s g^{sj} = -(h^{2p+2})_{ri} \ .$$

Therefore

$$T_{iabj}(\tau^p)^{ab} - (\tau^{p+1})_{ij} = 0 \ . \tag{5.9}$$

Thus, we can state

Theorem 4. *Let (M, g, J) be a complex hypersurface of a Kähler manifold of constant holomorphic sectional curvature. Then the relation (5.9) is valid for all integers $p = 1, 2, \dots$.*

In the case $c = 0$, i.e. for a complex hypersurface of the flat Kähler space, (5.9) reduces to

$$R_{iabj}(\rho^p)^{ab} - (\rho^{p+1})_{ij} = 0 \ ,$$

and is the intrinsic property of (M, g, J).

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