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SPECTRA OF COPIES OF BETHE TREES ATTACHED TO PATH AND APPLICATIONS

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A b s t r a c t. *The Bethe tree $B_{d,k}$ is the rooted tree of k levels whose root vertex has degree d, the vertices from level 2 to level $k-1$ have degree $d+1$, and the vertices at level k have degree 1. This paper gives a decomposition of the characteristic polynomial of the adjacency matrix of the tree $T(d,k,r)$, obtained by attaching copies of $B(d,k)$ to the vertices of the r-vertex path. Moreover, lower and upper bounds for the energy of $T(d,k,r)$ are obtained.*

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1. *Introduction*

Let G be a simple graph on n vertices. If we label its vertices by $1,\ldots,n$, then its adjacency matrix is given by $A(G)=(a_{i,j})$, where $a_{i,j}=1$ if the vertex i is connected, by an edge of G, to the vertex j, and $a_{i,j}=0$ otherwise. The characteristic polynomial of $A(G)$ is known as the characteristic polynomial of the graph G. The eigenvalues of $A(G)$, same as the zeros of the characteristic polynomial, form the spectrum of G [1]. The eigenvalues of G are denoted by $\lambda_j = \lambda_j(G)$, $j = 1, \ldots, n$, and labelled so that

$$\lambda_1 \leq \cdots \leq \lambda_n \ .$$

The concept of the energy of G , defined as

$$E(G) = \sum_{j=1}^{n} |\lambda_j|$$

was introduced by one of the present authors (see [5, 6] and the references cited therein). Nikiforov [10] defines the energy of a matrix M (square or not) as the sum of its singular values. Recall that if M is a symmetric matrix, then its singular values and the moduli of its eigenvalues coincide.

Let $k > 1$. A generalized Bethe tree B_k of k levels [13] is a rooted tree in which vertices at same level have the same degree. For $j = 1, \ldots k$, we denote by d_{k-j+1} and by n_{k-j+1} the degree of the vertices at the level j and their number, respectively. Thus, $d_1 = 1$ is the degree of the vertices at the level k (pendent vertices) and d_k is the degree of the root vertex. On the other hand, it is $n_k = 1$, pertaining to the single vertex at the first level, the root vertex. The ordinary Bethe tree $B_{d,k}$ is the rooted trees of k levels whose root vertex has degree d , the vertices from levels 2 to $k - 1$ have degree $d + 1$, and the vertices at level k have degree 1 .

Spectral properties of Bethe trees (both ordinary and generalized) were much studied in the past. Heilmann and Lieb [7] have determined the decomposition of the matching polynomial of $B_{d,k}$. Recall that the matching and characteristic polynomials of trees coincide [2].

Other spectral properties of Bethe and similar trees were considered in [4, 12, 14, 15]. Eventually, Rojo and one of the present authors found explicit formulas for the eigenvalues of the Bethe trees [16]. With these results the authors in [11] obtained an explicit formula for the energy of the Bethe tree $B_{d,k}$.

Let P_r and C_r be, respectively, the r-vertex path and the r-vertex cycle. In the paper [13] the spectrum of the graph obtained by attaching copies of $B_{d,k}$ to the vertices of C_r was determined. In this paper we obtain an analogous decomposition of the characteristic polynomial of the tree $T(d, k, r)$, obtained by attaching copies of $B_{d,k}$ to the vertices of P_r . Using this result, lower and upper bounds for the energy of $T(d, k, r)$ are established.

The tree $T(2, 4, 3)$ is depicted in Fig. 1.

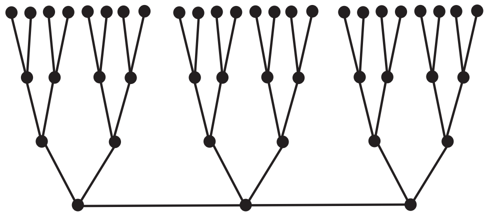

Fig. 1. The tree $T(2,4,3)$

In connection with the graphs considered in [13] and in this article, one needs to recall the following result of Godsil and McKay [3].

Denote by $\phi(H,x)$ the characteristic polynomial of a graph H. Let G be a graph on n vertices. Let R be a rooted graph and v its root. Construct the graph product $G[R]$ by attaching a copy of R, via its root, to each vertex of G.

Then $\phi(G[R],x)$, the characteristic polynomial of $G[R]$, is equal to

$$\phi(R-v,x)^n\,\phi\left(G,\frac{\phi(R,x)}{\phi(R-v,x)}\right) . \tag{1}$$

Evidently, the above formula is applicable to the trees $T(d,k,r)$. Some of the results of the present paper could have been obtained by using the Godsil–McKay formula. Yet, our reasoning follows a somewhat different path.

The present paper has five sections. In this section we recall some results from Matrix Theory. The main result in the second section is

Theorem 1. *The characteristic polynomial of the tree $T = T(d,k,r)$ has the following decomposition:*

$$\det(\lambda I - A(T)) = \left(\prod_{j\in\Omega} Q_j^{n_j-n_{j+1}}(\lambda)\right)^r \left(\prod_{\ell=1}^{r} Q_{\ell,k}(\lambda)\right) \tag{2}$$

where $Q_{\ell,k}(\lambda)$, $\ell = 1,\dots,r$, is the characteristic polynomial of the matrix $T_{\ell,k}$ in Eq. (12), whereas $Q_j(\lambda)$, $j = 1,\dots,k-1$, is the characteristic polynomial of the $j\times j$ leading principal submatrix of $T_{\ell,k}$.

In the third section, as an application of Theorem 1, we show that the energy of the tree $T(d,2,r)$ is equal to

$$4\sum_{\ell=1}^{\lfloor r/2\rfloor}\sqrt{d+\cos^2\frac{\ell\pi}{r+1}} \qquad \text{if } r \text{ is even} \tag{3}$$

$$2\sqrt{d}+4\sum_{\ell=1}^{\lfloor r/2\rfloor}\sqrt{d+\cos^2\frac{\ell\pi}{r+1}} \qquad \text{if } r \text{ is odd .} \tag{4}$$

We denote by $E(N)$ the energy of any matrix N.

Let $M=M(\alpha,h)$ be the $k\times k$ tridiagonal symmetric matrix, specified in Eq. (17). In an earlier work [11] we proved the following

Theorem 2. *The energy of $M(\alpha,0)$ is equal to*

$$E(M(\alpha,0))=2\alpha\left(\frac{\sin(\lfloor k/2\rfloor+1/2)\frac{\pi}{k+1}}{\sin\frac{\pi}{2(k+1)}}-1\right) .$$

In the fourth section we obtain lower and upper bounds for the energy of a $M=M(\alpha,h)$, for $\alpha>0$, $h\geq 0$.

Let $\alpha,h>0$. Let $k\geq 3$. Let $b:=b(k)$ be defined by

$$b:=\frac{\sin(\lfloor k/2\rfloor+1/2)\frac{\pi}{k}}{\sin\frac{\pi}{2k}}-1 . \tag{5}$$

We denote by $B(\mu)$ an upper bound for the greatest eigenvalue μ of the matrix $M(\alpha,h)$.

Theorem 3. *The energy $E(M(\alpha,h))$ of the matrix $M(\alpha,h)$, defined via Eq. (17), is bounded by*

$$-2\alpha\cos\frac{\pi}{k}<E(M(\alpha,h))-2\alpha\left(b+\cos\frac{\pi}{k+1}\right)<B(\mu)$$

whenever k is even and

$$-2\alpha\cos\frac{\pi}{k}<E(M(\alpha,h))-2\alpha\left(b+\cos\frac{\pi}{k+1}\right)<B(\mu)-2\alpha\cos\frac{\lfloor k/2\rfloor\pi}{k}$$

whenever k is odd, where b is given by Eq. (5).

Finally, in the last section we search for an upper bound $B(\mu)$. We prove

Theorem 4. *Let $\alpha,h>0$. The greatest eigenvalue μ of the matrix $M(\alpha,h)$, defined via Eq. (17), is bounded by $B(\mu)=\min\{B_1(\mu),B_2(\mu)\}$, where*

$$B_1(\mu)=\max\left\{2\alpha\cos\frac{\pi}{2k+1},\ 2h\cos\frac{\pi}{2k+1}\right\} \ \text{ and } \ B_2(\mu)=\max\{2\alpha,\ \alpha+h\} .$$

In the fifth section we also obtain lower and upper bounds for the energy of the tree $T(d,k,r)$.

* * * *

We recall that the Kronecker product $A \otimes B$ of a pair of matrices $A = (a_{i,j})$ and $B = (b_{i,j})$ of orders $r \times s$ and $p \times q$, respectively, is the $rp \times sq$ matrix, defined by [9]

$$A \otimes B = \begin{pmatrix} a_{11}B & \dots & a_{1s}B \\ \vdots & \ddots & \vdots \\ a_{r1}B & \dots & a_{rs}B \end{pmatrix}.$$

This binary operation has the following properties:

1. $(A \otimes B)^T = A^T \otimes B^T$.
2. $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$ provided that the matrices A^{-1} and B^{-1} exist.
3. $(A \otimes B)(C \otimes D) = (AC \otimes BD)$ provided that the products AC and BD exist.
4. $(I_r \otimes I_s) = I_{rs}$, where for a positive integer ℓ, I_ℓ denotes the identity matrix of order ℓ.
5. $(I_r \otimes B) = diag(B, \dots, B)$.

We denote by $\lambda_p(N)$ the $p-th$ eigenvalue of any matrix N and we recall the following Lemmas.

Lemma 5 *(Monotonicity Theorem) [8]. Let A, B be $k \times k$ real and symmetric matrices. Let*

$$C = A + B$$

and let $\lambda_1(A) \le \lambda_2(A) \le \dots \le \lambda_k(A)$, $\lambda_1(B) \le \lambda_2(B) \le \dots \le \lambda_k(B)$, and $\lambda_1(C) \le \lambda_2(C) \le \dots \le \lambda_k(C)$ be the ordered eigenvalues of A, B, and C, respectively. Then

$$\lambda_j(A) + \lambda_{i-j+1}(B) \le \lambda_i(C)$$

whenever $i \ge j$ and

$$\lambda_i(C) \le \lambda_j(A) + \lambda_{i-j+k}(B)$$

whenever $i \leq j$ *.*

Lemma 6. *(Interlacing Cauchy Theorem) [8]. Let*

$$A = \begin{bmatrix} B & C \\ C^T & D \end{bmatrix}$$

be a $k \times k$ *symmetric matrix, where* B *is a* $j \times j$ *principal submatrix of* A *. Let* $\lambda_1(A) \leq \lambda_2(A) \leq \cdots \leq \lambda_k(A)$ *,* $\lambda_1(B) \leq \cdots \leq \lambda_j(B)$ *be the ordered eigenvalues of* A *and* B *respectively. Then for* $\ell = 1, \ldots j$ *,*

$$\lambda_\ell(A) \leq \lambda_\ell(B) \leq \lambda_{\ell+k-j}(A) \ .$$

Lemma 7. *Let* A *be an* $m \times m$ *symmetric tridiagonal matrix with nonzero codiagonal entries. Then the eigenvalues of any* $(m-1) \times (m-1)$ *principal submatrix strictly interlace the eigenvalues of* A *.*

The following result is known as the three–term recursion formula for tridiagonal symmetric matrices. The characteristic polynomials, $\widetilde{Q}_j(\lambda)$, of the $j \times j$ leading principal submatrices of the symmetric tridiagonal matrix

$$A_k = \begin{bmatrix} a_1 & b_1 & & \\ b_1 & \ddots & \ddots & \\ & \ddots & \ddots & b_{k-1} \\ & & b_{k-1} & a_k \end{bmatrix}$$

satisfy the following three–term recursion formula

$$\widetilde{Q}_j(\lambda) = (\lambda - a_j)\widetilde{Q}_{j-1}(\lambda) - b_{j-1}^2 \widetilde{Q}_{j-2}(\lambda) \ , \ j = 2, \ldots, k \tag{6}$$

where

$$\widetilde{Q}_0(\lambda) = 1 \qquad \text{and} \qquad \widetilde{Q}_1(\lambda) = \lambda - a_1 \ .$$

Let $L > 0$. In [11] it was shown that

$$\sum_{k=1}^{n} \left(2\cos\frac{k\pi}{L} \right) = \frac{\sin(n+1/2)\frac{\pi}{L}}{\sin\frac{\pi}{2L}} - 1 \ . \tag{7}$$

2. *Characteristic polynomial of* $T(d,k,r)$

We observe that on the level j , $j = 1, \ldots, k$ of the tree $T := T(d,k,r)$, there are $r\, n_{k-j+1}$ vertices. Hence and if n denotes the order of adjacency matrix, we see that $n = r \sum\limits_{j=1}^{k} n_j$.

From our notation for the tree B_k it is clear that

$$n_{k-j} = (d_{k-j+1} - 1) n_{k-j+1} \ , \ j = 1, \ldots, k-1 \qquad \text{and} \qquad n_{k-1} = d_k \ .$$

Here and in what follows $\mathbf{0}$ denotes the all–zero matrix; its order is clear from the context. For a positive integer ℓ let $\mathbf{e}_\ell = (\mathbf{1}, \ldots, \mathbf{1})^{\mathbf{T}}$, the all–one vector of order ℓ . Let

$$m_j = \frac{n_j}{n_{j+1}} \ , \ j = 1, \ldots, k-1 \ .$$

Hence,

$$m_j = d_{j+1} - 1 \ , \ j = 1, \ldots, k-2$$

and

$$m_{k-1} = d_k \ .$$

Let B_j be the block diagonal matrix defined by

$$B_j = I_{n_{j+1}} \otimes \mathbf{e}_{\mathbf{m_j}}$$

with n_{j+1} diagonal blocks equal to $\mathbf{e}_{\mathbf{m_j}}$. We observe that B_j has order $n_j \times n_{j+1}$ and that $B_{k-1} = \mathbf{e}_{\mathbf{n_{k-1}}}$. For $j = 1, \ldots, k-1$, let C_j be the $(r\, n_j) \times (r\, n_{j+1})$ block diagonal matrix,

$$C_j = I_r \otimes B_j$$

with r diagonal blocks equal to B_j . Hence C_{k-1} is the $r\, n_{k-1} \times r$ block diagonal matrix $C_{k-1} = I_r \otimes \mathbf{e}_{\mathbf{n_{k-1}}}$, with r diagonal blocks equal to $\mathbf{e}_{\mathbf{n_{k-1}}}$.

Consider the tridiagonal symmetric $r \times r$ matrices

$$F_r = \begin{bmatrix} 0 & 1 & & \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & 0 \end{bmatrix} \tag{8}$$

and

$$M_r = \alpha\, F_r \; . \tag{9}$$

Observe that F_r is the adjacency matrix of the path P_r. It is well known that [1]

$$\lambda_\ell(M_r) = 2\alpha \cos\frac{(r+1-\ell)\pi}{r+1} \;,\; \ell = 1,\ldots,r \; .$$

Therefore, $\lambda_\ell(M_r) = -\lambda_{r+1-\ell}(M_r)$, $\ell = 1,\ldots,r$. Moreover if r is odd, then $\lambda_{(r+1)/2}(M_r) = 0$.

In view of our labelling, the adjacency matrix of $T(d,k,r)$ becomes equal to the following block tridiagonal matrix:

$$A(T(d,k,r)) = \begin{bmatrix} 0 & C_1 & & \\ C_1^T & \ddots & \ddots & \\ & \ddots & 0 & C_{k-1} \\ & & C_{k-1}^T & F_r \end{bmatrix} .$$

Lemma 8. *For $j = 1,\ldots,k$, let $\alpha_j := \alpha_j(\lambda)$ be a polynomial with real coefficients, and*

$$M_1 = \begin{bmatrix} \alpha_1\, I_{r\, n_1} & -C_1 & & \\ -C_1^T & \ddots & \ddots & \\ & \ddots & \alpha_{k-1}\, I_{r\, n_{k-1}} & -C_{k-1} \\ & & -C_{k-1}^T & \alpha_k\, I_r - F_r \end{bmatrix} .$$

Moreover let

$$\beta_1 = \alpha_1$$

and $\beta_{j-1}(\lambda) \neq 0$,

$$\beta_j = \alpha_j - \frac{n_{j-1}}{n_j}\frac{1}{\beta_{j-1}}$$

for $j = 2,3,\ldots,k$. Then for all real λ, such that $\beta_j(\lambda) \neq 0$, $j = 1,2,\ldots,k-1$,

$$\det M_1 = \left(\beta_1^{n_1}\beta_2^{n_2}\cdots\beta_{k-1}^{n_{k-1}}\right)^r \prod_{\ell=1}^{r}\left(\beta_k - 2\cos\frac{\ell\pi}{r+1}\right) \; . \tag{10}$$

P r o o f. For the matrix M_1 we proceed as in Gaussian elimination, but this time we use the blocks. Thus for $j = 1, \ldots, k-1$ and $\beta_j \neq 0$, the matrix C_j^T is eliminated with $\beta_j \, I_{rn_j}$, in consequence $\alpha_{j+1} \, I_{r\, n_{j+1}}$ is replaced by $\alpha_{j+1} \, I_{r\, n_{j+1}} - \beta_j^{-1} \, C_j^T \, C_j \, I_{r\, n_j}$. Via definition of matrices B_j and C_j and the above properties of Kronecker product, we prove directly that $C_j^T \, C_j \, I_{r\, n_j} = m_j \, I_{r\, n_{j+1}} = \frac{n_j}{n_{j+1}} I_{r\, n_{j+1}}$. Then

$$\alpha_{j+1} \, I_{r\, n_{j+1}} - \beta_j^{-1} \, C_j^T \, C_j \, I_{r\, n_j} = \beta_{j+1} \, I_{r\, n_{j+1}} \ .$$

Finally this process yields the matrix

$$\alpha_k \, I_r - F_r - n_{k-1} \, I_r = \beta_k \, I_r - F_r \ .$$

By a similar reasoning as in the Gaussian elimination, it is possible to show that the determinants of the obtained block upper triangular matrix and of M_1 coincide. Eq. (10) follows. $\square$

Let $\Omega = \{ j = 1, \ldots, k-1 \ : \ n_j > n_{j+1} \}$.
Consider the polynomials $Q_0(\lambda) = 1$, $Q_1(\lambda) = \lambda$ and

$$Q_j(\lambda) = \lambda \, Q_{j-1}(\lambda) - \frac{n_{j-1}}{n_j} \, Q_{j-2}(\lambda) \ , \ j = 2, \ldots, k-1 \ .$$

Finally let

$$Q_{\ell,k}(\lambda) = \left(\lambda - 2\cos\frac{\ell\pi}{r+1} \right) Q_{k-1}(\lambda) - \frac{n_{k-1}}{n_k} \, Q_{k-2}(\lambda) \ , \ \ell = 1, \ldots, r \ .$$

Theorem 1 can now be derived from Lemma 8.

P r o o f of Theorem 1. We apply Lemma 8 to the matrix $M_1 = \lambda I - A(T)$. For this matrix,

$$\alpha_j = \lambda \ , \ j = 1, \ldots, k \ .$$

Suppose that $\lambda \in \mathbb{R}$ is such that $Q_j(\lambda) \neq 0$ for all $j = 1, \ldots, k-1$. We have

$$\begin{aligned}
\beta_1 &= \lambda = \frac{Q_1}{Q_0} \neq 0, \\
\beta_2 &= \lambda - \frac{1}{\beta_1} \frac{n_1}{n_2} = \frac{\lambda \, Q_1 - \frac{n_1}{n_2} Q_0}{Q_1} = \frac{Q_2}{Q_1} \neq 0,
\end{aligned}$$

$$\begin{aligned}
&\cdots \quad \cdots \quad \cdots \\
\beta_{k-1} &= \lambda - \frac{1}{\beta_{k-2}} \frac{n_{k-2}}{n_{k-1}} = \frac{Q_{k-1}}{Q_{k-2}} \neq 0 \\
\beta_k - 2\cos\frac{\pi\ell}{r+1} &= \lambda - 2\cos\frac{\pi\ell}{r+1} - \frac{n_{k-1}}{n_k}\frac{1}{\beta_{k-1}} \\
&= \left(\lambda - 2\cos\frac{\pi\ell}{r+1}\right) - \frac{n_{k-1}}{n_k}\frac{Q_{k-2}}{Q_{k-1}} = \frac{Q_{\ell,k}}{Q_{k-1}} \ .
\end{aligned}$$

From Eq. (10) we obtain

$$\det(\lambda\, I - A(T) = \left(\prod_{j=1}^{k-1} \beta_j^{n_j}\right)^r \prod_{\ell=1}^{r} \left(\beta_k - 2\cos\frac{\pi\ell}{r+1}\right) = \left(\prod_{j\in\Omega} Q_j^{n_j - n_{j+1}}\right)^r \prod_{\ell=1}^{r} Q_{\ell,k} \ . \tag{11}$$

Thus, (2) is proved for all $\lambda \in \mathbb{R}$, such that $Q_j(\lambda) \neq 0$, $j = 1, \dots, k-1$.

Suppose now that $Q_j(\lambda_0) = 0$ for some j and for some $\lambda_0 \in \mathbb{R}$. Since the zeros of any nonzero polynomial are isolated, there exists a neighborhood $N(\lambda_0)$ of λ_0 such that $Q_\ell(\lambda) \neq 0$ for all $\lambda \in N(\lambda_0) \setminus \{\lambda_0\}$ and for all $\ell = 1, \dots, k-1$. Hence (11) is proved for all $\lambda \in N(\lambda_0) \setminus \{\lambda_0\}$. By continuity, taking the limit as $\lambda \to \lambda_0$, we have

$$\det(\lambda_0\, I - A(T)) = \left(\prod_{j\in\Omega} Q_j^{n_j - n_{j+1}}(\lambda_0)\right)^r \prod_{\ell=1}^{r} Q_{\ell,k}(\lambda_0) \ .$$

Therefore (2) holds for all $\lambda \in \mathbb{R}$. □

For $\ell = 1, \dots, r$, let $T_{\ell,k}$ be the following $k \times k$ symmetric tridiagonal matrix

$$\begin{bmatrix}
0 & \sqrt{d_2 - 1} & & & \\
\sqrt{d_2 - 1} & 0 & \ddots & & \\
 & \ddots & \ddots & \sqrt{d_{k-1} - 1} & \\
 & & \sqrt{d_{k-1} - 1} & 0 & \sqrt{d_k} \\
 & & & \sqrt{d_k} & 2\cos\frac{\ell\pi}{r+1}
\end{bmatrix} . \tag{12}$$

For $j = 1, \dots, k-1$, let T_j be the $j \times j$ leading principal submatrix of $T_{1,k}$. The following result is a direct consequence of relation (6).

Lemma 9. *For* $j = 1, 2, \dots, k-1$,

$$\det(\lambda I - T_j) = Q_j(\lambda) \tag{13}$$

and for $\ell = 1, 2, \dots, r$,

$$\det(\lambda I - T_{\ell,k}) = Q_{\ell,k}(\lambda)\ . \tag{14}$$

In particular, when we attach r copies of the tree $B_{d,k}$ to the vertices of P_r , then the matrices in (12) (pertaining to $A(T(d,k,r))$) become the following irreducible matrices, [8]:

$$S(\ell,k,r) = \begin{bmatrix} 0 & \sqrt{d} & & \\ \sqrt{d} & \ddots & \ddots & \\ & \ddots & 0 & \sqrt{d} \\ & & \sqrt{d} & 2\cos\frac{\ell\pi}{r+1} \end{bmatrix}\ ,\ \ell = 1, \dots, r\ . \tag{15}$$

Moreover for $j = 1, \dots, k-1$, the submatrices T_j , are equal to the matrices $S_j = \sqrt{d}\, F_j$ where the matrix F_j is as in (8). The energy of these matrices is given by [11]

$$E(S_j) = 2\sqrt{d}\left(\frac{\sin(\lfloor j/2\rfloor + 1/2)\frac{\pi}{j+1}}{\sin\frac{\pi}{2(j+1)}} - 1\right)\ ,\ j = 1, \dots, k-1\ .$$

Moreover [7],

$$n_j - n_{j+1} = d^{k-1-j}\,(d-1)\ . \tag{16}$$

We denote by S_k the matrix $\sqrt{d}\, F_k$, where F_k is as in (8). For matrices in (15) we observe,

$$\lambda_p(S(\ell,k,r)) = -\lambda_{k+1-p}(S(r+1-\ell,k,r))\ ,\ p = 1, \dots, k\ ,\ \ell = 1, \dots, r\ .$$

For $r \geq 2$ this property and the definition of energy imply

$$E(S(\ell,k,r)) = E(S(r+1-\ell,k,r))\ ,\ \ell = 1, \dots, \lfloor r/2\rfloor\ .$$

For odd r and $\ell = 1 + \lfloor r/2\rfloor$, it is $2\cos\frac{\ell\pi}{r+1} = 0$, which implies $S(1 + \lfloor r/2\rfloor, k, r) = S_k$. Therefore we obtain [11],

$$E(S(1+\lfloor r/2\rfloor,k,r)) = 2\sqrt{d}\left(\frac{\sin(\lfloor k/2\rfloor + 1/2)\frac{\pi}{k+1}}{\sin\frac{\pi}{2(k+1)}} - 1\right)\ .$$

The following Lemma is a consequence of Theorem 1 and Lemmas 5, 6 and 7.

Lemma 10. *Let* $1 \leq \ell_1 \leq \ell_2 \leq \lfloor r/2 \rfloor$ *. Then,*

a) $\lambda_p(S(\ell_2,k,r)) \leq \lambda_p(S(\ell_1,k,r))$, $p = 1,\ldots,k$,

b) $\lambda_p(S(\ell,k,r)) < \lambda_p(S_j) < \lambda_{p+k-j}(S(\ell,k,r))$, $p = 1,\ldots,j$, $\ell = 1,\ldots,\lfloor r/2 \rfloor$.

c) The greatest $\lfloor r/2 \rfloor$ *eigenvalues of the tree* $T(d,k,r)$ *are the greatest eigenvalues of the matrices* $S(\ell,k,r)$, $\ell = 1,\ldots,\lfloor r/2 \rfloor$, *whenever* r *is an even number, and the greatest* $\lfloor r/2 \rfloor + 1$ *eigenvalues of the tree* $T(d,k,r)$ *are the greatest* $\lfloor r/2 \rfloor + 1$ *eigenvalues of* $S(\ell,k,r)$, $\ell = 1,\ldots,\lfloor r/2 \rfloor + 1$, *whenever* r *is an odd number.*

d) The greatest eigenvalue of $T(d,k,r)$ *is* $\lambda_k(S(1,k,r))$.

P r o o f. Note that $S(\ell_1,k,r) = S(\ell_2,k,r) + B$, where $B = \begin{bmatrix} \mathbf{0} & 0 \\ 0^T & g \end{bmatrix}$, $g > 0$. By Lemma 5 we have

$$\lambda_q(S(\ell_2,k,r)) + \lambda_{p-q+1}(B) \leq \lambda_p(S(\ell_1,k,r)) \ , \ p \geq q \ .$$

We obtain the first result by taking $q = p$. The second result is obtained by observing that S_j is a $j \times j$ submatrix of the matrix $S(\ell,k,r)$ and by applying Lemma 6. The facts c) and d) are obtained as consequence of the Theorem 1 and of a) and b). □

3. *Example: The energy of* $T(d,2,r)$

In this section, in order to exemplify the general theory from the previous section, we compute the spectrum and energy of $T(d,2,r)$. The same results could have been obtained also by using the Godsil–McKay formula (1). The figure below corresponds to $T(2,2,8)$.

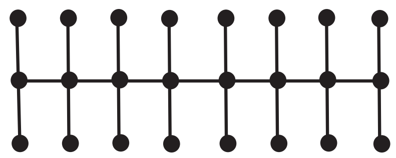

Fig. 2. The tree $T(2,2,8)$.

For the tree $T(d,2,r)$ the matrices in (15) reduce to the following 2×2 matrices:

$$S(\ell,2,r)=\begin{bmatrix} 0 & \sqrt{d} \\ \sqrt{d} & 2\cos\frac{\ell\pi}{r+1}\end{bmatrix}\ ,\ \ell=1,\ldots,\lfloor r/2\rfloor$$

with eigenvalues

$$\cos\frac{\ell\pi}{r+1}\pm\sqrt{d+\cos^2\frac{\ell\pi}{r+1}}\quad ,\quad \ell=1,\ldots,\lfloor r/2\rfloor\ .$$

All other eigenvalues of $T(d,2,r)$ are equal to zero. Thus

$$\begin{aligned} E(S(\ell,2,r)) &= \left|\cos\frac{\ell\pi}{r+1}+\sqrt{d+\cos^2\frac{\ell\pi}{r+1}}\right|+\left|\cos\frac{\ell\pi}{r+1}-\sqrt{d+\cos^2\frac{\ell\pi}{r+1}}\right| \\ &= 2\sqrt{d+\cos^2\frac{\ell\pi}{r+1}}\ . \end{aligned}$$

Then the decomposition in (2) and the matrices in (15) imply

Theorem 11. *The energy $E(T(d,2,r))$ is given by Eqs. (3) and (4).*

4. *Bounds for the energy of certain matrices*

Let $\alpha>0$ and $h\geq 0$. Consider the tridiagonal symmetric $k\times k$ matrix

$$M:=M(\alpha,h)=\begin{bmatrix} M_{k-1} & (0,0,\ldots,0,\alpha)^T \\ (0,0,\ldots,0,\alpha) & h\end{bmatrix} \tag{17}$$

where $M_{k-1}=\alpha\,F_{k-1}$ is given by Eq. (9). Then,

$$\lambda_j(M_{k-1})=2\alpha\cos\frac{(k-j)\pi}{k}\ ,\ j=1,\ldots,k-1\ . \tag{18}$$

Lemma 12a. *Let $k=2p$. Then for $i=2,\ldots,k-1$,*

$$2\alpha\cos\frac{i\pi}{k}<|\lambda_i(M)|<2\alpha\cos\frac{(i-1)\pi}{k}\ ,\ i=2,\ldots,p \tag{19}$$

and

$$2\alpha\cos\frac{(k-i+1)\pi}{k} < |\lambda_i(M)| < 2\alpha\cos\frac{(k-i)\pi}{k}\ ,\ i=p{+}1,\ldots,k{-}1\ . \quad (20)$$

P r o o f. We consider the ordered eigenvalues

$$\lambda_1(M) \le \lambda_2(M) \le \cdots \le \lambda_k(M)\ .$$

By Lemmas 5 and 6, and by using the eigenvalues in (18), we derive,

$$2\alpha\cos\frac{(k-i+1)\pi}{k} < \lambda_i(M) < 2\alpha\cos\frac{(k-i)\pi}{k}\ ,\ i=2,\ldots,k-1\ . \quad (21)$$

Hence, if $i \le p$ then $2\alpha\cos\frac{(k-i)\pi}{k} \le 0$ implying $\lambda_i(M) < 0$. By multiplying in (21) by (-1) the inequalities in (19) are obtained. In a same way, if $i \ge p+1$, then by $2\alpha\cos\frac{(k-i+1)\pi}{k} > 0$ we have $\lambda_i(M) \ge 0$, and (20) follows from (21). □

Lemma 12b. *Let* $k = 2p+1$. *Then*

$$2\alpha\cos\frac{i\pi}{k} < |\lambda_i(M)| < 2\alpha\cos\frac{(i-1)\pi}{k}\ ,\ i=2,\ldots,p \quad (22)$$

$$0 \le |\lambda_{p+1}(M)| < 2\alpha\cos\frac{p\,\pi}{k} \quad (23)$$

and

$$2\alpha\cos\frac{(k-i+1)\pi}{k} < |\lambda_i(M)| < 2\alpha\cos\frac{(k-i)\pi}{k}\ ,\ i=p{+}2,\ldots,k{-}1\ . \quad (24)$$

P r o o f. As before, the inequalities (21) imply (22). For $i = p+1$ in (21) we obtain

$$2\alpha\cos\frac{(p+1)\pi}{2p+1} < \lambda_{p+1}(M) < 2\alpha\cos\frac{p\,\pi}{2p+1}\ .$$

Hence, by using the identity

$$\cos\frac{(p+1)\pi}{2p+1} = -\cos\frac{p\pi}{2p+1}$$

we arrive at (23). If $i \geq p+2$, then $2\alpha \cos \frac{(k-i+1)\pi}{k} > 0$. Therefore, (24) follows directly from (21). $\square$

Let $b := b(k)$ be defined as

$$b := \sum_{i=1}^{\lfloor k/2 \rfloor} \left(2 \cos \frac{i\pi}{k} \right) = \frac{\sin(\lfloor k/2 \rfloor + 1/2)\frac{\pi}{k}}{\sin \frac{\pi}{2k}} - 1 \tag{25}$$

where the last identity is implied by (7).

Lemma 13. *Let k be an even number. Let b be as in (25). Consider the tridiagonal symmetric $k \times k$ matrix $M = M(\alpha, h)$, Eq. (17), and suppose that $a_1 < |\lambda_1(M)| < b_1$ and $a_k < |\lambda_k(M)| < b_k$. Then*

$$a_1 + a_k - 4\alpha \cos \frac{\pi}{k} < E(M) - 2\alpha b < b_1 + b_k \ .$$

P r o o f. Let $k = 2p$. Thus $\lfloor k/2 \rfloor = p$. Clearly

$$E(M) = |\lambda_1(M)| + |\lambda_k(M)| + \sum_{i=2}^{p} |\lambda_i(M)| + \sum_{i=p+1}^{k-1} |\lambda_i(M)| \ . \tag{26}$$

By summation from $i = 2$ to $i = p$ in the inequalities in (19) we obtain

$$\alpha \sum_{i=2}^{p} 2 \cos \frac{i\pi}{k} < \sum_{i=2}^{p} |\lambda_i(M)| < \alpha \sum_{i=2}^{p} 2 \cos \frac{(i-1)\pi}{k} \ .$$

For this case

$$\sum_{i=2}^{p} 2 \cos \frac{(i-1)\pi}{k} = \sum_{i=1}^{p} 2 \cos \frac{i\pi}{k} = b \ .$$

Thus from the above inequalities and (25) we have

$$\alpha \left(b - 2 \cos \frac{\pi}{k} \right) < \sum_{i=2}^{p} |\lambda_i(M)| < \alpha b \ . \tag{27}$$

In (20) we take the sum from $i = p+1$, to $i = k-1$, thus obtaining

$$\alpha \sum_{i=p+1}^{k-1} 2 \cos \frac{(k-i+1)\pi}{k} < \sum_{i=p+1}^{k-1} |\lambda_i(M)| < \alpha \sum_{i=p+1}^{k-1} 2 \cos \frac{(k-i)\pi}{k} \ . \tag{28}$$

By a change of variable,

$$\alpha \sum_{i=p+1}^{k-1} 2\cos\frac{(k-i+1)\pi}{k} = \alpha \sum_{j=2}^{p} 2\cos\frac{j\pi}{k} = \alpha\left(b - 2\cos\frac{\pi}{k}\right)$$

and

$$\alpha \sum_{i=p+1}^{k-1} 2\cos\frac{(k-i)\pi}{k} = \alpha \sum_{j=1}^{p-1} 2\cos\frac{j\pi}{k} = \alpha b\ .$$

Thus, inequalities in (28) and (25) imply

$$\alpha\left(b - 2\cos\frac{\pi}{k}\right) < \sum_{i=p+1}^{k-1} |\lambda_i(M)| < \alpha b\ . \tag{29}$$

Now the result is obtained directly from (26), bounds in (27) and (29) and the bounds given for $|\lambda_1(M)|$ and $|\lambda_k(M)|$. □

Lemma 14. *Let k be an odd integer. Let b be as in (25) and $M = M(\alpha, h)$ as in (17). Suppose that $a_1 < |\lambda_1(M)| < b_1$ and $a_k < |\lambda_k(M)| < b_k$. Then*

$$a_1 + a_k - 4\alpha\cos\frac{\pi}{k} < E(M) - 2\alpha b < b_1 + b_k - 2\alpha\cos\frac{\lfloor k/2\rfloor\pi}{k}\ .$$

P r o o f. Let $k = 2p+1$. Thus $\lfloor k/2\rfloor = p$. Evidently,

$$E(M) = |\lambda_1(M)| + |\lambda_k(M)| + \sum_{i=2}^{p} |\lambda_i(M)| + \sum_{i=p+2}^{k-1} |\lambda_i(M)| + |\lambda_{p+1}(M)|\ . \tag{30}$$

By summation from $i = 2$ to $i = p$ in (22) we obtain

$$\alpha \sum_{i=2}^{p} 2\cos\frac{i\pi}{k} < \sum_{i=2}^{p} |\lambda_i(M)| < \alpha \sum_{i=2}^{p} 2\cos\frac{(i-1)\pi}{k}\ .$$

Hence by (25) we directly obtain,

$$\alpha\left(b - 2\cos\frac{\pi}{k}\right) < \sum_{i=2}^{p} |\lambda_i(M)| < \alpha\left(b - 2\cos\frac{p\,\pi}{k}\right)\ . \tag{31}$$

By summing from $i = p+2$ to $i = k-1$ in (24) we have

$$\alpha \sum_{i=p+2}^{k-1} 2\cos\frac{(k-i+1)\pi}{k} < \sum_{i=p+2}^{k-1} |\lambda_i(M)| < \alpha \sum_{i=p+2}^{k-1} 2\cos\frac{(k-i)\pi}{k} \ . \quad (32)$$

By a change of variable,

$$\alpha \sum_{i=p+2}^{k-1} 2\cos\frac{(k-i+1)\pi}{k} = \alpha \sum_{j=2}^{p} 2\cos\frac{j\pi}{k} = \alpha\left(b - 2\cos\frac{\pi}{k}\right)$$

and

$$\alpha \sum_{i=p+2}^{k-1} 2\cos\frac{(k-i)\pi}{k} = \alpha \sum_{j=1}^{p-1} 2\cos\frac{j\pi}{k} = \alpha\left(b - 2\cos\frac{p\pi}{k}\right) \ .$$

Thus from the inequalities in (32) and by (25) we get

$$\alpha\left(b - 2\cos\frac{\pi}{k}\right) < \sum_{i=p+2}^{k-1} |\lambda_i(M)| < \alpha b - 2\alpha\cos\frac{p\pi}{k} \ . \quad (33)$$

The result is directly obtained from (30), bounds in (31), (33), and (23) as well as the bounds given for $|\lambda_1(M)|$ and $|\lambda_k(M)|$. $\square$

We recall that for the extreme eigenvalues of the matrix $M(\alpha, 0)$ in (17),

$$|\lambda_1(M(\alpha,0))| = |\lambda_k(M(\alpha,0))| = 2\alpha\cos\frac{\pi}{k+1} \ .$$

Lemma 15. *Let* $h > 0$ *. Then*

$$2\alpha\cos\frac{\pi}{k} < |\lambda_1(M(\alpha,h))| < 2\alpha\cos\frac{\pi}{k+1} \quad (34)$$

and

$$2\alpha\cos\frac{\pi}{k+1} \le |\lambda_k(M(\alpha,h))| \ . \quad (35)$$

P r o o f. Consider the ordered eigenvalues of $M(\alpha, h)$,

$$\lambda_1(M(\alpha,h)) \le \lambda_2 M(\alpha,h)) \le \cdots \le \lambda_k(M(\alpha,h))$$

and observe that

$$M(\alpha,h) = \alpha\, F_k + B \quad \text{ where } \; B = \begin{bmatrix} \mathbf{0} & 0 \\ 0^T & h \end{bmatrix} .$$

By Lemma 5 we have

$$\lambda_j(\alpha F_k) + \lambda_{i-j+1}(B) \le \lambda_i(M(\alpha,h)) \ , \ i \ge j \ . \tag{36}$$

In (36), $i = j = 1$ imply

$$2\alpha \cos\frac{k\pi}{k+1} + \lambda_1(B) \le \lambda_1(M(\alpha,h)) \ .$$

In this case $\lambda_1(B) = 0$. Therefore

$$2\alpha \cos\frac{k\pi}{k+1} \le \lambda_1(M(\alpha,h)) \ .$$

On the other hand, by using Lemma 7 we have

$$\lambda_1(M(\alpha,h)) < \lambda_1(M_{k-1}) = 2\alpha \cos\frac{(k-1)\pi}{k} \ .$$

Therefore

$$2\alpha \cos\frac{k\pi}{k+1} \le \lambda_1(M(\alpha,h)) < 2\alpha \cos\frac{(k-1)\pi}{k}$$

and (34) is obtained by multiplying the above inequalities by (-1). In (36) we consider $i = j = k$ and obtain

$$\lambda_k(M(\alpha,h)) \ge \lambda_k(\alpha F_k) + \lambda_1(B) = 2\alpha \cos\frac{\pi}{k+1}$$

which leads to (35). □

Theorem 3 is now obtained as an immediate corollary of Lemmas 13, 14, and 15.

5. *Estimating the energy of* $T(d,k,r)$

P r o o f of Theorem 4. Let

$$P = \begin{bmatrix} F_{k-1} & (0,0,\ldots,0,1)^T \\ (0,0,\ldots,0,1) & 1 \end{bmatrix}$$

with F_{k-1} given in Eq. (8). Let $\upsilon = \max\{\alpha, h\}$. Then $M(\alpha, h) \leq \upsilon P$. Both $M(\alpha, h)$ and υP are irreducible matrices. Then by comparing their spectral radii [9] we have

$$\lambda_k(M(\alpha, h)) \leq \lambda_k(\upsilon P) = 2\upsilon \cos \frac{\pi}{2k+1} \ .$$

Thus $B_1(\mu)$ is an upper bound of μ. By the Gershgorin theorem [9], $B_2(\mu)$ is an other upper bound of μ. Theorem 4 follows. □

Example. For the 20×20 matrix $M(\sqrt{2}, 2)$ defined by taking $k = 20$ in (17) we have $E(M(\sqrt{2}, 2)) = 35.9051$. Our lower and upper bounds are 31.4536 and 37.6614, respectively.

By applying Theorem 1 to the tree $T(d, k, r)$, the matrices in (15) are obtained. By means of the decomposition (2), and the relations (13), (14), (16), we obtain

$$E(T(d,k,r)) = r \sum_{j=1}^{k-1} d^{k-1-j} \, (d-1) E(S_j) + 2 \sum_{\ell=1}^{\lfloor r/2 \rfloor} E(S(\ell, k, r))$$

for r even, and

$$E(T(d,k,r)) =$$
$$r \sum_{j=1}^{k-1} d^{k-1-j} \, (d-1) E(S_j) + \sum_{\ell=1}^{\lfloor r/2 \rfloor} 2E(S(\ell, k, r)) + E(S(1 + \lfloor r/2 \rfloor, k, r))$$

for odd r. We recall that $E(B_{d,k})$ denotes the energy of the Bethe tree $B_{d,k}$. In [11] it was proven that

$$\sum_{j=1}^{k-1} d^{k-1-j} \, (d-1) E(S_j) = E(B_{d,k}) - E(S_k) \ .$$

Therefore

$$E(T(d,k,r)) - r\,E(B_{d,k}) + r\,E(S_k) = 2 \sum_{\ell=1}^{\lfloor r/2 \rfloor} E(S(\ell, k, r)) \tag{37}$$

for even r and

$$E(T(d,k,r)) - r\,E(B_{d,k}) + (r-1) E(S_k) = 2 \sum_{\ell=1}^{\lfloor r/2 \rfloor} E(S(\ell, k, r)) \tag{38}$$

whenever r is odd. For the matrices $M(\alpha,h)$ in (17) and $S(\ell,k,r)$ we obtain

$$S(\ell,k,r) = M\left(\sqrt{d}, 2\cos\frac{\ell\pi}{r+1}\right) \ , \ \ell = 1,\dots,\lfloor r/2\rfloor$$

and by the results from Section 4, their energies are bounded as follows, for $\ell = 1,\dots,\lfloor r/2\rfloor$,

$$-2\sqrt{d}\cos\frac{\pi}{k} < E(S(\ell,k,r)) - 2\sqrt{d}\left(b+\cos\frac{\pi}{k+1}\right) < B(\ell,k,r)$$

for $\ell = 1,\dots,\lfloor r/2\rfloor$, and even k , and

$$-2\sqrt{d}\cos\frac{\pi}{k} < E(S(\ell,k,r)) - 2\sqrt{d}\left(b+\cos\frac{\pi}{k+1}\right) < B(\ell,k,r) - 2\sqrt{d}\cos\frac{\lfloor k/2\rfloor\pi}{k}$$

for $\ell = 1,\dots,\lfloor r/2\rfloor$, and odd k , By Theorem 4 we have

$$B(\ell,k,r) = \min\{B_1(\ell,k,r) \ , \ B_2(\ell,k,r)\} \ , \ \ell = 1,\dots,\lfloor r/2\rfloor$$

where

$$B_1(\ell,k,r) = \max\left\{2\sqrt{d}\cos\frac{\pi}{2k+1} \ , \ 4\cos\frac{\ell\pi}{r+1}\cos\frac{\pi}{2k+1}\right\} \ , \ \ell = 1,\dots,\lfloor r/2\rfloor$$

and

$$B_2(\ell,k,r) = \max\left\{2\sqrt{d} \ , \ \sqrt{d} + 2\cos\frac{\ell\pi}{r+1}\right\} \ , \ \ell = 1,\dots,\lfloor r/2\rfloor \ .$$

From this we obtain

$$2\sum_{\ell=1}^{\lfloor r/2\rfloor} E(S(\ell,k,r)) < 4\sqrt{d}\left\lfloor\frac{r}{2}\right\rfloor\left(b+\cos\frac{\pi}{k+1}\right) + 2\sum_{\ell=1}^{\lfloor r/2\rfloor} B(\ell,k,r) \tag{39}$$

and

$$4\left\lfloor\frac{r}{2}\right\rfloor\sqrt{d}\left(b+\cos\frac{\pi}{k+1} - \cos\frac{\pi}{k}\right) < 2\sum_{\ell=1}^{\lfloor r/2\rfloor} E(S(\ell,k,r)) \tag{40}$$

for even k , and

$$2\sum_{\ell=1}^{\lfloor r/2\rfloor} E(S(\ell,k,r)) < 4\sqrt{d}\left\lfloor\frac{r}{2}\right\rfloor\left(b+\cos\frac{\pi}{k+1} - \cos\frac{\lfloor k/2\rfloor\pi}{k}\right) + 2\sum_{\ell=1}^{\lfloor r/2\rfloor} B(\ell,k,r) \tag{41}$$

and

$$4\left\lfloor\frac{r}{2}\right\rfloor\sqrt{d}\left(\cos\frac{\pi}{k+1}+b-\cos\frac{\pi}{k}\right)<2\sum_{\ell=1}^{\lfloor r/2\rfloor}E(S(\ell,k,r))\,. \qquad (42)$$

for odd k.

Then for the energy $E(T(d,k,r))$, from (37), (39), and (40) we obtain

$$E(T(d,k,r))-rE(B_{d,k})+rE(S_k)<4\sqrt{d}\left\lfloor\frac{r}{2}\right\rfloor\left(\cos\frac{\pi}{k+1}+b\right)+2\sum_{\ell=1}^{\lfloor r/2\rfloor}B(\ell,k,r)$$

and

$$4\sqrt{d}\left\lfloor\frac{r}{2}\right\rfloor\left(\cos\frac{\pi}{k+1}+b-\cos\frac{\pi}{k}\right)<E(T(d,k,r))-rE(B_{d,k})+rE(S_k)$$

whenever k and r are even numbers. Moreover from (37), (41), and (42) we obtain

$$\begin{aligned}&E(T(d,k,r))-rE(B_{d,k})+rE(S_k)\\<\ &4\sqrt{d}\left\lfloor\frac{r}{2}\right\rfloor\left(\cos\frac{\pi}{k+1}+b-\cos\frac{\lfloor k/2\rfloor\pi}{k}\right)+2\sum_{\ell=1}^{\lfloor r/2\rfloor}B(\ell,k,r)\end{aligned}$$

and

$$4\sqrt{d}\left\lfloor\frac{r}{2}\right\rfloor\left(\cos\frac{\pi}{k+1}+b-\cos\frac{\pi}{k}\right)<E(T(d,k,r))-rE(B_{d,k})+rE(S_k)$$

whenever r is an even number and k is an odd number.

From (38), (39), and (40) we obtain,

$$E(T(d,k,r))-rE(B_{d,k})+(r-1)E(S_k)<4\sqrt{d}\left\lfloor\frac{r}{2}\right\rfloor\left(\cos\frac{\pi}{k+1}+b\right)+2\sum_{\ell=1}^{\lfloor r/2\rfloor}B(\ell,k,r)$$

and

$$4\sqrt{d}\left\lfloor\frac{r}{2}\right\rfloor\left(\cos\frac{\pi}{k+1}+b-\cos\frac{\pi}{k}\right)<E(T(d,k,r))-rE(B_{d,k})+(r-1)E(S_k)$$

whenever r is odd and k is even.

From (38), (41), and (42) we obtain,

$$E(T(d,k,r)) - rE(B_{d,k}) + (r-1)E(S_k)$$

$$< \; 4\sqrt{d}\left\lfloor \frac{r}{2} \right\rfloor \left(\cos\frac{\pi}{k+1} + b - \cos\frac{\lfloor k/2 \rfloor \pi}{k} \right) + 2\sum_{\ell=1}^{\lfloor r/2 \rfloor} B(\ell,k,r)$$

and

$$4\left\lfloor \frac{r}{2} \right\rfloor \sqrt{d}\left(\cos\frac{\pi}{k+1} + b - \cos\frac{\pi}{k} \right) < E(T(d,k,r)) - rE(B_{d,k}) + (r-1)E(S_k)$$

whenever r is odd and k is odd.

Example.

d	k	r	$E(T(d,k,r))$	lower bound	upper bound
2	5	4	105.00	97.59	114.74
2	4	3	29.78	24.90	34.21
4	4	2	130.25	124.18	137.36
4	5	3	861.58	856.63	868.31

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