Bulletin T.CXXXIV de l'Académie serbe des sciences et des arts – 2007
Classe des Sciences mathématiques et naturelles
Sciences mathématiques, № 32

REGULARLY VARYING SOLUTIONS TO FUNCTIONAL DIFFERENTIAL EQUATIONS WITH DEVIATING ARGUMENT

K. TAKAŜI, V. MARIĆ

(Presented at the 4th Meeting, held on May 24, 2006)

A b s t r a c t. The existence and the asymptotic behaviour of regularly varying solutions (in the sense of Karamata) of the functional differential equation presented below are studied.

AMS Mathematics Subject Classification (2000): Primary 34K06, Secondary 26A12
Key Words: functional differential equations, deviating arguments, regularly varying functions

1. *Introduction*

The first attempt of investigating functional differential equations with deviating argument in the framework of regular variation in the sense of Karamata was made by the present authors in the paper [4] in which a sharp condition was presented for differential equations of the type

$$x''(t) = q(t)x(g(t)) \tag{A}$$

to possess slowly varying solutions.

The purpose of this paper is to study the existence of regularly varying solutions (of nonzero regularity indices) of (A).

For readers convenience we recall that a measurable function $L : [0,\infty) \to (0,\infty)$ is said to be slowly varying if it satisfies

$$\lim_{t\to\infty} \frac{L(\lambda t)}{L(t)} = 1 \quad \text{for any} \quad \lambda > 0.$$

Furthermore, the function

$$f(t) = t^{\rho} L(t)$$

is said to be regularly varying of regularity index $\rho \in \mathbf{R}$. The totality of these functions is denoted by $RV(\rho)$ and in particular $RV(0)$ (or SV) stands for the totality of slowly varying functions.

In the paper, among many basic properties of slowly varying functions, we emphasize the representation theorem which asserts that $L(t) \in SV$ if and only if it is expressed in the form

$$L(t) = c(t) \exp\left\{ \int_{t_0}^{t} \frac{\varepsilon(s)}{s} ds \right\}, \quad t \geq t_0, \tag{1.1}$$

for some $t_0 > 0$ and some measurable functions $c(t)$ and $\varepsilon(t)$ such that

$$\lim_{t\to\infty} c(t) = c_0 \in (0,\infty) \quad \text{and} \quad \lim_{t\to\infty} \varepsilon(t) = 0.$$

If especially $c(t) = c_0$ then $L(t)$ is called normalized.

The most comprehensive text on the theory and numerous applications of regularly (slowly) varying functions is [1]. In particular, applications to the asymptotics of solutions to some classes of differential equations are presented in [5].

Here and hereafter it is assumed that the functions $q(t)$ and $g(t)$ are positive and continuous on $[0,\infty)$, that $g(t)$ is an increasing function such that $g(t) < t$ and $\lim_{t\to\infty} g(t) = \infty$ and that $q(t)$ is integrable over $(0,\infty)$.

We prove

Theorem A. *Let $c > 0$ and denote by λ_i, $i = 0,1$, $\lambda_0 < \lambda_1$, the two roots of the quadratic equation*

$$\lambda^2 - \lambda - c = 0. \tag{1.2}$$

Suppose that

$$\lim_{t\to\infty} \frac{g(t)}{t} = 1; \tag{1.3}$$

then equation (A) has two regularly varying solutions $x_i(t)$ of indices λ_i, i.e. possessing the form

$$x_i(t) = t^{\lambda_i} L_i(t)$$

where $L_i(t)$ are some normalized slowly varying functions, if and only if

$$Q(t) := t \int_t^{\infty} q(s)ds - c \to 0, \quad as \quad t \to \infty. \tag{1.4}$$

The proof will be given in section 3. It heavily depends on a similar result for the equation without deviating argument

$$x''(t) = q(t)x(t) \tag{B}$$

([5], Theorem 1.11). In section 2 we partially frame it as Lemma 2.1 and prove it using the procedure from [3]. The later is modified here in such a way that the explicit expression obtained for the regularly varying solution of (B) can be directly applicable to the construction of the desired solution of equation (A) by means of the fixed point technique.

In order to point out some possible applications of the existence Theorem A we prove in section 3 the following result which is in fact a corollary of a known one to be quoted below.

Theorem B. *Let conditions of Theorem A hold with*

$$\frac{g(t)}{t} = 1 + O\left(\frac{1}{t}\right) \quad as \quad t \to \infty \tag{1.5}$$

instead of (1.3).

If

$$\int_a^{\infty} t \left| q(t) - \frac{c}{t^2} \right| dt < \infty \tag{1.6}$$

then there exist two solutions $x_i(t)$, $i = 0, 1$ of equation (A) such that

$$x_i(t) \sim t^{\lambda_i} \quad as \quad t \to \infty.$$

2. *A Lemma*

We prove

Lemma 2.1. *Let $c > 0$ and λ_i be as in Theorem A, then there exist two regularly varying solutions $x_i(t)$, $i = 0, 1, 2$ of (B) possessing the form*

$$x_i(t) = \exp\left\{\int_T^t \frac{\lambda_i - Q(s) + v_i(s)}{s} ds\right\}, \quad t \geq T \tag{2.1}$$

where $v_i(t)$ are solutions of the integral equations

$$v_0(t) = t^{1-2\lambda_0} \int_t^\infty s^{2(\lambda_0 - 1)}\{[v(s) - Q(s)]^2 - 2\lambda_0 Q(s)\}ds, \quad t \geq T. \quad (2.2-0)$$

$$v_1(t) = t^{1-2\lambda_1} \int_T^t s^{2(\lambda_1 - 1)}\{2\lambda_1 Q(s) - [v(s) - Q(s)]^2\}ds, \quad (2.2-1)$$

respectively, if and only if (1.4) holds.

P r o o f of **Lemma 2.1.**

The "only if" part is proved in [2]. We will present here the detailed proof of the "if" part for the case $i = 0$, and then point out the alterations needed for the case $i = 1$. To simplify the notation the subscripts $0, 1$ in the functions v_i and x_i will be omitted throughout the proof.

Since by (1.4), $Q(t) \to 0$ as $t \to \infty$, for a given $0 < m < 1$ one can choose $T > 0$ such that $|Q(t)| \leq m^2$ for $t \geq T$ and

$$\frac{2(2 + |\lambda_0|)}{1 + 2|\lambda_0|} m \leq 1. \tag{2.3}$$

Define the set

$$V := \{v \in C_0[T, \infty) : |v(t)| \leq m, \quad t \geq T\}, \tag{2.4}$$

where $C_0[T, \infty)$ denotes the set of all continuous functions on $[T, \infty)$ that tend to 0 as $t \to \infty$. It is obvious that $C_0[T, \infty)$ is a Banach space with the norm $\|v\|_0 = \sup_{s \geq T} |v(t)|$ and that V is a closed subset of $C_0[T,\infty)$. Consider the integral operator $\mathcal{F}$ defined by

$$\mathcal{F}v(t) = t^{1-2\lambda_0} \int_t^\infty s^{2(\lambda_0-1)}\{[v(s) - Q(s)]^2 - 2\lambda_0 Q(s)\}ds, \quad t \geq T. \qquad (2.5)$$

If $v \in V$, then

$$|\mathcal{F}v(t)| \leq \frac{(2(2+|\lambda_0|)}{1+2|\lambda_0|} m \cdot m \leq m, \quad t \geq T,$$

so that $\mathcal{F}v \in V$, and if $v, w \in V$, then

$$\begin{aligned} |\mathcal{F}v(t) - \mathcal{F}w(t)| &\leq t^{1-2\lambda_0} \int_t^\infty s^{2(\lambda_0-1)}\{[|v(s)| + |w(s)| + 2m^2]|v(s) - w(s)|\}ds \\ &\leq \frac{4m}{1+2|\lambda_0|}\|v - w\|_0, \quad t \geq T, \end{aligned}$$

which implies that

$$\|\mathcal{F}v - \mathcal{F}w\|_0 \leq \frac{4m}{1+2|\lambda_0|}\|v - w\|_0.$$

In view of (2.3) this shows that $\mathcal{F}$ is a contraction mapping on V. Therefore, there exists a unique $v \in V$ such that $v = \mathcal{F}v$, which is equivalent to the integral equation (2.2).

Using this function $v(t)$ we construct the function (2.1). We claim that this function $x(t)$ becomes a regularly varying solution of index λ_0 of equation (B). That $x(t)$ is a regularly varying function of index λ_0 follows from the fact that $\lambda_0 - Q(t) + v(t) \to \lambda_0$ as $t \to \infty$. To show that $x(t)$ is a solution of (B) it suffices to verify that the function $u(t) = (\lambda_0 - Q(t) + v(t))/t$ satisfies the Riccati equation $u'(t) + u(t)^2 - q(t) = 0$ on $[T,\infty)$ associated with (B). But this is almost trivial, since the Riccati equation in question can be rewritten in the form

$$(t^{2\lambda_0-1}v(t))' + t^{2(\lambda_0-1)}\{[v(t) - Q(t)]^2 - 2\lambda_0 Q(t)\} = 0, \quad t \geq T,$$

which can readily be seen to be the one that follows from differentiation of the integral equation (2.2) for $v(t)$. This completes the proof for the case $i = 0$.

The proof of the case $i = 1$ mimics the preceeding one, making use of the operator

$$\mathcal{G}v(t) = t^{1-2\lambda_1} \int_T^t s^{2(\lambda_1-1)}\{2\lambda_1 Q(s) - [v(s) - Q(s)]^2\}ds,$$

instead of $\mathcal{F}v(t)$ defined by (2.5). Also the constant $0 < m < 1$ should satisfy

$$\frac{2(\lambda_1+2)}{2\lambda_1 - 1} m \le 1 \tag{2.6}$$

instead of (2.3).

3. *Proofs*

3.1. *P r o o f* of **Theorem A**.

Here again, we present a detailed proof for the case $i = 0$ and then point out the alterations needed for the case $i = 1$. Also the subscripts $0, 1$, in the functions v_i, x_i will be omitted.

Sufficiency. Let $\tau = g(T)$ for some $T > 0$. We define Ξ to be the set consisting of those functions $\xi \in C[\tau,\infty) \cap C^1[T,\infty)$ such that

$$\xi(t) = 1, \quad \tau \le t \le T \tag{3.1}$$

and for $t \ge T$ have the representation

$$\xi(t) = \exp\left\{\int_T^t \frac{\lambda_0 + \delta_\xi(s)}{s} ds\right\} \tag{3.2}$$

for some continuous functions $\delta_\xi(t)$ satisfying for all $\xi \in \Xi$

$$\lim_{t\to\infty} \delta_\xi(t) = 0 \tag{3.3}$$

and

$$-R(t) - m \le \delta_\xi(t) \le -\lambda_0, \quad t \ge T. \tag{3.4}$$

Here we put

$$R(t) = 2Q(t) + c \tag{3.5}$$

which is positive for sufficiently large t due to (1.4). The number $0 < m < 1$ will be conveniently chosen.

In view of (3.2) and (3.3) all the members of Ξ are decreasing regularly varying functions of index $\lambda_0 < 0$, i.e. $\Xi \subset RV(\lambda_0)$. Obviously, Ξ can be regarded as a subset of the locally convex space $C^1[T,\infty)$ endowed with the topology of uniform convergence of functions and their derivatives on compact subintervals of $[T,\infty)$. Moreover Ξ is a closed convex subset of $C^1[T,\infty)$: Let $\{\xi_n\}$ be a sequence of elements of Ξ converging to ξ_0 in $C^1[T,\infty)$. Use is made of the representation (3.2) for ξ_n:

$$\xi_n(t) = \exp\left\{\int_T^t \frac{\lambda_0 + \delta_n(s)}{s} ds\right\}, \quad t \geq T, \tag{3.6}$$

(where, to simplify, $\delta_n(s)$ stands for $\delta_{\xi_n}(s)$).

From (3.6) we have

$$t\frac{\xi_n'(t)}{\xi_n(t)} = \lambda_0 + \delta_n(t), \quad t \geq T, n = 1, 2, \ldots. \tag{3.7}$$

Letting $n \to \infty$ and noting that $\xi_n(t) \to \xi_0(t)$ and $\xi_n'(t) \to \xi_0'(t)$ as $n \to \infty$ uniformly on compact subintervals of $[T,\infty)$, we see that $\xi_0(t)$ satisfies

$$t\frac{\xi_0'(t)}{\xi_0(t)} = \lambda_0 + \delta_0(t), \quad t \geq T, \tag{3.8}$$

for some continuous function $\delta_0(t) = \lim_{n\to\infty} \delta_n(t)$ with the property that

$$R(t) - m \leq \delta_0(t) \leq -\lambda_0, \quad \lim_{t\to\infty} \delta_0(t) = 0. \tag{3.9}$$

This implies that $\xi_0(t)$ is expressed in the form

$$\xi_0(t) = \exp\left\{\int_T^t \frac{\lambda_0 + \delta_0(s)}{s} ds\right\}, \quad t \geq T, \tag{3.10}$$

with $\delta_0(t)$ satisfying (3.9). This guarantees that ξ_0 is a member of Ξ, showing that Ξ is closed in $C^1[T,\infty)$.

To prove the convexity of Ξ we proceed as follows. Let ξ_n, $n = 1, \ldots, N$, be any elements of Ξ and let c_n, $n = 1, \ldots, N$, be any positive constants such that $\sum_{n=1}^{N} c_n = 1$. Put $\eta = \sum_{n=1}^{N} c_n\xi_n$. Using the representations (3.8)-(3.9), we obtain

$$t\frac{\eta'(t)}{\eta(t)} = \lambda_0 + \Delta(t), \tag{3.11}$$

where $\Delta(t)$ is given by

$$\Delta(t) = \sum_{n=1}^{N} c_n \delta_n(t) \xi_n(t) / \sum_{n=1}^{N} c_n \xi_n(t).$$

As easily verified, $\Delta(t)$ satisfies

$$-R(t) - m \leq \Delta(t) \leq -\lambda_0 \quad \text{and} \quad \lim_{t\to\infty} \Delta(t) = 0. \tag{3.12}$$

From (3.11) it follows that $\eta(t)$ has the representation

$$\eta(t) = \exp\left\{ \int_T^t \frac{\lambda_0 + \Delta(s)}{s} ds \right\}, \quad t \geq T,$$

with $\Delta(t)$ satisfying (3.12). This means that $\eta \in \Xi$, implying that Ξ is a convex subset of $C^1[T, \infty)$.

In the sequel some properties of functions ξ are needed. To derive these an auxiliary function is of a crucial importance.

Define for some $a > 0$ and $m > 0$

$$\Lambda(t) = \exp\left\{ \int_a^t \frac{\lambda_0 - R(s) - m}{s} ds \right\}.$$

Then, since $\Lambda(t)$ is decreasing and $g(t) < t$, one has

$$1 \leq \frac{\Lambda(g(t))}{\Lambda(t)} = \exp\left\{ \int_{g(t)}^t \frac{-\lambda_0 + R(s) + m}{s} ds \right\}. \tag{3.13}$$

But $0 < R(t) + m - \lambda_0 \leq k$ for some $k > 0$ and sufficiently large t, so that in view of (1.3), there follows for $t \to \infty$

$$\int_{g(t)}^t \frac{R(s) + m - \lambda_0}{s} ds \leq k \log \frac{t}{g(t)} \to 0.$$

Thus, due to (3.13)

$$\lim_{t\to\infty} \frac{\Lambda(g(t))}{\Lambda(t)} = 1 \tag{3.14}$$

which implies that there exists $T > 0$ such that for $t \geq T$

$$1 \leq \frac{\Lambda(g(t))}{\Lambda(t)} \leq 2. \tag{3.15}$$

Two preceeding formulas lead to

$$\lim_{t\to\infty} \frac{\xi(g(t))}{\xi(t)} = 1 \quad \text{and} \quad 1 \leq \frac{\xi(g(t))}{\xi(t)} \leq 2 \tag{3.16}$$

for all $\xi \in \Xi$ and $t \geq T$. (Observe here that T is independent of ξ).

Indeed by the definition of the function ξ one has

$$\begin{aligned} 1 \leq \tfrac{\xi(g(t))}{\xi(t)} &= \exp\left\{ \int_{g(t)}^{t} \tfrac{-\lambda_0 - \delta(s)}{s} ds \right\} \\ &\leq \exp\left\{ \int_{g(t)}^{t} \tfrac{-\lambda_0 + R(s) + m}{s} ds \right\} = \tfrac{\Lambda(g(t))}{\Lambda(t)}, \end{aligned} \tag{3.17}$$

and (3.16) follows by (3.14), (3.15).

Now, for any $\xi \in \Xi$ put for $t \geq T$

$$q_\xi(t) = q(t)\frac{\xi(g(t))}{\xi(t)}, \tag{3.18}$$

and

$$Q_\xi(t) = t\int_t^\infty q_\xi(s)ds - c. \tag{3.19}$$

Because of (3.17) we have for all ξ and for $t \geq T$, $q(t) \leq q_\xi(t) \leq q_\Lambda(t)$ which due to (3.17) implies

$$Q(t) \leq Q_\xi(t) \leq Q_\Lambda(t). \tag{3.20}$$

Observe that condition (1.4) combined with (3.14) gives that for $t \to \infty$, $Q_\Lambda(t) \to 0$ and so $Q_\xi(t) \to 0$ for all $\xi \in \Xi$. Consequently, there exist $0 < m < 1$ and $T = T(m)$ such that for $t \geq T$ and all $\xi \in \Xi$

$$-m^2 \leq Q(t) \leq Q_\xi(t) \leq Q_\Lambda(t) \leq m^2$$

implying that

$$|Q(t)| \leq m^2, \quad |Q_\Lambda(t)| \leq m^2, \quad |Q_\xi(t)| \leq m^2. \tag{3.21}$$

Now we choose m such that for $t \geq T$

$$\frac{2(2+|\lambda_0|)m}{1+2|\lambda_0|} \leq 1. \tag{3.22}$$

Let us consider the family of ordinary differential equations

$$x''(t) = q_\xi(t)x(t), \quad \xi \in \Xi. \tag{3.23}$$

Because of (3.21) and (3.22), we are able to apply Lemma 2.1 to each member of the family (3.23), concluding that each equation of the family possesses a regularly varying solution of index λ_0 having the form

$$x_\xi(t) = \exp\left\{\int_T^t \frac{\lambda_0 - Q_\xi(s) + v_\xi(s)}{s} ds\right\}, \tag{3.24}$$

where $v_\xi(t)$ denote the function satisfying the integral equation

$$v_\xi(t) = t^{1-2\lambda_0} \int_t^\infty s^{2(\lambda_0-1)}\{[v_\xi(s) - Q_\xi(s)]^2 - 2\lambda_0 Q_\xi(s)\}ds, \tag{3.25}$$

Formulas (3.24) and (3.25) hold for $t \geq T$, where T is determined in (3.21) and (3.22). Unless otherwise stated the same is true for the formulas that follow and the adjective will be omitted.

Let $\tau = g(T)$ and denote by Φ the mapping which associates to each $\xi \in \Xi$ the function $\Phi\xi(t)$ defined by

$$\Phi\xi(t) = 1 \quad \text{for} \quad \tau \leq t \leq T, \quad \Phi\xi(t) = x_\xi(t), \quad t \geq T. \tag{3.26}$$

We claim that Φ is a self-map on Ξ and sends Ξ into a relatively compact subset of $C^1[\tau,\infty)$.

i) Φ maps Ξ into itself: Let $\xi \in \Xi$. Rewrite (3.24) as

$$x_\xi(t) = \exp\left\{\int_T^t \frac{\lambda_0 + \delta_\xi(s)}{s} ds\right\} \quad \text{with} \quad \delta_\xi(t) = -Q_\xi(t) + v_\xi(t).$$

It is clear that $\lim\limits_{t\to\infty} \delta_\xi(t) = 0$. Since $x_\xi(t)$ is a decreasing solution of (3.23), being convex and tending to zero, one has $\delta_\xi(t) \leq -\lambda_0$ for $t \geq T$. On the other hand, from (3.20) and since $|v_\xi(t)| \leq m$, it follows that

$$\delta_\xi(t) \geq -Q_\xi(t) - m \geq -Q_\Lambda(t) - m \geq -R(t) - m,$$

where $R(t)$ is defined by (3.5). This shows that $\Phi\xi \in \Xi$.

ii) $\Phi(\Xi)$ is relatively compact in $C^1[T,\infty)$: This is an immediate consequence of the Arzela-Ascoli lemma applied to the following inequalities holding for all $\xi \in \Xi$ and $t \geq T$:

$$1 \geq \Phi\xi(t) \geq \exp\left\{\int_T^t \frac{\lambda_0 - R(s) - m}{s} ds\right\},$$

$$0 \geq (\Phi\xi)'(t) \geq \Phi\xi(t) \cdot \frac{\lambda_0 + \delta_\xi(t)}{t} \geq \frac{\lambda_0 - R(t) - m}{t}$$

and

$$(\Phi\xi)''(t) = q_\xi(t)\Phi\xi(t) \leq q_\Lambda(t).$$

iii) Φ is a continuous mapping: Let $\{\xi_n\}$ be a sequence in Ξ converging to $\eta \in \Xi$ in $C^1[T,\infty)$. It should be proved that $\{\Phi\xi_n\}$ converges to $\Phi\eta$ in $C^1[T,\infty)$, that is,

$$\Phi\xi_n(t) \to \Phi\eta(t) \quad \text{and} \quad (\Phi\xi_n)'(t) \to (\Phi\eta)'(t) \quad \text{as} \quad n \to \infty, \tag{3.27}$$

uniformly on compact subintervals of $[T,\infty)$. Using (3.24), we have for $t \geq T$

$$|\Phi\xi_n(t) - \Phi\eta(t)| \leq \int_T^t s^{-1}\{|Q_{\xi_n}(s) - Q_\eta(s)| + |v_{\xi_n}(s) - v_\eta(s)|\}ds,$$

and

$$\begin{aligned} |(\Phi\xi_n)'(t) - (\Phi\eta)'(t)| &\leq |\Phi\xi_n(t) - \Phi\eta(t)|t^{-1}\{|\lambda_0 - Q_{\xi_n}(t) + v_{\xi_n}(t)|\} + \\ &\quad +|\Phi\eta(t)|t^{-1}\{|Q_{\xi_n}(t) - Q_\eta(t)| + |v_{\xi_n}(t) - v_\eta(t)|\} \\ &\leq |\Phi\xi_n(t) - \Phi\eta(t)|t^{-1}\{|\lambda_0| + 2m\} \\ &\quad +t^{-1}\{|Q_{\xi_n}(t) - Q_\eta(t)| + |v_{\xi_n}(t) - v_\eta(t)|\}, \end{aligned}$$

and so to establish the desired convergence (3.27) it suffices to show that the two sequences

$$\frac{1}{t}|Q_{\xi_n}(t) - Q_\eta(t)| \quad \text{and} \quad \frac{1}{t}|v_{\xi_n}(t) - v_\eta(t)| \tag{3.28}$$

converge to zero as $n \to \infty$ on any compact subinterval of $[T, \infty)$. The first sequence in (3.28) is easier to deal with, since we have by (3.19)

$$\frac{1}{t}|Q_{\xi_n}(t) - Q_\eta(t)| \le \int_t^\infty q(s)\left|\frac{\xi_n(g(s))}{\xi(s)} - \frac{\eta(g(s))}{\eta(s)}\right| ds, \quad t \ge T, \tag{3.29}$$

which converges to zero as desired by means of the Lebesgue dominated convergence theorem. Turning to the second sequence in (3.28), in order to evaluate $|v_{\xi_n}(t) - v_\eta(t)|$ with the help of (3.25), we first observe that

$$\begin{aligned}
&|\{[v_{\xi_n}(t) - Q_{\xi_n}(t)]^2 - 2\lambda_0 Q_{\xi_n}(t)\} - \{[v_\eta(t) - Q_\eta(t)]^2 - 2\lambda_0 Q_\eta(t)\}| \\
&\le [v_{\xi_n}(t) + v_\eta(t) + Q_{\xi_n}(t) + Q_\eta(t)][|v_{\xi_n}(t) - v_\eta(t)| + |Q_{\xi_n}(t) - Q_\eta(t)|] + \\
&\qquad\qquad + 2|\lambda_0||Q_{\xi_n}(t) - Q_\eta(t)| \\
&\le 4m|v_{\xi_n}(t) - v_\eta(t)| + (4m + 2|\lambda_0|)|Q_{\xi_n}(t) - Q_\eta(t)|.
\end{aligned}$$

Using the above inequality, we obtain

$$|v_{\xi_n}(t) - v_\eta(t)| \le t^{1-2\lambda_0}\int_t^\infty \frac{\theta|v_{\xi_n}(s) - v_\eta(s)| + \rho|Q_{\xi_n}(s) - Q_\eta(s)|}{s^{2(1-\lambda_0)}} ds, \tag{3.30}$$

where $\theta = 4m < 1$ and $\rho = 4m + 2|\lambda_0| < 1 + 2|\lambda_0|$.

Putting

$$w_n(t) = \int_t^\infty s^{2(\lambda_0-1)}\{|v_{\xi_n}(s) - v_\eta(s)|\} ds. \tag{3.31}$$

(3.30) can be transformed into the following differential inequality for $w_n(t)$:

$$(t^\theta w_n(t))' \ge -\frac{\rho}{t^{1-\theta}}\int_t^\infty s^{2(\lambda_0-1)}\{|Q_{\xi_n}(s) - Q_\eta(s)|\} ds. \tag{3.32}$$

We note by (3.31) that $t^\theta w_n(t) \to 0$ as $t \to \infty$. Integrating (3.32) from t to ∞ then yields

$$w_n(t) \le \frac{\rho}{\theta t^\theta}\int_t^\infty s^{2(\lambda_0-1)}\{|Q_{\xi_n}(s) - Q_\eta(s)|\} ds, \tag{3.33}$$

Combining (3.33) with (3.30), we finally obtain the inequality

$$\frac{1}{t}|v_{\xi_n}(t) - v_\eta(t)| \leq \frac{\rho}{t^{\theta+2\lambda_0}} \int_t^\infty s^{2(\lambda_0-1)+\theta}\{|Q_{\xi_n}(s) - Q_\eta(s)|\}ds +$$

$$+\frac{\rho}{t^{2\lambda_0}} \int_t^\infty s^{2(\lambda_0-1)}\{|Q_{\xi_n}(s) - Q_\eta(s)|\}ds$$

$$\leq \frac{\rho}{t^\theta} \int_t^\infty s^{\theta-2}\{|Q_{\xi_n}(s) - Q_\eta(s)|\}ds + \rho \int_t^\infty s^{-2}\{|Q_{\xi_n}(s) - Q_\eta(s)|\}ds,$$

from which it readily follows that $|v_{\xi_n}(t) - v_\eta(t)|/t \to 0$ as $n \to \infty$ uniformly on compact subintervals of $[T, \infty)$. Thus the mapping Φ defined by (3.26) is continuous in the topology of $C^1[T, \infty)$.

Therefore, applying the Schauder-Tychonoff fixed point theorem to Φ, we conclude that there exists a $\xi \in \Xi$ such that $\xi = \Phi\xi$, which clearly means that $\xi(t)$ is a regularly varying function of index λ_0 and satisfies the ordinary differential equation

$$\xi''(t) = q_\xi(t)\xi(t),$$

which, in view of (3.18), reduces to the functional differential equation

$$\xi''(t) = q(t)\xi(g(t)).$$

This establishes the existence of a desired regularly varying solution for the equation (A), and the proof is completed for the root λ_0.

To prove the case $i = 1$, we again put $\tau = g(T)$ and define Ξ to be the set of those functions $\xi \in C[\tau, \infty) \cap C^1[T, \infty)$ which this time satisfy

$$\xi(t) = 1, \quad \tau \leq t \leq T, \tag{3.34}$$

and

$$\xi(t) = \exp\left\{\int_T^t \frac{\lambda_1 + \delta_\xi(s)}{s} ds\right\}, \quad t \geq T, \tag{3.35}$$

for some continuous functions $\delta_\xi(t)$ such that for all $\xi \in \Xi$.

$$\lim_{t\to\infty} \delta_\xi(t) = 0 \tag{3.36}$$

and

$$0 \leq \delta_\xi(t) \leq \lambda_1, \quad t \geq T. \tag{3.37}$$

Notice that all the members of Ξ are nondecreasing regularly varying functions of index $\lambda_1 > 0$, i.e., $\Xi \subset RV(\lambda_1)$ and that Ξ can be regarded as a subset of locally convex space $C^1[T,\infty)$ endowed with the topology of uniform convergence of functions and their first derivatives on compact subintervals of $[T,\infty)$. Moreover, exactly as in the case $i = 0$ one shows that Ξ is a closed convex subset of $C^1[T,\infty)$.

The auxillary function here replacing Λ from the case $i = 0$, is

$$\rho(t) = \left(\frac{t}{a}\right)^{2\lambda_1}, \quad t > a, \tag{3.38}$$

satisfying by (1.3),

$$\frac{\rho(g(t))}{\rho(t)} = \left(\frac{g(t)}{t}\right)^{2\lambda_1} \to 1 \quad \text{as} \quad t \to \infty. \tag{3.39}$$

From (3.35) and (3.39) one readily derives that for every $\xi \in \Xi$ one has

$$\frac{\xi(g(t))}{\xi(t)} \to 1, \quad \text{as} \quad t \to \infty \tag{3.40}$$

and

$$\frac{\rho(g(t))}{\rho(t)} \leq \frac{\xi(g(t))}{\xi(t)}, \quad t \geq T. \tag{3.41}$$

Now we define for each $\xi \in \Xi$ and for ρ given by (3.38)

$$q_\xi(t) = q(t)\frac{\xi(g(t))}{\xi(t)}, \quad Q_\xi(t) = t\int_t^\infty q_\xi(s)ds - c, \tag{3.42}$$

$$q_\rho(t) = q(t)\frac{\rho(g(t))}{\rho(t)}, \quad Q_\rho(t) = t\int_t^\infty q_\rho(s)ds - c. \tag{3.43}$$

Due to (3.41) and since $\xi(t)$ is increasing, one has for $t \geq T$

$$Q_\rho(t) \leq Q_\xi(t) \leq Q(t)$$

and also, because of (1.4) and (3.40) (implying $Q_\rho(t) \to 0$, as $t \to \infty$) that for all $\xi \in \Xi$ and $t \geq T$

$$|Q_\xi(t)| \leq m^2. \tag{3.44}$$

Once more, let us consider the family of linear ordinary differential equations

$$x'' = q_\xi(t)x, \quad \xi \in \Xi. \tag{3.45}$$

Because of (3.44) we are able to apply Lemma 2.1 to (3.45), concluding that each of its members possesses an $RV(\lambda_1)$-solution $x_\xi(t)$ expressed in the form

$$x_\xi(t) = \exp\left\{\int_T^t \frac{\lambda_1 - Q_\xi(s) + v_\xi(s)}{s} ds\right\}, \quad t \geq T, \tag{3.46}$$

where $v_\xi(t)$ satisfies the integrasl equation

$$v_\xi(t) = t^{1-2\lambda_1} \int_T^t s^{2(\lambda_1 - 1)}\{2\lambda_1 Q_\xi(s) - (v_\xi(s) - Q_\xi(s))^2\}ds, \tag{3.47}$$

and

$$|v_\xi(t)| \leq m \quad \text{for} \quad t \geq T.$$

Let Ψ denote the mapping which associate to each $\xi \in \Xi$ the function $\Psi\xi$ defined by

$$\Psi\xi(t) = 1 \quad \text{for} \quad \tau \leq t \leq T, \quad \Psi\xi(t) = x_\xi(t) \quad \text{for} \quad t \geq T, \tag{3.48}$$

where $x_\xi(t)$ is given by (3.46) and (3.47). It is now a matter of routine to repeat the procedure used in the case of λ_0 to establish the conditions on Ψ needed for the Schauder-Tychonoff theorem which guarantees the existence of a $\xi \in \Xi$ such that $\xi = \Psi\xi$, which according to (3.46) and (3.48) means that $\xi(t)$ is a regularly varying solution of index λ_1 satisfying (3.45) and so, in view of (3.42), the equation (A). This completes the proof of the sufficiency part of Theorem A.

Necessity. (for both cases $i = 0, 1$). Assume $x(t) = t^{\lambda_i} L_i(t)$ where L_i are some normalized SV functions (hence positive by definition).

Put

$$q_x(t) = q(t)\frac{x(g(t))}{x(t)}$$

and write equation (A) as

$$\left(\frac{x'(t)}{x(t)}\right)' + \left(\frac{tx'(t)}{x(t)}\right)^2 \frac{1}{t^2} = q_x(t),$$

integrate over (t, ∞), multiply throughout by t to obtain

$$\frac{-tx'(t)}{x(t)} + \int_t^\infty \left(\frac{sx'(s)}{x(s)}\right)^2 \frac{ds}{s^2} = t\int_t^\infty q_x(s)ds.$$

A direct calculation, using (1.1) with $c(t) = c_0$, gives

$$\frac{tx'(t)}{x(t)} \to \lambda_i \quad \text{as} \quad t \to \infty$$

so that

$$\lambda_i^2 - \lambda_i = \lim_{t\to\infty} t\int_t^\infty q_x(s)ds$$

which gives

$$\lim_{t\to\infty} t\int_t^\infty q_x(s)ds = c,$$

since λ_i satisfies (1.2).

This, however, implies (1.4) since $\frac{x(g(t))}{x(t)} \to 1$ as $t \to \infty$ due to the representation (2.1) in which $Q(s)$ and $v_i(s)$ tend to zero, and using (1.3).

P r o o f of **Theorem B.**

Write equation (A) as

$$x''(t) = q_x(t)x(t)$$

where $x(t)$ is positive. Such solutions exist in virtue of Theorem A. Now apply Theorem 2.7 in [5] asserting that there exist solutions $x_i(t)$ such that

$$x_i(t) \sim t^{\lambda_i} \quad \text{as} \quad t \to \infty$$

if

$$I := \int_a^\infty t|q_x(t) - \frac{c}{t^2}|dt < \infty.$$

But, because of (1.5) and (2.1)

$$t\left|q_x(t) - \frac{c}{t^2}\right| \le t\left|q(t) - \frac{c}{t^2}\right| + kq(t),$$

for some $k > 0$, so that I converges due to (1.6) and the integrability condition on $q(t)$.

Example 3.1. Consider the retarded differential equation

$$x''(t) = \left(\frac{c}{t^2} + \frac{d}{t^2\varphi(t)}\right) x(t - \psi(t)), \tag{3.49}$$

where c and d are positive constants, and $\varphi(t)$ is a continuous slowly varying function on $[0, \infty)$ such that $\lim\limits_{t\to\infty} \varphi(t) = \infty$, and $\psi(t)$ is a continuous regularly varying function of index $\mu < 1$ defined on $[0, \infty)$.

Since $\psi(t)/t$ decreases to 0 as $t \to \infty$, $g(t) = t - \psi(t)$ is eventually increasing and satisfies $\lim\limits_{t\to\infty} \frac{g(t)}{t} = 1$. On the other hand, by the Karamata theorem for integrals (see [5, Proposition 1]), we have

$$\int_t^\infty \frac{ds}{s^2\varphi(s)} ds \sim \frac{1}{t\varphi(t)} \quad \text{as} \quad t \to \infty,$$

which implies that $q(t) = \frac{c}{t^2} + \frac{d}{t^2\varphi(t)}$ satisfies $\lim\limits_{t\to\infty} t \int_t^\infty q(s)ds = c$. Therefore, from Theorem A we see that equation (3.49) possesses regularly varying solutions of indices λ_i.

Example 3.2. Let $g(t)$ be a retarded argument satisfying (1.3) and consider the functional differential equation

$$x''(t) = \frac{2\log t + 3}{g(t)^2 \log g(t)} x(g(t)). \tag{3.50}$$

Since $g(t) \sim t$, again by Karamata theorem one has

$$t \int_t^\infty q(s)ds \sim \frac{2\log t + 3}{\log t}, \quad \text{as} \quad t \to \infty,$$

so that $t \int_t^\infty q(s)ds \to 2$ as $t \to \infty$. The quadratic equation $\lambda^2 - \lambda - 2 = 0$ has a positive root $\lambda_1 = 2$. Therefore, Theorem A ensures the existence of an $RV(2)$-solution of equation (2.38). One such solution is $x(t) = t^2 \log t$.

4. *The special case* $c = 0$

In the preceding section we have established the existence of regularly varying solutions under the assumption that the constant c is positive. What will happen if c is zero? In that case the quadratic equation (1.2) has the roots $\lambda_0 = 0$ and $\lambda_1 = 1$.

When $\lambda_0 = 0$, it is shown in [4] that equation (A) has slowly varying solutions both for the retarded case ($g(t) < t$) and for the advanced one ($g(t) > t$) under the conditions $g(t)$ that are much more general then (1.3).

The purpose of this section is to show that, when $\lambda_1 = 1$, the existence of RV(1) solutions can be established for quite and extensive class of retarded functional differential equations of the form (A). Namely, we will prove the following

Theorem 4.1. *Equation (A) possesses an RV(1)-solution, i.e., of the form $x(t) = tL(t)$, where $L(t)$ is some normalized slowly varying solution for every retarded argument $g(t)$ satisfying*

$$g(t) < t, \quad g(t) \to \infty, \quad \textit{as} \quad t \to \infty \tag{4.1}$$

if

$$Q(t) := t \int_t^\infty q(s)ds \to 0, \quad \textit{as} \quad t \to \infty. \tag{4.2}$$

P r o o f of **Theorem 4.1.** Let $m \in (0, 1/4)$ be any fixed constant and choose $T > \max\{a, 1\}$ so that $g(T) \geq a$ and

$$Q(t) \leq m^2, \quad t \geq T. \tag{4.3}$$

By [5, Theorem 1.1] the linear ordinary differential equation

$$x'' = q(t)x \tag{4.4}$$

possesses an RV(1)-solution $X(t)$ having the expression

$$X(t) = \frac{16}{15} T \exp\left\{\int_T^t \frac{1 - Q(s) + v(s)}{s} ds\right\}, \tag{4.5}$$

where $v(t)$ is a solution of the integral equation

$$v(t) = \frac{1}{t} \int_T^t \left\{2Q(s) - (Q(s) - v(s))^2\right\} ds. \tag{4.6}$$

Relations (4.5), (4.6) and the following ones are valid for $t \geq T$. Hence the adjective will mostly be omitted. It should be noticed that $X(t)$ satisfies the inequality

$$t \leq X(t) \leq \frac{16}{15} T \exp\left\{\int_T^t \frac{1+v^*(s)}{s} ds\right\}, \tag{4.7}$$

where $v^*(t)$ solves the integral equation

$$v^*(t) = \frac{1}{t} \int_T^t 2Q(s) \{v^*(s)+1\} \, ds. \tag{4.8}$$

Since $X'(t)$ is increasing and

$$X(T) = \frac{16}{15} T > T, \qquad X'(T) = \frac{16}{15}(1 - Q(T)) > 1,$$

we have $X(t) > t$ for $t \geq T$, which is the left inequality in (4.7). The proof of the right inequality proceeds as follows. First note that the solution $v^*(t)$ of (4.8) is a fixed point of the integral operator

$$\mathcal{G}^* v(t) = \frac{1}{t} \int_T^t 2Q(s) \{v(s)+1\} \, ds, \qquad t \geq T,$$

which, as easily seen, is a contraction mapping on the set

$$V^* = \{v \in C_0[T, \infty) : \ 0 \leq v(t) \leq m, \qquad t \geq T\}.$$

Consequently, $v^*(t)$ is obtained as the limit of the sequence $\{v_n^*(t)\}$:

$$v_n^*(t) = \mathcal{G}^* v_{n-1}^*(t), \quad n = 1, 2, \cdots, \quad v_0^*(t) = 0.$$

This fact enables us to compare $v^*(t)$ with the solution $v(t)$ of (4.6), which is constructed as the limit of the sequence $\{v_n(t)\}$:

$$v_n(t) = \mathcal{G} v_{n-1}(t), \quad n = 1, 2, \cdots, \quad v_0(t) = 0.$$

In fact, it can be shown by an inductive argument that $v_n(t) \leq v_n^*(t)$ for all n and $t \geq T$. This implies that $v(t) \leq v^*(t)$ for $t \geq T$, and hence the right inequality in (4.7).

We denote by Ξ the set of all continuous nondecreasing functions on $[g(T), \infty)$ that satisfy $\xi(t) = 1$ for $g(T) \leq t \leq T$ and

$$t \leq \xi(t) \leq \frac{16}{15} T \exp\left\{\int_T^t \frac{1+v^*(s)}{s} ds\right\}, \qquad t \geq T. \tag{4.9}$$

Clearly, Ξ is a closed convex subset of the locally convex space $C[g(T),\infty)$ with the topology of uniform convergence on compact subintervals of $[g(T),\infty)$. For any $\xi \in \Xi$ define

$$q_\xi(t) = q(t)\frac{\xi(g(t))}{\xi(t)}, \qquad Q_\xi(t) = t\int_t^\infty q_\xi(s)ds. \tag{4.10}$$

Since $\xi(t)$ is nondecreasing and $g(t) < t$, we have $\xi(g(t))/\xi(t) \le 1$, and hence

$$q_\xi(t) \le q(t), \qquad Q_\xi(t) \le Q(t), \qquad t \ge T, \tag{4.11}$$

which implies that

$$Q_\xi(t) \le m^2, \qquad t \ge T, \quad \text{for all} \;\; \xi \in \Xi. \tag{4.12}$$

Let us consider the infinite family of linear ordinary differential equations

$$x'' = q_\xi(t)x, \qquad \xi \in \Xi. \tag{4.13}$$

Because of (4.12) each of the equations (4.13) has an RV(1)-solution

$$X_\xi(t) = \frac{16}{15}T\exp\left\{\int_T^t \frac{1 - Q_\xi(s) + v_\xi(s)}{s}ds\right\}, \tag{4.14}$$

where $v_\xi(t)$ satisfies

$$v_\xi(t) = \frac{1}{t}\int_T^t \left\{2Q_\xi(s) - (v_\xi(s) - Q_\xi(s))^2\right\}ds. \tag{4.15}$$

Let $\Psi : \Xi \to C[g(T),\infty)$ be the mapping which assigns to each $\xi \in \Xi$ the function $\Psi\xi$ defined by

$$\Psi\xi(t) = 1 \quad \text{for} \;\; g(T) \le t \le T, \qquad \Psi\xi(t) = X_\xi(t) \quad \text{for} \;\; t \ge T. \tag{4.16}$$

Much as before, we will show that Ψ is continuous and maps Ξ into a relatively compact subset of Ξ.

(i) If $\xi \in \Xi$ then, proceeding exactly as in deriving (4.7), we see that

$$t \le X_\xi(t) \le \frac{16}{15}T\exp\left\{\int_T^t \frac{1 + v^*(s)}{s}ds\right\}, \tag{4.17}$$

wihch gurantees that $\Psi\xi \in \Xi$.

(ii) Since $\Psi(\Xi) \subset \Xi$, $\Psi(\Xi)$ is locally uniformly bounded on $[g(T),\infty)$. For any $\xi \in \Xi$ we have for $t \ge T$

$$0 \le (\Psi\xi)'(t) = X_\xi(t)\frac{1 - Q_\xi(t) + v_\xi(t)}{t} \le \frac{16}{15}\{1+v^*(t)\}\exp\left\{\int_T^t \frac{1 + v^*(s)}{s}ds\right\}$$

which implies that $\Psi(\Xi)$ is locally equi-continuous on $[g(T), \infty)$. The relative compactness of $\Psi(\Xi)$ then follows from the Arzela-Ascoli theorem.

(iii) Let $\{\xi_n\}$ be a sequence in Ξ converging to ξ as $n \to \infty$. This means that the sequence of functions $\{\xi_n(t)\}$ converges to $\xi(t)$ uniformly on any compact subinterval of $[g(T), \infty)$. It suffices to verify that $\{\Psi\xi_n(t)\}$ converges to $\Psi\xi(t)$ on compact subintervals of $[T, \infty)$. Using (4.14) and (4.16), we have

$$\begin{aligned}
&|\Psi\xi_n(t) - \Psi\xi(t)| = |X_{\xi_n}(t) - X_\xi(t)| \\
&\le \frac{16}{15}T\left|\exp\left\{\int_T^t \frac{1 - Q_{\xi_n}(s) + v_{\xi_n}(s)}{s}ds\right\} - \exp\left\{\int_T^t \frac{1 - Q_\xi(s) + v_\xi(s)}{s}ds\right\}\right| \\
&\le \tfrac{16}{15}T\exp\left\{\textstyle\int_T^t s^{-1}[1 + v^*(s)]ds\right\}\textstyle\int_T^t s^{-1}(|v_{\xi_n}(s) - v_\xi(s)| + |Q_{\xi_n}(s) - Q_\xi(s)|)ds,
\end{aligned}$$

and so we need only to prove that

$$\frac{1}{t}|v_{\xi_n}(t) - v_\xi(t)| \to 0 \quad \text{and} \quad \frac{1}{t}|Q_{\xi_n}(t) - Q_\xi(t)| \to 0 \quad \text{as } n \to \infty \tag{4.18}$$

uniformly on any compact subinterval of $[T, \infty)$. The desired convergence of the second sequence in (4.18) follows immediately from the Lebesgue dominated convergence theorem applied to the inequality

$$\frac{1}{t}|Q_{\xi_n}(t) - Q_\xi(t)| \le \int_t^\infty q(s)\left|\frac{\xi_n(g(s))}{\xi_n(s)} - \frac{\xi(g(s))}{\xi(s)}\right|ds.$$

To deal with the first sequence in (4.18), we first use (4.6) to obtatin

$$|v_{\xi_n}(t) - v_\xi(t)| \le \frac{\theta}{t}\int_T^t |Q_{\xi_n}(s) - Q_\xi(s)|ds + \frac{\sigma}{t}\int_T^t |v_{\xi_n}(s) - v_\xi(s)|ds, \tag{4.19}$$

where $\theta = 2(m^2 + m) \le 5/8$ and $\sigma = 2(m^2 + m + 1) \le 21/8$. Putting

$$z(t) = \int_T^t |v_{\xi_n}(s) - v_\xi(s)|ds, \tag{4.20}$$

we transform (4.19) into

$$\left(\frac{z(t)}{t^\theta}\right)' \le \frac{\sigma}{t^{\theta+1}}\int_T^t |Q_{\xi_n}(s) - Q_\xi(s)|ds,$$

from which, after integration on $[T, t]$, it follows that

$$z(t) \le \frac{\sigma}{\theta}t^\theta \int_T^t s^{-\theta}(|Q_{\xi_n}(s) - Q_\xi(s)|)ds. \tag{4.21}$$

Combining (4.21) with (4.19) yields

$$\begin{aligned}\frac{1}{t}|v_{\xi_n}(t)-v_\xi(t)| &\le \frac{\sigma}{t^2}\int_T^t |Q_{\xi_n}(s)-Q_\xi(s)|ds + \frac{\sigma}{t^{2-\theta}}\int_T^t |Q_{\xi_n}(s)-Q_\xi(s)|ds \\ &\le \frac{2\sigma}{t}\int_T^t |Q_{\xi_n}(s)-Q_\xi(s)|ds, \qquad t \ge T,\end{aligned} \tag{4.22}$$

which clearly ensures the uniform convergence of the first sequence (4.18).

Thus we are able to apply the Schauder-Tychonoff fixed point theorem to conclude that there exists $\xi \in \Xi$ such that $\xi = \Psi\xi$, that is, $\xi(t) = X_\xi(t)$ for $t \ge T$. This shows that $\xi(t)$ satisfies

$$\xi''(t) = q_\xi(t)\xi(t) \quad \text{or equivalently} \quad \xi''(t) = q(t)\xi(g(t)) \quad \text{on} \quad [T,\infty),$$

establishing the existence of an RV(1)-solution for equation (A). This completes the proof.

Example 4.1. Consider the retarded differential equation

$$x''(t) = \frac{\lambda}{t^2\log(\mu t)}x(g(t)), \qquad t \ge e, \tag{4.23}$$

where λ and μ are positive constants. The function $q(t) = \frac{\lambda}{t^2\log(\mu t)}$ satisfies (4.2), and so by Theorem 4.1 there exists an RV(1)-solution of (4.23) for any retarded argument $g(t)$. If in particular $g(t) = \mu t$ and $\frac{1}{\lambda} = \mu$ with $0 < \mu < 1$, one such solution is $x(t) = t\log t$.

Example 4.2. Theorem 4.1 does not apply to the retarded equation

$$x''(t) = \frac{1}{\theta t^{1+\theta}\log t}x(t^\theta), \qquad t \ge e, \tag{4.24}$$

where $0 < \theta < 1$, since the function $q(t) = \frac{1}{\theta t^{1+\theta}\log t}$ does not satisfy (4.2). Yet equation (4.24) has an RV(1)-solution $x(t) = t\log t$.

This example shows that for the rather general class of functions $g(t)$ satisfying (4.1), condition (4.2) is not a necessary one for the existence of an $RV(1)$ solution. However, there holds

Theorem 4.2. *If in addition to (4.1), $g(t)$ satisfies*

$$\limsup_{t\to\infty} \frac{t}{g(t)} < \infty. \tag{4.25}$$

Then, equation (A) *possesses an* RV(1)- *solution i.e. of the form* $x(t) = tL(t)$ *where* $L(t)$ *is some normalized SV function if and only if* (4.2) *is satisfied.*

P r o o f of **Theorem 4.2.** We need only to prove the "only if" part of the theorem. Let $x(t)$ be an RV(1)-solution of (A). By (1.1), $x(t)$ has the representation

$$x(t) = ct\exp\left\{\int_T^t \frac{\delta(s)}{s}ds\right\}, \qquad t \geq T, \tag{4.26}$$

for some $T > a$, some constant $c > 0$ and some continuous function $\delta(t) \geq 0$ such that $\delta(t) \to 0$, as $t \to \infty$.

Write equation (A) as

$$x'' = q_x(t)x(t)$$

where

$$q_x(t) = q(t)\frac{x(g(t))}{x(t)} \quad \text{or} \quad q(t) = q_x(t)\frac{x(t)}{x(g(t))}.$$

Then again by [5. Theorem 1.1]

$$t\int_t^\infty q_x(s)ds \to 0, \quad \text{as} \quad t \to \infty. \tag{4.27}$$

By (4.26) and (4.27) for some $k, m > 0$ and any $\varepsilon > 0$

$$\begin{aligned} 0 < \frac{x(t)}{x(g(t))} &= \frac{t}{g(t)}\exp\left(\int_{g(t)}^t \frac{\delta(s)}{s}ds\right) \\ &\leq m\exp\varepsilon\log\frac{t}{g(t)} < k, \end{aligned}$$

whence $q(t) \leq kq_x(t)$ and condition (4.2) follows.

Remark 4.1. Recently T. Tanigawa proved in [6] a result for the half-linear retarded functional differential equation which (among other things) contains our Theorem 4.2 but not its sufficency part in the full generality of our Theorem 4.1.

REFERENCES

[1] N. H. B i n g h a m, C. M. G o l d i e, J. L. T e u g e l s, *Regular Variation*, Encyclopedia of Mathematics and its Applications, Vol. **27**, Cambridge University Press, 1987.

[2] H. C. H o w a r d, V. M a r i ć, *Regularity and nonoscillation of solutions of second order lilnear differential equations*, Bull. T. CXIV de Acad. Serbe Sci. et Arts, Classe Sci. Nat. Sci. Math. **22** (1997), 85–98.

[3] J. J a r o š, T. K u s a n o, *Remarks on the existence of regularly varying solutions for second order linear differential equations*, Publ. Inst. Math. (Beograd) **72** (86) (2002), 113–118.

[4] T. K u s a n o, V. M a r i ć, *On a class of functional differential equations having slowly varying solutions*, Publ. Inst. Math. (Beograd), **82** (94) (2006), 207–217.

[5] V. M a r i ć, *Regular Variation and Differential Equations*, Lecture Notes in Mathematics, Vol. 1726, Springer, Berlin, 2000.

[6] T. T a n i g a w a, *Regularly varying solutions of half-linear functional differential equations with retarded arguments*, (to appear).

Department of Applied Mathematics
Faculty of Science
Fukuoka University
Fukuoka, 841-0180, Japan
tkusano@cis.fukuoka-u.ac.jp

Serbian Academy of Sciences and Arts
Kneza Mihaila 35
11000 Beograd, Serbia
vojam@uns.ns.ac.yu