Bulletin T.CXXXVII de l'Académie serbe des sciences et des arts − 2007
Classe des Sciences mathématiques et naturelles
Sciences mathématiques, № 33

ON RICCI H-PSEUDOSYMMETRIC H-HYPERSURFACES OF SOME ANTI-KÄHLER MANIFOLDS

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(Presented at the 1st Meeting, held on February 29, 2008)

A b s t r a c t. *We adopt the notion of the pseudosymmetry and Ricci pseudosymmetry to the case of the anti-Kähler manifolds and then we extend the results of the paper [1] to the h-hypersurfaces of the anti-Kähler manifolds of the constant totally real sectional curvatures.*

AMS Mathematics Subject Classification (2000): 53C56
Key Words: anti-Kähler manifold, h-hypersurface, h-pseudosymmetry, Ricci h-pseudosymmetry

1. *The object of the paper*

Let (M, g) be a semi-Riemannian manifold of dimension ≥ 3. The manifold (M, g) is locally symmetric if $\nabla R = 0$, on M, where ∇ is its Levi-Civita connection and R the curvature tensor. The proper generalization of locally symmetric manifolds form semi-symmetric manifolds. They are characterized by the condition

$$R \cdot R = 0,$$

which holds on M, where R acts as a derivation. Some of the investigations of such manifolds gave rise to the next generalization, namely to the pseudosymmetric manifolds, i.e., manifolds satisfying on some set $\mathcal{U} \subset M$ the condition

$$R \cdot R = \mathcal{L}\, Q(g, R), \tag{1.1}$$

where $\mathcal{L}$ is a function on U and Q is a special operator (see section 2).

A manifold (M, g), dim $M \geq 3$, is said to be Ricci pseudosymmetric, resp. Ricci semi-symmetric, if

$$R \cdot \rho = \mathcal{L}\, Q(g, \rho), \quad \text{resp. } R \cdot \rho = 0, \tag{1.2}$$

holds on the appropiate set $\mathcal{U} \subset M$, and ρ is the Ricci tensor.

For a survey of results on different aspects of pseudosymmetric manifolds, we refer to [3]; see also [2], [10], [11], [14]. Among other problems there were studied the extrinsic characterizations of Ricci pseudosymmetric hypersurfaces of semi-Riemannian spaces of constant curvature in terms of the shape operator. Namely, in [1] (see Theorems 3.1 and 3.2) the following result is proved

Let M be a hypersurface of a semi-Reimannian space of constant curvature and dimension $n \geq 3$. Then M is Ricci pseudosymmetric if and only if at every point $p \in M$, the second fundamental form h satisfies one of the following conditions

$$h^2 = \alpha h + \beta q, \qquad \alpha, \beta \in R, \tag{1.3}$$

or

$$h^3 = \mathrm{tr}\, h\, h^2 + \lambda h, \qquad \lambda \in R.$$

In practicular, for semi-Euclidean space, the previous result imply

A hypersurface M of semi-Euclidean space of dimension $n \geq 3$ is Ricci pseudosymmetric if and only if for every point $p \in M$ the tensor $R \cdot \rho$ vanishes at p, or (1.3) holds.

In section 4 of the present paper, we adopt the notion of pseudosymmetry and Ricci pseudosymmetry to the complex structure of the anti-Kähler manifolds and then we extend the above theorems for the h-hypersurface of anti-Kähler manifold of constant totally real sectional curvature. To do this, we use two formulas proved in section 3, valid for h-hypersurface of the anti-Kähler manifold of constant totally real sectional curvature. In section 2, we explain notations used in the paper.

2. *Preliminaries*

Let $\widetilde{M}$ be a connected differentiable manifold endoved with pseudo-Riemannian metric G and a $(1,1)$ tensor field F such that, with respect to the local coordinates, holds

$$F_B^A F_C^B = -\delta_C^A, \quad F_A^E F_B^D G_{ED} = -G_{AB}, \quad \widetilde{\nabla}_D F_B^A = 0. \tag{2.1}$$

Here $\widetilde{\nabla}$ is the Levi-Civita connection of $(\widetilde{M}, G)$ and $A, B, C, D \in \{1, 2, \ldots, 2m\}$, $2m = \dim \widetilde{M}$. The manifold $(\widetilde{M}, G, F)$ is said to be anti-Kähler manifold [12]. In some papers $(\widetilde{M}, G, F)$ is named B-manifold ([6],[7],[13]) and in some others - the Kähler manifold with the Norden metric ([8],[9]).

The manifold $(\widetilde{M}, G, F)$ is orientable and evendimensional. The metric G of such a manifold is indefinite and the signature is (m, m). Also, tr $F = 0$.

We denote by

$\widetilde{R}_{ABCD}$ - the Riemannian curvature tensor,

$\widetilde{\rho}_{AB} = \widetilde{R}^C{}_{ABC}$ - the Ricci tensor,

$\widetilde{\overset{*}{\rho}}_{AB} = F_A^D \widetilde{\rho}_{DB}$ - the second Ricci tensor,

$\widetilde{\kappa} = G^{AB} \widetilde{\rho}_{AB}$ - the scalar curvature,

$\widetilde{\overset{*}{\kappa}} = G^{AB} \widetilde{\overset{*}{\rho}}_{AB}$ - the second scalar curvature.

Since $\widetilde{\nabla} F = 0$, the curvature tensor and the Ricci tensors satisfy

$$\left.\begin{aligned} F_A^L F_B^M \widetilde{R}_{LMCD} &= -\widetilde{R}_{ABCD}\,, \\ F_A^L F_B^M \widetilde{\rho}_{LM} &= -\widetilde{\rho}_{AB}\,, \\ F_A^L F_B^M \widetilde{\overset{*}{\rho}}_{LM} &= -\widetilde{\overset{*}{\rho}}_{AB}\,. \end{aligned}\right\} \tag{2.2}$$

The manifold $(\widetilde{M}, G, F)$ is of pointwise constant totally real sectional curvature if at $p \in M$, ([6], [7]):

$$\begin{aligned} \widetilde{R}_{ABCD} = &\\ \frac{\widetilde{\kappa}(p)}{4m(m-1)}&(G_{AD}G_{BC} - G_{AC}G_{BD} - F_A^L G_{LD} F_B^M G_{MC} + F_A^L G_{LC} F_B^M G_{MD}) \\ -\frac{\widetilde{\overset{*}{\kappa}}(p)}{4m(m-1)}&(G_{AD}F_B^L G_{LC} + G_{BC}F_A^L G_{LD} - G_{AC}F_B^L G_{LD} - G_{BD}F_A^L G_{LC}). \end{aligned} \tag{2.3}$$

If $m \geq 3$, both functions $\widetilde{\kappa}$ and $\widetilde{\overset{*}{\kappa}}$ are constants.

Now, we consider a differentiable submanifold M of $\widetilde{M}$, $\dim M = 2n$, $n = m - 1$. Suppose that M is expressed in each neighbourhood $\widetilde{U}$ of $\widetilde{M}$ by the equations

$$x^A = x^A(u^a) \ ,$$

where x^A are the local coordinates of $\widetilde{M}$ in $\widetilde{U}$ and u^a are the local coordinates in $U = \widetilde{U} \bigcap M$. Lowercase Latin indices $a, b, c, \ldots, i, j, k, \ldots$ run over the range $\{1, 2, \ldots, 2n\}$. M is said to be a <u>h-hypersurface</u> (<u>holomorphic hypersurface</u>) of $\widetilde{M}$ if the restriction g of G on M has the maximal rank and the complex structure F leaves invariant the tangent space of M at each point $p \in M$. F induces on M the complex structure f such that (M, g, f) itself is an anti-Kähler manifold [4]. Similarly to (2.1) and (2.2), we have

$$f_i^a f_a^j = -\delta_i^j, \quad f_i^a f_j^b g_{ab} = -g_{ij}, \quad \nabla_i f_j^k = 0 \ ,$$

$$\left.\begin{array}{ll} f_i^a f_j^b R_{ablm} = -R_{ijlm}, & \overset{*}{\rho}_{ij} = f_i^a \rho_{aj} \ , \\ f_i^a f_j^b \rho_{ab} = -\rho_{ij} \ , & f_i^a f_j^b \ \overset{*}{\rho}_{ab} = - \ \overset{*}{\rho}_{ij}, \end{array}\right\} \tag{2.4}$$

where ∇ is the Levi-Civita connection with respect to the metric g, and R_{ijlm}, ρ_{ij} and $\overset{*}{\rho}_{ij}$ denote the local components of the Riemannian curvature tensor, Ricci tensor and the second Ricci tensor, respectively. We denote by κ and $\overset{*}{\kappa}$ the scalar curvature and the second scalar curvature of (M, g, f).

Because F leaves invariant the tangent space of M, it leaves invariant the normal space, too. There exist locally vector fields $N_{1|}$ and $N_{2|}$ normal to M, such that ([4]):

$$G_{AB}{N_{1|}}^A {N_{1|}}^B = -G_{AB}{N_{2|}}^A {N_{2|}}^B = 1, \quad G_{AB}{N_{1|}}^A {N_{2|}}^B = 0,$$

$$F_B^A {N_{1|}}^B = -{N_{2|}}^A, \qquad F_B^A {N_{2|}}^B = {N_{1|}}^A.$$

Denoting by h and k the second fundamental forms corresponding to $N_{1|}$ and $N_{2|}$ respectively, we have

$$h_{ij} = f_i^a k_{aj}, \qquad k_{ij} = -f_i^a h_{aj} \ . \tag{2.5}$$

Also, we shall use

$$h^2{}_{ij} = h_i{}^a h_{aj}, \qquad h^3{}_{ij} = h_i{}^a h^2{}_{aj}.$$

It is easy to see that the following conditions are satisfied

$$\left.\begin{aligned}
&f_i^a f_j^b h_{ab} = -h_{ij}, \qquad f_i^a f_j^b k_{ab} = -k_{ij}\ ,\\
&f_i^a h_{aj} = f_j^a h_{ai}, \qquad f_i^a k_{aj} = f_j^a k_{ia},\\
&h^2_{ij} = h^2_{ji}, \quad f_i^a f_j^b h^2_{ab} = -h^2_{ij}, \quad f_i^a h^2_{aj} = f_j^a h^2_{ia},\\
&h^3_{ij} = h^3_{ji}, \quad f_i^a f_j^b h^3_{ab} = -h^3_{ij}, \quad f_i^a h^3_{aj} = f_j^a h^3_{ia}.
\end{aligned}\right\} \tag{2.6}$$

Let at $p \in M$, A and D be two symmetric $(0,2)$ tensors and B the curvature like tensor, satisfying

$$f_i^a f_j^b A_{ab} = -A_{ij}, \qquad f_i^a f_j^b D_{ab} = -D_{ij}\ , \tag{2.7}$$

$$f_i^a f_j^b B_{ablm} = -B_{ijlm} \tag{2.8}$$

Let T be a $(0,4)$ tensor. We define the tensors $B \cdot A$, $B \cdot T$, $Q(A,D)$, $Q(A,B)$ by the formulas

$$(B \cdot A)_{rsij} = A_{aj} B^a{}_{irs} + A_{ia} B^a{}_{jrs}, \tag{2.9}$$

$$(B \cdot T)_{rsijlm} = T_{ajlm} B^a{}_{irs} + T_{ialm} B^a{}_{jrs} + T_{ijam} B^a{}_{lrs} + T_{ijla} B^a{}_{mrs}, \tag{2.10}$$

$$\begin{aligned}
Q(A,D)_{rsij} = {}& A_{ri} D_{sj} + A_{rj} D_{si} - A_{si} D_{rj} - A_{sj} D_{ri}\\
& - f_r^a f_s^b (A_{ai} D_{bj} + A_{aj} D_{bi} - A_{bi} D_{aj} - A_{bj} D_{ai}),
\end{aligned} \tag{2.11}$$

$$\begin{aligned}
Q(A,B)_{rsijlm} = {}& A_{ri} B_{sjlm} + A_{rj} B_{islm} + A_{rl} B_{ijsm} + A_{rm} B_{ijls}\\
& - A_{si} B_{rjlm} - A_{sj} B_{irlm} - A_{sl} B_{ijrm} - A_{sm} B_{ijlr}\\
& - f_r^a f_s^b \left(A_{ai} B_{bjlm} + A_{aj} B_{iblm} + A_{al} B_{ijbm} + A_{am} B_{ijlb}\right.\\
& \left. - A_{bi} B_{ajlm} - A_{bj} B_{ialm} - A_{bl} B_{ijam} - A_{bm} B_{ijla}\right).
\end{aligned} \tag{2.12}$$

Remark. The operator Q of a semi-Riemannian manifold (M,g) is defined in the following way (e.g. see [1],[2],[3]):

$$Q(A,D)_{rsij} = A_{ri} D_{sj} + A_{rj} D_{si} - A_{si} D_{rj} - A_{sj} D_{ri},$$

$$\begin{aligned}
Q(A,B)_{rsijlm} = {}& A_{ri} B_{sjlm} + A_{rj} B_{islm} + A_{rl} B_{ijsm} + A_{rm} B_{ijls}\\
& - A_{si} B_{rjlm} - A_{sj} B_{irlm} - A_{sl} B_{ijrm} - A_{sm} B_{ijlr}.
\end{aligned}$$

Thus, (2.11) and (2.12) are the same operators, but adopted to the complex structure of the manifold.

We note that

$$\left.\begin{aligned}&Q(A,D)=-Q(D,A)\quad\text{and therefore}\quad Q(A,A)=0,\\&Q(fA,fD)=-Q(A,D)\quad\text{and therefore}\quad Q(fA,D)=Q(A,fD),\\&Q(A,fA)=0,\quad Q(fD,B)=Q(D,fB).\end{aligned}\right\}\tag{2.13}$$

For the latter use, we present

Lemma 2.1 *([4]) Let as a point $p\in M$, A and D be two symmetric* $(0,2)$ *tensors satisfying (2.7). If*

$$Q(A,D)=0\ ,\tag{2.14}$$

then

$$D=\delta A+\bar{\delta}fA,\qquad \delta,\bar{\delta}\in R.\tag{2.15}$$

P r o o f. Let X be a vector such that

$$X^aX^bA_{ab}=\omega\neq 0,\qquad X^a\bar{X}^bA_{ab}=\bar{\omega}\neq 0,$$

where $\bar{X}^i=f^i_aX^a$. We put

$$\eta=X^aX^bD_{ab},\qquad \bar{\eta}=X^a\bar{X}^bD_{ab}.$$

Transvecting (2.14) with X^iX^r, and symmetrizing the resulting equality, we get

$$\omega D_{sj}-\eta A_{sj}-\bar{\omega}f^a_sD_{aj}+\bar{\eta}f^a_sA_{aj}=0\ ,$$

from which it follows that

$$\omega f^a_iD_{aj}-\eta f^a_iA_{aj}+\bar{\omega}D_{ij}-\bar{\eta}A_{ij}=0\ .$$

These two relations imply

$$D_{ij}=\frac{\omega\eta+\bar{\omega}\bar{\eta}}{\omega^2+\bar{\omega}^2}A_{ij}-\frac{\omega\bar{\eta}-\bar{\omega}\eta}{\omega^2+\bar{\omega}^2}f^a_iA_{aj}\ .$$

But this is just the relation (2.15). □

3. *H-hypersurface of an anti-Kähler manifold of constant totally real sectional curvatures*

The Gauss equation for an h-hypersurface (M, g, f) reads

$$\widetilde{R}_{ABCD}\frac{\partial x^A}{\partial u^i}\frac{\partial x^B}{\partial u^j}\frac{\partial x^C}{\partial u^l}\frac{\partial x^D}{\partial u^m} = R_{ijlm} - (h_{im}h_{jl} - h_{il}h_{jm}) + (k_{im}k_{jl} - k_{il}k_{jm}).$$

Now, we suppose that the ambient manifold $(\widetilde{M}, G, F)$ is a manifold of constant totally real sectional curvatures. Then, substituting (2.3) into above Gauss equation, and taking into account that $m = n+1$, we get

$$R_{ijlm} = K\, G_{ijlm} + \overset{*}{K} f_i^a G_{ajlm} + E_{ijlm} \tag{3.1}$$

where

$$G_{ijlm} = g_{im}g_{jl} - g_{il}g_{jm} - f_{im}f_{jl} + f_{il}f_{jm}\ , \tag{3.2}$$

$$E_{ijlm} = h_{im}h_{jl} - h_{il}h_{jm} - k_{im}k_{jl} + k_{il}k_{jm} \tag{3.3}$$

$$K = \frac{\widetilde{\kappa}}{4n(n+1)}\ , \qquad \overset{*}{K} = -\frac{\widetilde{\overset{*}{\kappa}}}{4n(n+1)}\ , \tag{3.4}$$

and $f_{ij} = f_i^a g_{aj}$.

The relation (3.1) yields

$$\left.\begin{aligned} \rho_{im} &= 2(n-1)(Kg_{im} + \overset{*}{K} f_{im}) + \mathrm{tr}\, h\, h_{im} + \mathrm{tr}\, k\, f_i^a h_{am} - 2h^2_{im}\ , \\ \overset{*}{\rho}_{im} &= 2(n-1)(Kf_{im} - \overset{*}{K} g_{im}) + \mathrm{tr}\, h\, f_i^a h_{am} - \mathrm{tr}\, k\, h_{im} - 2f_i^a h^2_{am}\ , \end{aligned}\right\} \tag{3.5}$$

and therefore

$$\left.\begin{aligned} \kappa &= 4n(n-1)K + (\mathrm{tr}\, h)^2 - (\mathrm{tr}\, k)^2 - 2\mathrm{tr}\,(h^2), \\ \overset{*}{\kappa} &= -4n(n-1)\,\overset{*}{K} - 2\mathrm{tr}\, h\ \mathrm{tr}\, k - 2\mathrm{tr}\,(fh^2). \end{aligned}\right\} \tag{3.6}$$

We note that, because of $k_{ij} = -f_i^a h_{aj}$, we have $\mathrm{tr}\, k = -\mathrm{tr}\,(fh)$. In view of (3.1), we have

$$R \cdot R = K\, G \cdot R + \overset{*}{K}(fG) \cdot R + E \cdot R\ .$$

Using (2.12), we can easy to see that

$$G \cdot R = Q(g, R), \qquad (fG) \cdot R = Q(fg, R)\ .$$

Therefore

$$R \cdot R = K\, Q(g,R) + \overset{*}{K} Q(fg,R) + E \cdot R\,. \tag{3.7}$$

On the other hand

$$E \cdot R = K\, (E \cdot G) + \overset{*}{K} (E \cdot fG) + E \cdot E\,.$$

But

$$\begin{aligned}
(E \cdot G)_{rsijlm} &= \\
&= G_{ajlm} E^a{}_{irs} + G_{ialm} E^a{}_{jrs} + G_{ijam} E^a{}_{lrs} + G_{ijla} E^a{}_{mrs} \\
&= g_{jl}\,(E_{mirs} + E_{imrs}) - g_{jm}\,(E_{lirs} + E_{ilrs}) \\
&\quad + g_{im}\,(E_{ljrs} + E_{jlrs}) - g_{il}\,(E_{mjrs} + E_{jmrs}) \\
&\quad - f_{im}\left(f^a_l E_{ajrs} + f^a_j E_{alrs}\right) + f_{il}\left(f^a_m E_{ajrs} + f^a_j E_{amrs}\right) \\
&\quad - f_{jl}\,(f^a_m E_{airs} + f^a_i E_{amrs}) + f_{jm}\,(f^a_l E_{airs} + f^a_i E_{alrs}) = 0\,,
\end{aligned}$$

because of

$$E_{mirs} = -E_{mirs} \qquad \text{and} \qquad f^a_m E_{airs} = -f^a_i E_{amrs}\,.$$

Similary we have $E \cdot fG = 0$, and therefore (3.7) reduces to

$$R \cdot R = K\, Q(g,R) + \overset{*}{K}\, Q(fg,R) + E \cdot E\,.$$

Finally

$$\begin{aligned}
(E \cdot E)_{rsijlm} &= \\
&= -\Big[h^2_{ri} E_{sjlm} + h^2_{rj} E_{islm} + h^2_{rl} E_{ijsm} + h^2_{rm} E_{ijls} \\
&\quad - h^2_{si} E_{rjlm} - h^2_{sj} E_{irlm} - h^2_{sl} E_{ijrm} - h^2_{sm} E_{ijlr} \\
&\quad - f^a_r f^b_s \Big(h^2_{ai} E_{bjlm} + h^2_{aj} E_{iblm} + h^2_{al} E_{ijbm} + h^2_{am} E_{ijlb} \\
&\quad - h^2_{bi} E_{ajlm} - h^2_{bj} E_{ialm} - h^2_{bl} E_{ijam} - h^2_{bm} E_{ijla}\Big)\Big] \\
&= -Q(h^2, E)_{rsijlm}.
\end{aligned}$$

Thus, we can state

Proposition 3.1. *The relation*

$$(R \cdot R)_{rsijlm} = KQ(g,R)_{rsijlm} + \overset{*}{K} Q(fg,R)_{rsijlm} - Q(h^2,E)_{rsijlm} \tag{3.8}$$

holds good for any h-hypersurface of an anti-Kähler manifold of constant totally real sectional curvatures.

Transvecting (3.8) with g^{jl} we get

$$R \cdot \rho = KQ(g, \rho) + \overset{*}{K}(fg, \rho) + Q(h, \mathrm{tr}\, h\, h^2 + \mathrm{tr}\, k\, fh^2 - 2h^3). \tag{3.9}$$

Thus, we have

Proposition 3.2. *The relation (3.9) holds good for any h-hypersurface of an anti-Kähler manifold of constant totally real sectional curvatures.*

4. *H-pseudosymmetry*

In the case of anti-Kähler manifolds, we adopt the conditions (1.1) and (1.2) to the complex structure of the manifold introducing the following

Definition. *The anti-Kähler manifold (M, g, f) is said to be h-pseudo-symmetric if the condition*

$$R \cdot R = \mathcal{L}_1 Q(g, R) + \mathcal{L}_2 Q(fg, R) \tag{4.1}$$

is satisfied on some set $U \subset M$, where $\mathcal{L}_1$ and $\mathcal{L}_2$ are some scalar function on U.

The manifold (M, g, f) is said to be Ricci h-pseudosymmetric if the condition

$$R \cdot \rho = \mathcal{L}_1 Q(g, \rho) + \mathcal{L}_2 Q(fg, \rho) \tag{4.2}$$

is satisfied on U.

Now, we consider h-hypersurface (M, g, f) of the anti-Kähler manifold of constant totally real sectional curvatures. Then, according to the Proposition 3.2, the relation (3.9) holds good. Thus, if (M, g, f) is also Ricci h-pseudosymmetric, then we have

$$(\mathcal{L}_1 - K)Q(g, \rho) + (\mathcal{L}_2 - \overset{*}{K})Q(fg, \rho) = Q(h, \mathrm{tr}\, h\, h^2 + \mathrm{tr}\, k\, fh^2 - 2h^3). \tag{4.3}$$

We shall examine two cases.

Case (1). If

$$\mathcal{L}_1 = K \qquad \text{and} \qquad \mathcal{L}_2 = \overset{*}{K}, \tag{4.4}$$

then (4.3) reduces to

$$Q(h, \mathrm{tr}\, h\, h^2 + \mathrm{tr}\, k\, fh^2 - 2h^3) = 0$$

and, in view of Lemma 2.1, we have

$$h^3 = \frac{1}{2}\mathrm{tr}\, h\, h^2 + \frac{1}{2}\mathrm{tr}\, k\, fh^2 + \delta h + \bar{\delta} fh \, . \tag{4.5}$$

Conversely, if (4.5) holds, then

$$Q(h, \mathrm{tr}\, h\, h^2 + \mathrm{tr}\, k\, fh^2 - 2h^3) =$$
$$= -2\delta Q(h,h) - 2\bar{\delta} Q(h, fh) = 0 \, ,$$

and (3.9) reduces to

$$R \cdot \rho = KQ(g, \rho) + \overset{*}{K} Q(fg, \rho) \, ,$$

i.e., (4.4) holds.

Thus, we can state

Theorem 4.1. *Let (M, g, f) be h-hypersurface of the anti-Kähler manifold (M, G, F) of constant totally real sectional curvatures. Then (4.5) is the necessary and the sufficient condition for (M, g, f) to be Ricci h-pseudosymmetric on the appropriate set $U \subset M$ such that (4.4) holds.*

Remark. According to (3.4), (4.4) turns into

$$\mathcal{L}_1 = \frac{\widetilde{\kappa}}{4n(n+1)} \, , \qquad \mathcal{L}_2 = -\frac{\widetilde{\overset{*}{\kappa}}}{4n(n+1)} \, ,$$

where $\widetilde{\kappa}$ and $\widetilde{\overset{*}{\kappa}}$ are the first and the second scalar curvatures of $\widetilde{M}$ and $\dim M = 2n$.

Corollary. *Let (M, g, f) be h-hypersurface of the flat anti-Kähler manifold. Then (4.5) is the necessary and the sufficient condition for (M, g, f) to be Ricci semisymmetric.*

Case (2) If

$$\lambda_1 = \mathcal{L}_1 - \frac{\widetilde{\kappa}}{4n(n+1)} \neq 0 \quad \text{and} \quad \lambda_2 = \mathcal{L}_2 + \frac{\widetilde{\overset{*}{\kappa}}}{4n(n+1)} \neq 0 \, ,$$

then (4.3) gives

$$\lambda_1 Q(g,\rho) + \lambda_2 Q(fg,\rho) = Q(h, \mathrm{tr}\ h\ h^2 + \mathrm{tr}\ k\ fh^2 - 2h^3)\ , \tag{4.6}$$

from which it follows that

$$\lambda_1 Q(fg,\rho) - \lambda_2 Q(g,\rho) = -Q(h, \mathrm{tr}\ k\ h^2 - \mathrm{tr}\ h\ fh^2 + 2fh^3)\ . \tag{4.7}$$

In the local coordinates, the left hand side of (4.6) is the following

$$\lambda_1(g_{ri}\rho_{sj} + g_{rj}\rho_{si} - g_{si}\rho_{rj} - g_{sj}\rho_{ri} - f_{ri}\ \overset{*}{\rho}_{sj} - f_{rj}\ \overset{*}{\rho}_{si} + f_{si}\ \overset{*}{\rho}_{rj} + f_{sj}\ \overset{*}{\rho}_{ri})$$
$$+\lambda_2(f_{ri}\rho_{sj} + f_{rj}\rho_{si} - f_{si}\rho_{rj} - f_{sj}\rho_{ri} + g_{ri}\ \overset{*}{\rho}_{sj} + g_{rj}\ \overset{*}{\rho}_{si} - g_{si}\ \overset{*}{\rho}_{rj} - g_{sj}\ \overset{*}{\rho}_{ri}),$$

from which, by transvection with g^{ri} we get

$$\lambda_1(2n\rho - \kappa g + \overset{*}{\kappa} fg) + \lambda_2(2n\ \overset{*}{\rho} - \kappa fg - \overset{*}{\kappa} g).$$

In the similar way, we obtain from

$$Q(h, \mathrm{tr}\ h\ h^2 + \mathrm{tr}\ k\ fh^2 - 2h^3)$$

the following expression

$$-\left[\mathrm{tr}\ h\ \mathrm{tr}\ h^2 + \mathrm{tr}\ k\ \mathrm{tr}(fh^2) - 2\mathrm{tr}\ h^3\right] h + \left[\mathrm{tr}\ h\ \mathrm{tr}(fh^2) - \mathrm{tr}\ k\ \mathrm{tr}\ h^2 - 2\mathrm{tr}(fh^3)\right] fh$$
$$+\left[(\mathrm{tr}\ h)^2 - (\mathrm{tr}\ k)^2\right] h^2 + 2\mathrm{tr}\ h\ \mathrm{tr}\ k\ fh^2 - 2\mathrm{tr}\ h\ h^3 - 2\mathrm{tr}\ k\ fh^3\ .$$

Thus, as a consequence of (4.6), we have

$$\begin{aligned} &\lambda_1 Q(2n\rho - \kappa g + \overset{*}{\kappa} fg, h) + \lambda_2 Q(2n\ \overset{*}{\rho} - \kappa fg - \overset{*}{\kappa} g, h) \\ &= -\left[(\mathrm{tr}h)^2 - (\mathrm{tr}k)^2\right] Q(h,h^2) - 2\mathrm{tr}h\ \mathrm{tr}k\ Q(h,fh^2) \\ &\qquad +2\mathrm{tr}h\ Q(h,h^3) + 2\mathrm{tr}k\ Q(h,fh^3)\ . \end{aligned} \tag{4.8}$$

But, the right hand side of (4.8) can be written in the form

$$\begin{aligned} &\left[-(\mathrm{tr}h)^2 + (\mathrm{tr}k)^2\right] Q(h,h^2) - 2\mathrm{tr}h\ \mathrm{tr}k\ Q(h,fh^2) \\ &\qquad +2\mathrm{tr}h\ Q(h,h^3) + 2\mathrm{tr}k\ Q(h,fh^3) \\ &= -\mathrm{tr}h\ \left[\mathrm{tr}h\ Q(h,h^2) + \mathrm{tr}k\ Q(h,fh^2) - 2Q(h,h^3)\right] \\ &\quad +\mathrm{tr}k\ \left[\mathrm{tr}k\ Q(h,h^2) - \mathrm{tr}h\ Q(h,fh^2) + 2Q(h,fh^3)\right] \\ &= -\mathrm{tr}h\ Q(h, \mathrm{tr}h\ h^2 + \mathrm{tr}k\ fh^2 - 2h^3) + \mathrm{tr}k\ Q(h, \mathrm{tr}k\ h^2 - \mathrm{tr}h\ fh^2 + 2fh^3) \\ &= -\mathrm{tr}h\ \left[\lambda_1 Q(g,\rho) + \lambda_2 Q(fg,\rho)\right] + \mathrm{tr}k\ \left[-\lambda_1 Q(fg,\rho) + \lambda_2 Q(g,\rho)\right]\ , \end{aligned}$$

because of (4.6) and (4.7). Thus, (4.8) is of the form

$$\begin{aligned}
&\lambda_1 Q(2n\rho - \kappa g + \overset{*}{\kappa} fg, h) + \lambda_2 Q(2n \overset{*}{\rho} - \kappa fg - \overset{*}{\kappa} g, h) \\
&= -\lambda_1 \left[\mathrm{tr} h\, Q(g,\rho) + \mathrm{tr} k\, Q(fg,\rho)\right] + \lambda_2 \left[\mathrm{tr} k\, Q(g,\rho) - \mathrm{tr} h\, Q(fg,\rho)\right] .
\end{aligned} \tag{4.9}$$

On the other hand, using (3.5), we have

$$\begin{aligned}
&Q(2n\rho - \kappa g + \overset{*}{\kappa} fg, h) \\
&= \left[4n(n-1)K - \kappa\right] Q(g,h) + \left[4n(n-1)\overset{*}{K} + \overset{*}{\kappa}\right] Q(fg,h) + 4nQ(h,h^2) .
\end{aligned}$$

Similarly

$$\begin{aligned}
&Q(2n \overset{*}{\rho} - \kappa fg - \overset{*}{\kappa} g, h) \\
&= \left[4n(n-1)K - \kappa\right] Q(fg,h) - \left[4n(n-1)\overset{*}{K} + \overset{*}{\kappa}\right] Q(g,h) + 4nQ(h,fh^2) ,
\end{aligned}$$

such that (4.9) becomes

$$\begin{aligned}
&\lambda_1 \{[4n(n-1)K - \kappa] Q(g,h) + [4n(n-1)\overset{*}{K} + \overset{*}{\kappa}] Q(fg,h) \\
&\qquad +4nQ(h,h^2) + \mathrm{tr} h\, Q(g,\rho) + \mathrm{tr} k\, Q(fg,\rho)\} \\
&+\lambda_2 \{[4n(n-1)K - \kappa] Q(fg,h) - [4n(n-1)\overset{*}{K} + \overset{*}{\kappa}] Q(g,h) \\
&\qquad +4nQ(h,fh^2) - \mathrm{tr} k\, Q(g,\rho) + \mathrm{tr} h\, Q(fg,\rho)\} = 0 .
\end{aligned} \tag{4.10}$$

If we set

$$\begin{aligned}
P = \; & [4n(n-1)K - \kappa] Q(g,h) + [4n(n-1)\overset{*}{K} + \overset{*}{\kappa}] Q(fg,h) \\
& +4nQ(h,h^2) + \mathrm{tr} h\, Q(g,\rho) + \mathrm{tr} k\, Q(fg,\rho) ,
\end{aligned}$$

then

$$\begin{aligned}
fP = \; & [4n(n-1)K - \kappa] Q(fg,h) - [4n(n-1)\overset{*}{K} - \overset{*}{\kappa}] Q(g,h) \\
& +4nQ(fh,h^2) + \mathrm{tr} h\, Q(fg,\rho) - \mathrm{tr} k\, Q(g,\rho) .
\end{aligned}$$

But

$$Q(h,fh^2) = Q(fh,h^2) .$$

This means that (4.10) can be expressed in the form

$$\lambda_1 P + \lambda_2 fP = 0 .$$

This relation, together with

$$-\lambda_2 P + \lambda_1 f P = 0\ ,$$

yields

$$(\lambda_1^2 + \lambda_2^2) P = 0\ ,$$

and $P = 0$ if at last one of the conditions $\lambda_1 \neq 0$ and $\lambda_2 \neq 0$ is satisfied. Thus we have

$$\begin{aligned}
&[4n(n-1)K - \kappa]Q(g,h) + [4n(n-1)\overset{*}{K} + \overset{*}{\kappa}]Q(fg,h) \\
&+4nQ(h,h^2) + \mathrm{tr}h\ Q(g,\rho) + \mathrm{tr}k\ Q(fg,\rho) = 0\ ,
\end{aligned}$$

or

$$\begin{aligned}
&[4n(n-1)K - \kappa + (\mathrm{tr}h)^2 - (\mathrm{tr}k)^2]Q(g,h) \\
&+[4n(n-1)\overset{*}{K} + \overset{*}{\kappa} + 2\mathrm{tr}h\ \mathrm{tr}k]Q(fg,h) \\
&-2\mathrm{tr}h\ Q(g,h^2) - 2\mathrm{tr}k\ Q(g,fh^2) + 4nQ(h,h^2) = 0\ ,
\end{aligned} \tag{4.11}$$

because of

$$\begin{aligned}
&Q(g,\rho) = \mathrm{tr}h\ Q(g,h) + \mathrm{tr}k\ Q(g,fh) - 2Q(g,h^2)\ , \\
&Q(fg,\rho) = \mathrm{tr}h\ Q(fg,h) + \mathrm{tr}k\ Q(g,h) - 2Q(g,fh^2)\ .
\end{aligned}$$

Finally, according to (3.6),

$$\begin{aligned}
&4n(n-1)K - \kappa + (\mathrm{tr}h)^2 - (\mathrm{tr}k)^2 = 2\mathrm{tr}(h^2), \\
&4n(n-1)\overset{*}{K} + \overset{*}{\kappa} + 2\mathrm{tr}h\ \mathrm{tr}k = -2\mathrm{tr}(fh^2)\ ,
\end{aligned}$$

and (4.11) becomes

$$\begin{aligned}
&\mathrm{tr}h^2 Q(g,h) - \mathrm{tr}(fh^2)Q(fg,h) \\
&-\mathrm{tr}h\ Q(g,h^2) - \mathrm{tr}k\ Q(g,fh^2) + 2nQ(h,h^2) = 0\ .
\end{aligned} \tag{4.12}$$

On the other hand,

$$\begin{aligned}
&Q(h - \frac{\mathrm{tr}h}{2n}g + \frac{\mathrm{tr}(fh)}{2n}fg,\ h^2 - \frac{\mathrm{tr}h^2}{2n}g + \frac{\mathrm{tr}(fh^2)}{2n}fg) \\
&= Q(h,h^2) + Q(g,\frac{\mathrm{tr}h^2}{2n}h) - Q(g,\ \frac{\mathrm{tr}(fh^2)}{2n}fh) \\
&\quad -Q(g,\ \frac{\mathrm{tr}h}{2n}h^2) + Q(g,\ \frac{\mathrm{tr}fh}{2n}fh^2)\ .
\end{aligned}$$

This, in view of (4.12), means that

$$Q(h - \frac{\mathrm{tr}\, h}{2n} g + \frac{\mathrm{tr}\,(fh)}{2n} fg,\ h^2 - \frac{\mathrm{tr}\, h^2}{2n} g + \frac{\mathrm{tr}\,(fh^2)}{2n} fg) = 0$$

from which, applying Lemma 2.1, we get

$$\begin{aligned} h^2 - \frac{\mathrm{tr}\, h^2}{2n} g + \frac{\mathrm{tr}\,(fh^2)}{2n} fg &= \delta(h - \frac{\mathrm{tr}\, h}{2n} g + \frac{\mathrm{tr}\,(fh)}{2n} fg) \\ &\quad + \bar{\delta}(fh - \frac{\mathrm{tr}\, h}{2n} fg - \frac{\mathrm{tr}\,(fh)}{2n} g) \ , \end{aligned}$$

or

$$h^2 = \delta h + \bar{\delta} fh + \mu g + \bar{\mu} fg \ , \tag{4.13}$$

where

$$\begin{aligned} \mu &= \frac{\mathrm{tr}\, h^2}{2n} - \delta \frac{\mathrm{tr}\, h}{2n} - \bar{\delta} \frac{\mathrm{tr}\,(fh)}{2n} \ , \\ \bar{\mu} &= -\frac{\mathrm{tr}\,(fh^2)}{2n} + \delta \frac{\mathrm{tr}\,(fh)}{2n} - \bar{\delta} \frac{\mathrm{tr}\, h}{2n} \ . \end{aligned}$$

Conversely, if (4.13) holds, then

$$\begin{aligned} Q(h^2, E) &= Q(\delta h + \bar{\delta} fh + \mu g + \bar{\mu} fg, E) \\ &= \delta Q(h, E) + \bar{\delta} Q(fh, E) + \mu Q(g, E) + \bar{\mu} Q(fg, E). \end{aligned}$$

But

$$Q(h, E) = Q(fh, E) = 0 \ ,$$

and therefore

$$Q(h^2, E) = \mu Q(g, E) + \bar{\mu} Q(fg, E) \ . \tag{4.14}$$

On the other hand, in view of $Q(g, G) = Q(fg, G) = 0$, we have

$$\mu Q(g,\ KG + \overset{*}{K} fG) = 0 \ , \qquad \bar{\mu} Q(fg,\ KG + \overset{*}{K} fG) = 0 \ ,$$

that is, the relation (4.14) is equivalent to

$$Q(h^2, E) = \mu Q(g, KG + \overset{*}{K} fG + E) + \bar{\mu} Q(fg, KG + \overset{*}{K} fG + E) \ .$$

In the other words

$$Q(h^2, E) = \mu Q(g, R) + \bar{\mu} Q(fg, R) \ . \tag{4.15}$$

According to the Proposition 3.1, for any h-hypersurface of the anti-Kähler manifold of constant totally real sectional curvatures, the relation (3.8) holds, which, in view of (4.15) becomes

$$R \cdot R = (K - \mu) Q(G, R) + (\overset{*}{K} - \bar{\mu}) Q(fg, R) \ .$$

This means that if (4.15) holds, then (M, g, f) is h-pseudosymmetric. But h-pseudosymmetric manifold is Ricci h-pseudosymmetric, too. Thus, we can state

Theorem 4.2. *Let (M, g, f), $\dim M = 2n$, be a h-hypersurface of the anti-Kähler manifold $(\widetilde{M}, G, F)$ of constant totally real sectional curvatures. Let $\widetilde{\kappa}$ and $\widetilde{\overset{*}{\kappa}}$ be the first and the second scalar curvatures of $(\widetilde{M}, G, F)$. Then (4.13) is the necessary and the sufficient condition for (M, g, f) to be, on the appropriate set $U \subset M$, Ricci h-pseudosymmetric such that at least one of the relations*

$$\mathcal{L}_1 \neq \frac{\widetilde{\kappa}}{4n(n+1)} \ , \qquad \mathcal{L}_2 \neq \frac{\widetilde{\overset{*}{\kappa}}}{4n(n+1)}$$

is satisfied.

5. Remark. H-pseudosymmetry is also considered in [5]. In that paper it is proved that every anti-Kähler manifold satisfying the Roter type equation

$$\begin{aligned} R(X,Y,Z,W) = \ & N_1 \Gamma(X,Y,Z,W) + N_2 \Gamma(fX,Y,Z,W) \\ & + N_3 G(X,Y,Z,W) + N_4 G(fX,Y,Z,W) \ , \end{aligned}$$

on some set $U \subset M$, is h-pseudosymmetric, where

$$\begin{aligned} \Gamma(X,Y,Z,W) = \ & \rho(X,W)\rho(Y,Z) - \rho(X,Z)\rho(Y,W) \\ & - \overset{*}{\rho}(X,W) \overset{*}{\rho}(Y,Z) + \overset{*}{\rho}(X,Z) \overset{*}{\rho}(Y,W) \ , \end{aligned}$$

and $N_1, \ldots, N_4$ are some scalar functions on U.

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